Appl. Math. and Comp. Sci., 1997, Vol.7, No.4, 737–774

CONTROL SYSTEMS THEORY FOR LINEAR REPETITIVE PROCESSES — RECENT PROGRESS AND OPEN RESEARCH PROBLEMS

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The unique feature of a repetitive process is a series of sweeps, termed passes, through a set of dynamics defined over a finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the next pass profile. This, in turn, is the source of the unique control problem for these processes in that the output sequence of pass profiles can contain oscillations that increase in amplitude in the pass to pass direction. It has long been recognised that this general problem cannot be removed by standard control action and hence the need for a rigorous control theory for these processes which are of both theoretical and applications interest. This paper critically reviews progress to date and identifies a number of key areas for short to medium term further research/development work.

1. Introduction

Repetitive, or multipass, processes are uniquely characterised by a series of sweeps, termed passes, through a set of dynamics defined over a finite fixed duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the next pass profile. Industrial examples include long-wall coal cutting (first 'identified' as a multipass process by Edwards (1974)) and metal rolling operations and algorithmic examples include classes of iterative learning control schemes. More recently, it has been shown by Roberts (1996) that a repetitive process problem formulation can be used to study properties of iterative solution algorithms for classes of dynamic nonlinear optimal control problems based on the maximum principle.

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It is necessary to use two co-ordinates to specify a variable in a repetitive process, i.e. the pass number, or index, $k \geq 0$, and the 'position' t along a given pass which, by definition, is finite and fixed and denoted here by α. Suppose also that $y_k(t)$, $0 \leq t \leq \alpha$, denotes the pass profile produced on pass k. Then in a repetitive process $y_k(t)$ acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile $y_{k+1}(t)$, $k \geq 0$.

Repetitive processes also exist where it is the previous $M > 1$ pass profiles which explicitly contribute to the current one. An example here is bench mining systems used to extract coal from 'relatively rich seam' coal mines (Smyth, 1992). Such processes are termed non-unit memory with memory length M. In this paper, however, it is only the unit memory case (i.e. $M = 1$) which is considered since the analysis presented generalises in a natural manner to non-unit memory examples.

The essential unique control problem for these processes arises directly from the explicit interaction between successive pass profiles. In essence, the sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass to pass direction. Such behaviour is easily generated in simulation studies and in experiments on scaled models of industrial examples (Smyth, 1992).

Early attempts to control these processes consisted, in effect, in direct application of standard linear systems techniques. The essential basis of this approach was to use the concept of the total distance traversed to write the process dynamics as an infinite length single pass process and then, for example, apply the Laplace transform to obtain a transfer function description of the resulting standard, or 1D, linear system. The basics of this approach can be found in (Edwards, 1974) and the relevant cited references.

Other work subsequently showed that this and other approaches based on direct application of 1D linear systems tools are incorrect (except in a few very restrictive special cases). The precise reason for this is that such an approach essentially ignores the key features of their underlying dynamics, i.e. the fact that they repeatedly operate over a finite duration and the influences/consequences of the resetting of the pass initial conditions. These features are similar to those of 2D linear systems recursive over the positive quadrant, i.e. systems which propagate information in two separate directions (usually termed 'horizontal' and 'vertical', respectively).

The fact that these processes cannot be analysed/controlled by direct application of existing theory/techniques has resulted in a considerable volume of work on the development of appropriate tools for their analysis and control, starting with a suitable stability theory which obviously defines the system to be stable if the sequence of pass profiles does not contain oscillations that increase in amplitude in the pass to pass direction (in a well-defined sense). The main purpose of this paper is to critically review the work to-date and identify areas for short to medium term further research effort. One key message to emerge is that direct application of 2D linear systems theory is not always possible, even for special cases which can be directly modelled by one of the well-known state-space models for 2D linear systems recursive over the positive quadrant.

2. Background

The essential unique feature of a repetitive, or multipass, process can be illustrated by considering machining operations where the material, or workpiece, involved is processed by a sequence of sweeps, or passes, of the processing tool. In particular, suppose that the necessarily finite pass length (or duration) α is constant and let $y_k(t)$, $0 \leq t \leq \alpha$, where t is the independent spatial or temporal variable, denote the output vector or pass profile produced on pass $k \geq 0$. Then in a repetitive process $y_k(t)$ acts as a forcing function on, and hence contributes to, the dynamics of the new pass profile $y_{k+1}(t)$, $k \geq 0$.

An industrial example is long-wall coal cutting which is the main method of extracting coal from deep cast mines in Great Britain. In long-wall coal cutting, roadways are machined and maintained open (a non-trivial task) at each end of the coal seam. The coal is then removed by a series of sweeps of a cutting machine along the coal face which is perpendicular to the roadways in plan view. During this operation, the machine (which can be up to 5 tonnes 'dead weight') is hauled along the coal face resting on the so-called armoured face conveyor (AFC)—a collection of loosely joined steel pans—which transports away the coal cut by the rotating drum. A nucleonic coal sensor, situated some distance behind the cutting head, provides the primary feedback control signal, where the basic objective is to steer the head within the undulating confines of the coal seam. Aside from the obvious objective of maximising the amount of coal cut, penetration of the stone-coal interface must be avoided on both economic and safety grounds.

At the end of each pass, the so-called pushover phase takes place in preparation for the next pass of the coal cutter. This phase begins by hauling back the machine in reverse at (relatively) high speed to its starting position. Hydraulic rams in the roof-floor support units are then used to 'snake over' the complete installation—machine, conveyor, and support units—such that it rests on the newly cut floor profile ready for the start of the next pass of the coal face. A simple consideration of the basic system geometry and dynamics confirms that long-wall coal cutting is indeed a repetitive process.

In this particular application, the pass length α can be up to 300 m in some cases and the pass profile is the height of the stone coal interface above a fixed datum. Then with the further assumption that the conveyor moulds itself exactly onto the newly cut floor profile (the so-called 'rubber conveyor' assumption) a simple model of the process dynamics is

$$y_{k+1}(t) = -k_1 y_{k+1}(t - X) + k_2 y_k(t) + k_1 R_{k+1}(t), \quad 0 \leq t \leq \alpha, \quad k \geq 0 \qquad (1)$$

with (assumed with no loss of generality) pass initial conditions

$$y_{k+1}(\tau) = 0, \qquad -X \leq \tau \leq 0 \qquad (2)$$

In (1), k_1 and k_2 are positive real constants, $R_{k+1}(t)$ is the control input signal (or the reference signal closed loop), X is the transport lag (time delay) by which the sensor lags the centre of the cutting drum in the along the pass direction, and $y_k(t)$—the pass profile—is the height of the stone/coal interface above a fixed datum.

In modelling studies on coal cutting operations, it is common to judge the system's response to the input signal (or reference signal closed loop)

$$R_{k+1}(t) = -1, \quad y_0(t) = 0, \quad 0 \leq t \leq \alpha, \;\; k \geq 0 \tag{3}$$

i.e. a downward unit step applied at $t = 0$ on each pass and zero initial pass profile. Using this signal it is possible (by suitable choices of k_1 and k_2) to generate examples where, for example, the first profile which is an acceptable 'classical' response, but oscillations are present in successive passes which increase in amplitude at a 'massive' rate in the pass-to-pass direction. This feature can only be caused by the interaction between successive pass profiles and is a simple demonstration of the essential unique control problem for repetitive processes.

Of course, an in-depth analysis based on the assumptions introduced above is unrealistic, since it completely ignores the considerable distortion caused to the previous pass profile by the machine's weight during the 'snake-over' phase. This is termed interpass interaction, and the modelling of this dynamic behaviour is a non-trivial task. Several promising models have been developed, however, for a range of coal cutting and other physical examples where significant interpass interaction also occurs (Smyth, 1992). In all these cases, the paramount task is to include the crucial interpass interaction explicitly.

Despite its simplicity, the model for the long wall coal cutting process given above has played a very useful role in the basic understanding of the underlying dynamics. In particular, it predicts the basic unique control problem for repetitive processes in that the output sequence can contain oscillations that increase in amplitude in the pass-to-pass direction. Physically this problem appears as severe undulations in the newly cut floor profile which means that cutting operations (i.e. productive work) must be suspended to enable their manual removal. This problem is one of the key factors behind the 'stop/start' cutting pattern in a typical working cycle in a coal mine.

Early attempts at stability analysis for repetitive processes represented by linear models basically first attempted to transform them into standard linear systems—here termed 1D linear systems. These were based on first writing repetitive process variables, such as $y_{k+1}(t)$, in terms of the so-called total distance traversed, using the single variable $V = k\alpha + t$. This converted the dynamics into those of an equivalent 'infinite length single pass' process to which it was claimed that standard techniques, such as Nyquist diagrams, could be used to predict stability and hence on to controller design. For example, in terms of the total distance traversed, the simplified model of the long wall coal cutting process is

$$y(V) = -k_1 y(V - X) + k_2 y(V - \alpha) + k_1 R(V) \tag{4}$$

and the claim then was that this repetitive process is stable if, and only if, the system of (4) is stable in the standard, or 1D, linear sense.

In fact, this claim is false (Smyth, 1992) except in a very few and highly restrictive special cases. The basic reasons for this is that it completely neglects

1. the 'stabilising influences' of resetting the initial conditions on each pass—for example, the long wall coal cutter starts each pass from a fixed height above the stone coal interface; and
2. the essential finite pass length 'repeatable' nature of these processes.

Given this most basic systems theoretic problem, there is clearly a pressing need to develop a rigorous stability theory for repetitive processes on which to base the development of a rigorous control theory and hence controller design algorithms. Smyth (1992) and the relevant cited references give a detailed treatment of the modelling and related analysis of long wall coal cutting and other physical examples of repetitive processes such as metal rolling operations. Also recent years have seen the emergence of so-called algorithmic examples which have the structure of a repetitive process. One such case is iterative learning control as discussed below.

Iterative learning control has its origins in the analysis and control of systems which are required to continually perform the same task. The general objective is to sequentially improve performance. Physical examples can be found across a wide range of areas and, in particular, the general area of robotics. A simple example in this last case is a robot manipulator which is required to repetitively follow the same geometric trajectory or path (Arimoto *et al.*, 1984).

Suppose now that a standard (or 'normal') feedback control law is applied to such a problem. Then clearly the resulting closed loop system would exhibit the same performance on each repetition or trial. Motivated by human learning, the basic idea of iterative learning control (denoted by ILC from this point onwards) is to use information from previous trials to improve performance from trial to trial. The ultimate aim (in theory) is to learn the precise input needed to achieve the desired output (which can, of course, be vector-valued) over the trial length which is, by definition, finite. In practice, the objective will be to reach the reference signal within a given accuracy or tolerance (suitably defined).

Intuitively, improvements to performance in this setting mean reductions (in some well-defined sense (such as point-wise or average)) in the error between the actual and desired system outputs over the trial length. Given the essential basis of ILC schemes, signals/measurements from previous trials are a natural choice of data for use in constructing the inputs for the current trial. The control system is hence said to 'learn' by remembering the effectiveness of previously employed inputs and uses data on their success or failure to construct new trial inputs—suggesting an inherent repetitive process structure. It is important to stress that the learning mechanism is by iteration and what is learned is the control input which, in theory, ensures that the system output equals the desired output at all points along the trial length. Hence in contrast to adaptive control, ILC schemes do not attempt to explicitly identify the plant dynamics, but only change or adapt the control input.

Generally speaking, the available results on ILC can be divided into two classes, the first of which is a special class of nonlinear dynamics which arise in particular application areas such as robotics (Arimoto *et al.*, 1984). The other class is linear systems treated in a general setting and hence well-known and extensively used analysis tools, such as frequency domain methods, are available. Considerable progress has been made using these methods but they do not allow the use of combinations of feedforward, i.e. from the results of previous trials, and feedback, i.e. from the current trial error, action which, intuitively, should yield much superior performance. Recent work has shown that treating ILC schemes in a repetitive process framework allows such action to great effect. The basics of this analysis is as follows.

The mathematical definition of the ILC schemes considered in this work is as follows.

Definition 1. Consider a dynamical system with input u and output y. Suppose also that U and Y denote the input and output function spaces respectively and let $r \in Y$ be a desired reference signal from this system. Then an ILC scheme for this system is said to be *successful* if, and only if, it constructs a sequence of control inputs $\{u_k\}_{k\geq 1}$ which when applied to the system (under the same experimental conditions) produces an output sequence $\{y_k\}_{k\geq 1}$ with the following so-called properties of convergent learning

$$\lim_{k\to\infty} y_k = r, \quad \lim_{k\to\infty} u_k = u_\infty \tag{5}$$

Here convergence is interpreted in terms of the topologies assumed on U and Y and u_∞ is termed the learned control. Note also that this general setting simultaneously describes linear and nonlinear dynamics, continuous or discrete systems, and time-varying or time-invariant dynamics. A detailed treatment of this aspect can be found in (Amann, 1996).

Suppose now that e_k denotes the tracking error on trial k, i.e. $e_k = r - y_k$. Then one class of ILC schemes whose properties have a natural repetitive process structure are those which construct the input u_{k+1} on trial $k+1$, $k \geq 0$, as a function of the current and previous errors and possibly the previous inputs, e.g. a recursive functional mapping of the form $u_{k+1} = f(e_{k+1}, e_k, u_k)$. Also Definition 1 essentially refines the intuitive concept of 'improving performance progressively' from trial to trial into a convergence condition on the learning error of the form

$$\lim_{k\to\infty} ||e_k(\cdot)|| = 0 \tag{6}$$

where $||(\cdot)||$ denotes the norm on the underlying function space.

Ideally the algorithm used should produce an error sequence $\{e_k\}_{k\geq 1}$ with the monotonic property that $||e_{k+1}|| \leq ||e_k||$, $k \geq 1$. It is possible to show that this condition can be achieved in certain cases but, in general, it is restrictive (Amann, 1996). A more practically relevant alternative is to ask that convergence is geometric in the sense that it can be bounded above by a geometric expression of the form $M\lambda^k$ for some choice of real numbers $M > 0$ and $\lambda \in (0,1)$.

The stability theory for linear repetitive processes finds direct application in characterising the convergence properties of the ILC schemes introduced above. Note also that the results obtained are, as shown in the next section, far superior to those available by any other approach. This analysis is, in effect, based on the application of the general stability theory for linear repetitive processes (Rogers and Owens, 1992a) to the sub-classes of so-called differential and discrete linear repetitive processes. Here we only consider the differential case (since the results for the discrete case follow naturally) where the state space description of a differential linear repetitive process has the structure

$$\begin{aligned}\dot{x}_{k+1}(t) &= Ax_{k+1}(t) + Bu_{k+1}(t) + B_0 y_k(t)\\ y_{k+1}(t) &= Cx_{k+1}(t) + D_0 y_k(t), \quad 0 \le t \le \alpha, \quad k \ge 0\end{aligned} \tag{7}$$

Here on pass k, $x_k(t)$ is the $n \times 1$ state vector, $y_k(t)$ is the $m \times 1$ vector pass profile, and $u_k(t)$ is the $l \times 1$ vector of control inputs.

To complete the description of a differential linear repetitive process, it is necessary to specify the 'initial conditions', i.e. the initial pass profile and the initial state vector on each pass. The simplest possible form for these is

$$\begin{aligned}x_{k+1}(0) &= d_{k+1}, \quad k \ge 0\\ y_0(t) &= f(t), \quad 0 \le t \le \alpha\end{aligned} \tag{8}$$

Here d_{k+1} is an $n \times 1$ column vector with constant entries and $f(t)$ is a known function of t.

Applications do exist, however, where it is necessary to consider pass initial conditions which are a function of the previous pass profile. An example here is the use of the theory of discrete linear repetitive processes to study the convergence properties of iterative solution algorithms to dynamic nonlinear optimal control problems based on the maximum principle (Roberts, 1996). A more general set of pass initial conditions for differential linear repetitive processes is

$$x_{k+1}(0) = d_{k+1} + K_0\, y_k(0) + \sum_{j=1}^{q} K_j y_k(t_j) + \int_0^{\alpha} K(t) y_k(t)\, \mathrm{d}t \tag{9}$$

Here $K_0, K_1, \ldots, K_q$ are constant $n \times m$ matrices, $K(t)$ is a piecewise-continuous $n \times m$ matrix function of t on the pass interval $0 \le t \le \alpha$, and $0 < t_1 \le t_2 \le \cdots \le t_q \le \alpha$ are q sample points.

A special case of (9) can be used to establish a link between differential linear repetitive processes and a class of 1D linear systems with a delay in the state. It will, however, be established later in this paper that the pass initial conditions alone can destabilise a differential (and discrete) linear repetitive process. Hence adequate modelling of the pass initial conditions is essential in the analysis of differential (and discrete) linear repetitive processes.

3. Stability Theory and Its Application

As noted in the previous section, the stability theory for linear repetitive processes (of constant pass length) has been developed (see (Rogers and Owens, 1992a) for the definitive treatment) using an abstract model of the underlying dynamics in a Banach space setting. A key fact is that this representation contains all such processes as special cases. Here we introduce this model using the differential linear repetitive process state space description (7)–(8) as a basis.

Let E_α denote the Banach space of bounded continuous mappings of the interval $0 \le t \le \alpha$ into the vector space $\mathbb{C}^m$ of complex m-vectors with norm

$$||y|| := \sup_{0 \le t \le \alpha} ||y(t)||_m \tag{10}$$

where $||\cdot||_m$ is any convenient norm in $\mathbb{C}^m$, e.g. $||p||_m = \max_{1 \le i \le m} |p|$. Now write (7)–(8) in the equivalent form

$$y_{k+1}(t) = Ce^{At}d_{k+1} + C\int_0^t e^{A(t-\tau)}\Big(B_0 y_k(\tau) + Bu_{k+1}(\tau)\Big)\,\mathrm{d}\tau$$
$$+ D_0 y_k(t), \quad 0 \le t \le \alpha, \quad k \ge 0 \tag{11}$$

Also define the bounded linear map L_α in E_α by

$$(L_\alpha y)(t) = D_0 y(t) + C\int_0^t e^{A(t-\tau)} B_0 y(\tau)\,\mathrm{d}\tau \tag{12}$$

and the vector b_{k+1} by

$$b_{k+1}(t) = Ce^{At}d_{k+1} + C\int_0^t e^{A(t-\tau)} Bu_{k+1}(\tau)\,\mathrm{d}\tau \tag{13}$$

Then the differential linear repetitive process (7)–(8) takes the abstract form

$$y_{k+1} = L_\alpha y_k + b_{k+1}, \quad k \ge 0 \tag{14}$$

In the case when the state initial conditions on each pass are given by (9) we have that

$$(L_\alpha y)(t) = Ce^{At}\hat{Y} + C\int_0^t e^{A(t-\tau)} B_0 y(\tau)\,\mathrm{d}\tau \tag{15}$$

where

$$\hat{Y} = K_0 y(0) + \sum_{j=1}^{q} K_j y(t_j) + \int_0^t K(\tau) y(\tau)\,\mathrm{d}\tau \tag{16}$$

and b_{k+1} is again given by (13).

Consider now any initial profile $y_0 \in E_\alpha$ and any disturbance sequence $\{b_k\}_{k \ge 1} \in E_\alpha$ with limit denoted by b_∞. Then the system of (14) is said to be (uniformly) asymptotically stable if, and only if, the resulting sequence of pass profiles $\{y_k\}_{k \ge 1}$ converges strongly to an equilibrium, or so-called limit, profile y_∞ defined by

$$y_\infty = L_\alpha y_\infty + b_\infty \tag{17}$$

and this property holds for all operators, say $\hat{L}_\alpha$ 'sufficiently close' to L_α. Following Rogers and Owens (1992a), it can be shown that this property holds if, and only if,

$$r(L_\alpha) < 1 \tag{18}$$

where $r(\cdot)$ denotes the spectral radius. Also if asymptotic stability holds, the resulting limit profile is the unique solution to (17).

To apply Theorem 1, it is necessary to calculate (or characterise) the spectral values of L_α. One way of doing this is to consider the equation

$$(\beta I - L_\alpha)\, y = z \tag{19}$$

and construct necessary and sufficient conditions on the complex scalar β such that this equation has a solution $\forall\, z \in E_\alpha$ which is bounded in the sense that $||y|| \le k_0 ||z||$ for some real scalar $k_0 > 0$ and $\forall\, z \in E_\alpha$. In the case of differential linear repetitive processes of the form (7)–(8), the following result characterises asymptotic stability.

Theorem 1. *The differential linear repetitive process of (7)–(8) is asymptotically stable if, and only if,*

$$r(D_0) < 1 \tag{20}$$

Corollary 1. *Suppose that the differential linear repetitive process (7)–(8) is asymptotically stable and the control input sequence applied $\{u_k\}_{k\ge1}$ and the sequence of state initial conditions $\{d_k\}_{k\ge1}$ converge strongly to u_∞ and d_∞, respectively. Then the resulting limit profile is simply the solution of the differential state space equations*

$$\dot{x}_\infty(t) = \left(A + B_0\,(I_m - D_0)^{-1}\, C\right) x_\infty(t) + Bu_\infty(t), \quad x_\infty(0) = d_\infty$$

$$y_\infty(t) = (I_m - D_0)^{-1}\, Cx_\infty(t), \quad 0 \le t \le \alpha, \quad k \ge 0 \tag{21}$$

This limit profile description is just a 1D linear systems state space model. Hence if a differential linear repetitive process (with pass initial conditions of the form given in (8)) is asymptotically stable then, after a 'sufficiently large' number of passes, its repetitive dynamics can be replaced by those of a standard linear system. This fact has obvious implications in terms of the structure and design of control schemes for these processes which is returned to in the conclusions section of this paper.

These asymptotic stability results are counter-intuitive in the sense that they are independent of the system state space model matrices A, B, B_0 and C and, in particular, the eigenvalues of A which clearly have a very significant influence on the dynamics produced along any pass. The reason for this is that the pass length α is finite by definition and it is easy to generate examples where asymptotic stability does not even guarantee that the resulting limit profile is stable in the 1D linear systems sense—for example, consider cases when $D_0 = 0$.

Clearly this 'weakness' will not be acceptable in all cases of interest. For such cases a stronger concept of stability along the pass is required which is discussed later in this section. Note, however, that asymptotic stability is a necessary condition for stability along the pass in all cases.

Asymptotic stability on its own turns out to be strong enough for a number of cases of interest. One example here is the convergence properties of iterative solution algorithms to nonlinear dynamic optimal control problems based on the maximum principle—see (Roberts, 1996) and related papers for a full treatment of this application. A second application is to certain classes of linear ILC schemes which is discussed next. The key point to emerge is that repetitive process based stability theory is far superior to other analysis settings for these schemes.

As noted previously here, many approaches to the design of linear ILC schemes use only feedforward action, i.e. information generated from the results of previous trials is used to construct the current trial input. The use of the repetitive process framework, however, uniquely allows learning laws which also take into account the current trial error. The result is learning laws which are a combination of feedforward (from the previous trial outputs and/or inputs) and feedback (from the current trial error) action. One immediate benefit from the presence of feedback action is that all the usual advantages of such action (i.e. improved robustness) are potentially available. The following discussion shows how one general class of feedforward/feedback linear ILC control schemes can be formulated such that their stability and convergence properties are equivalent, mathematically, to the asymptotic stability properties of discrete linear repetitive processes of the form (7)–(8).

In this paper the plant to which an ILC scheme is assumed to be strictly proper with plant dynamics defined by the state space triple (A, B, C) (see Amann *et al.*, 1996) for the extension of the results given here, plus other techniques, to more general model structures). Suppose also that the vector $r(t) \in \mathbb{R}^m$ is to be tracked over the trial interval $0 \leq t \leq \alpha < +\infty$. Then if $y_k(t) \in \mathbb{R}^m$ denotes the plant output on trial k, the corresponding error vector is

$$e_k(t) = r(t) - y_k(t) \tag{22}$$

and the state space model of the resulting error dynamics can be written as

$$\begin{aligned} \dot{x}_{k+1}(t) &= Ax_k(t) + Bu_k(t), \quad x_k(0) = 0 \\ e_k(t) &= r(t) - Cx_k(t), \quad 0 \leq t \leq \alpha, \quad k \geq 0 \end{aligned} \tag{23}$$

Here on trial k, $x_k(t)$ is the $n \times 1$ state vector, $u_k(t)$ is the $l \times 1$ vector of control inputs and the 'simple' state initial conditions on each trial incur no loss of generality.

Of all the candidate learning algorithms available, this paper only considers the following one which, in effect, is a (static and dynamic) combination of previous input vectors, the current error, and a finite number of previous errors:

$$u_k(t) = \sum_{i=1}^{N} \alpha_i u_{k-i}(t) + \sum_{i=1}^{N} K_i \left[e_{k-i}\right](t) + K_0 \left[e_k\right](t) \tag{24}$$

In addition to the 'memory' N, the design parameters are the static scalars α_i, $1 \leq i \leq N$, the linear operator $K_0 \left[e_k\right](t)$ which describes the current error contribution, and the linear operator $K_i \left[e_{k-i}\right](t)$, $1 \leq i \leq N$, which describes the contribution of the error on trial $k - i$.

The error dynamics of (23) can be written in convolution form as

$$e_k(t) = r(t) - G\left[u_k\right](t), \quad 0 \leq t \leq \alpha \tag{25}$$

where

$$G\left[y\right](t) = C\int_0^t e^{A(t-\tau)}By(\tau)\,\mathrm{d}\tau \tag{26}$$

Using this description, it is easily shown that the closed-loop error dynamics can be written as

$$\begin{aligned} e_k(t) = (I - GK_0)^{-1}\Bigg\{&\sum_{i=1}^{N}(\alpha_i I - GK_i)\left[e_{k-i}\right] \\ &+ \left(1 - \sum_{i=1}^{N}\alpha_i\right)r(t)\Bigg\}, \quad 0 \leq t \leq \alpha \end{aligned} \tag{27}$$

or, equivalently,

$$\hat{e}_{k+1} = L_\alpha \hat{e}_k + b \tag{28}$$

where

$$\hat{e}_k(t) = \left[e_{k+1-N}^T(t), \cdots, e_k^T(t)\right]^T \tag{29}$$

is the so-called error supervector, and

$$L_\alpha = \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & I \\ E_0E_N & \cdots & E_0E_2 & E_0E_1 \end{bmatrix} \tag{30}$$

with

$$\begin{aligned} E_0\left[y\right](t) &= (I + GK_0)^{-1}\left[y\right](t) \\ E_i\left[y\right](t) &= (\alpha_i - GK_i)\left[y\right](t), \quad 1 \leq i \leq N \end{aligned} \tag{31}$$

and

$$b = \left[0, 0, \cdots, \left(1 - \sum_{i=1}^{N}\alpha_i\right)r^T(t)\right]^T \tag{32}$$

It follows immediately from the structure of (28)–(32) that the closed-loop ILC error dynamics in this case can be studied as a linear repetitive process of the abstract form (14), where E_α is as before with $\mathbb{C}^m$ replaced by $\mathbb{C}^M$, $M = mN$. Hence the following stability result can now be stated.

Theorem 2. *The ILC scheme defined by (28)–(32) is stable in the sense of (18) if, and only if, the polynomial*

$$p_a(z) = z^N - \alpha_1 z^{N-1} - \alpha_2 z^{N-2} - \cdots - \alpha_{N-1} z - \alpha_N \tag{33}$$

satisfies

$$p_a(z) \neq 0, \quad \forall |z| \geq 1 \tag{34}$$

Note again the counter-intuitive basis of this result, i.e. stability is largely independent of the plant and the controllers. The reason for this is that the trial length is finite, over which a linear system can only produce a bounded output—even if it is 'unstable'. In the definition of stability used here, these 'unstable' outputs are still 'acceptable' (see also below).

Suppose now that Theorem 2 holds. Then the following corollary characterises the resulting limit profile (or the 'steady state' error dynamics).

Corollary 2. *Suppose that the ILC scheme of (28)–(32) is designed such that it satisfies Theorem 2. Then the error sequence generated* $\{e_k\}_{k\geq 1}$ *converges strongly to the limit profile*

$$e_\infty(t) = (I + GK_e)^{-1}[r](t) \tag{35}$$

where the so-called effective controller K_e *is given by*

$$K_e = \frac{K}{1-\hat{\beta}} \tag{36}$$

with

$$\hat{\beta} = \sum_{i=1}^{N} \alpha_i, \quad K = \sum_{i=0}^{N} K_i \tag{37}$$

As already noted, the stability theory leading to Theorem 2 does not absolutely guarantee that $e_\infty(t)$ is stable (in the 1D sense) and/or better than $e_1(t)$, i.e. that learning actually produces an improvement. To absolutely guarantee an 'acceptable' $e_\infty(t)$ (for all trial lengths) requires the stronger conditions of stability along the pass (see also below). In effect, this property imposes additional conditions on the operators E_i, $i = 0, 1, 2, \cdots, N$ which take into account the structure of the plant and the learning control law.

In a large number of cases, see (Amann, 1996) for more details on this point, the stability theory based on Theorem 2 suffices and in such cases further key aspects of closed-loop system performance can be established. This is a unique feature of the repetitive process theory for this application which is detailed fully in (Amann, 1996). Here we restrict attention to two key aspects, i.e. the magnitude of the limit error $e_\infty(t)$ and the convergence rate of the error sequence $\{e_k\}_{k\geq 1}$ to $e_\infty(t)$.

Consider first the magnitude of the limit error. Then further analysis using the repetitive process theory setting, see (Amann, 1996) for the details, shows that there is a 'trade off' between the magnitude of $e_\infty(t)$ and the rate of convergence, i.e. the ratio

$$\gamma = \frac{||e_{k+1} - e_\infty||}{||e_k - e_\infty||} \tag{38}$$

To detail this, it is convenient to introduce the parameter

$$\mu = \max_{1 \le i \le N} |\lambda_i| \tag{39}$$

where $\lambda_i, 1 \le i \le N$, is a root of $p_a(z)$ of (33). Then if μ is 'close to unity' a small $e_\infty(t)$ can be expected and if μ is 'close' to zero 'fast' convergence is enforced with a 'large' terminal error.

It is easy to show that zero terminal error can only be achieved if $b \equiv 0$ and hence only if $\sum_{i=1}^{N} \alpha_i = 1$. In which case it follows that $z = 1$ is a solution of $p_a(z) = 0$ and consequently the spectral radius of L_α can, at best, be equal to one. This situation is reminiscent of classical control where the inclusion of an integrator in a controller, which puts a forward path system pole on the stability boundary, results in zero error for constant disturbances.

In terms of the convergence rate to the limit error, the following result is proved by the use of standard bounding techniques for linear operators in Banach spaces and can again be found in (Amann, 1996):

Theorem 3. *Suppose that Theorem 2 holds for the ILC scheme defined by (28)–(32). Then the error sequence $\{e_k\}_{k \ge 1}$ converges in norm to e_∞ defined by (35)–(37) and the error sequence $\{e_k - e_\infty\}$ is bounded by an expression of the form*

$$||e_k - e_\infty|| \le M_1 \Big(\max\left(||e_0||, \dots, ||e_{N-1}||\right) + M_2 ||r|| \Big) \hat{\lambda}^k \tag{40}$$

for any $\hat{\lambda} \in (\mu, 1)$ where M_1 and M_2 are positive real scalars which depend on the choice of $\hat{\lambda}$.

Using this last result, it follows that the error sequence approaches e_∞ at a geometric rate governed by the parameter $\hat{\lambda}$. Also if $\hat{\lambda}$ is 'small' convergence is 'rapid' with a geometric upper bound of the form $M_3 \hat{\lambda}^k$ where M_3 is another positive scalar. Note also that the resulting limit error is, in general, non-zero and K_e (the effective controller) can have 'very high' gain—hence it cannot be argued that it should have been used on the first pass.

It is instructive to relate convergence here to classical concepts. Consider, for simplicity, the case of $N = 1$. Then if $\mu \to 0^+$ convergence is 'fast to, essentially, the first trial error. Hence ILC schemes are of little benefit in such cases and will simply lead to the normal 'large' errors encountered under 'standard' feedback control schemes.

This last conclusion suggests the selection of a μ 'close to' one with the result that K_e has a 'high gain' structure and, in general terms, a 'small error' will result. There is, however, a conflict in this argument which has major implications on the underlying systems theoretic/control structure both in terms of the underlying theory and its application. This is that 'small' learning errors require 'high' effective gain in the system which must remain stable under such gains.

One way of describing the essential detail of the high-gain aspect here is by the associated root locus. In particular, standard theory shows that for stability the plant must be minimum phase and have relative degree one or two. The first of these constraints cannot be removed and the second is such that it precludes the use of ILC schemes with 'high' effective gain for systems of 'high' relative gain. Such systems simply need a different class of ILC schemes.

Suppose now that the plant is minimum phase and has relative degree one or two. Then there is also a key difference between plants of relative degree one and those with relative degree two. This is most conveniently presented here by restricting attention to the single-input single-output (SISO) case.

If the system is SISO and relative degree one, it is easily shown, by pole-zero residue calculations that the convergence is uniform in both magnitude and derivative (particularly if the poles and zeros are interlaced when plotted on a pole-zero diagram). Conversely, if the system is SISO and relative degree two, the convergence is uniform in magnitude but non-uniform in the derivative. In particular, the limit error has a 'small' amplitude of the order of $(1-\hat{\mu})^{1/2}$ but oscillates at a ('high') frequency proportional to $(1-\hat{\mu})^{1/2}$, i.e. in practical terms, as $\hat{\mu} \to 1$ from below, the limit error exhibits a high frequency 'jitter' around the specified reference frequency.

As noted previously, asymptotic stability of a differential (or discrete) linear repetitive process guarantees the existence of a limit profile described by a standard (1D) linear systems state space model (21). It does not, however, also guarantee that this limit profile has 'acceptable' dynamics along the pass and, in particular, that the limit profile is stable, i.e.

$$\left| sI_n - A - B_0\left(I_m - D_0\right)^{-1} C \right| \neq 0, \quad \mathrm{Re}\,(s) \geq 0 \tag{41}$$

A simple example is the following single state process

$$\begin{aligned} \dot{x}_{k+1}(t) &= -x_{k+1}(t) + u_{k+1}(t) + (1+\rho) y_k(t), \quad x_{k+1}(0) = 0 \\ y_{k+1}(t) &= x_{k+1}(t), \quad 0 \leq t \leq \alpha, \quad k \geq 0 \end{aligned} \tag{42}$$

This process is clearly asymptotically stable with resulting limit profile

$$\dot{y}_\infty(t) = \rho y_\infty(t) + u_\infty(t), \quad 0 \leq t \leq \alpha \tag{43}$$

Setting $u_k(t) \equiv 1$ and $y_0(t) \equiv 0$, $0 \leq t \leq \alpha$, $k \geq 1$, gives

$$\begin{aligned} y_1(t) &= 1 - e^{-t}, \quad 0 \leq t \leq \alpha \\ y_\infty(t) &= \rho^{-1}\left(e^{\rho t} - 1\right), \quad 0 \leq t \leq \alpha \end{aligned} \tag{44}$$

Hence in this case the first pass profile $y_1(t)$ is a quite acceptable 'classical' response to the unit step input command. The limit profile can, however, have quite unacceptable dynamic characteristics. For example, if $\rho > 0$, the limit profile dynamics grow exponentially and can be said to be 'unstable along the pass' in an obvious intuitive sense.

The natural definition of stability along the pass for the above example is to ask whether (21) defining the limit profile is stable (in the standard linear systems sense), i.e. $\beta < 0$ if we let the pass length $\alpha \to \infty$. Unfortunately, this simple appealing idea does not apply in any simple manner to a wide range of cases, such as the coal cutting model. Consequently the definition of stability along the pass employed is, in effect, based on the rate of approach to the limit profile as the pass length becomes infinitely large.

Definition 2. The linear repetitive process (14) is *stable along the pass* if there exist finite real numbers $M_\infty > 0$ and $\lambda_\infty \in (0,1)$ such that $\forall\, \alpha > 0$ and for each constant disturbance sequence $b_{k+1} = b_\infty$ the output sequence $\{y_k\}_{k\geq 0}$ satisfies the inequality

$$||y_k - y_\infty|| \leq M_\infty \lambda_\infty^k \left(||y_0|| + \frac{||b_\infty||}{1-\lambda_\infty} \right), \quad k \geq 0 \tag{45}$$

In effect, this definition requires that the rate of approach to the limit profile has a guaranteed geometric upper bound independent of the pass length $\alpha > 0$. (This is in contrast to asymptotic stability which can be shown to require only that the rate of approach to the limit profile has a guaranteed upper bound for the given value of pass length.) Several equivalent sets of necessary and sufficient conditions for stability along the pass exist but here we only use the fact that this property is equivalent to the existence of real numbers (which are independent of α) $M_\infty > 0$ and $\lambda_\infty \in (0,1)$ such that

$$||L_\alpha^k|| \leq M_\infty \lambda_\infty^k, \quad \forall\, k \geq 0, \quad \forall\, \alpha > 0 \tag{46}$$

Using this last equation as a basis, it can be shown that necessary and sufficient conditions for stability along the pass of (14) are

$$r_\infty := \sup_{\alpha > 0} r(L_\alpha) < 1 \tag{47}$$

and

$$\hat{M} := \sup_{\alpha>0} \sup_{|\beta| \geq \gamma} ||\, (\beta I - L_\alpha)^{-1} \,|| < \infty \tag{48}$$

for some real number $\gamma \in (r_\infty, 1)$.

Note that condition (47) of this result is equivalent to asymptotic stability $\forall \alpha > 0$. Also condition (48) does, of course, imply (47) but in a large number of the cases considered to-date the former condition has proved much easier (in relative terms) to interpret than the latter. Hence the main reason for retaining their separate identities.

The boundedness condition (48) of this last theorem is equivalent to the existence of a $\gamma \in (r_\infty, 1)$ such that (19) has a uniformly bounded, with respect to α, solution $y \in E_\alpha$ for all choices of $z \in E_\alpha$ satisfying $\sup_\alpha ||z|| < +\infty$ and $\forall\, |\beta| \geq \gamma$. In general, this condition could prove 'very difficult' to interpret but for differential linear repetitive processes with 'simple' pass initial conditions the following result can be obtained.

Theorem 4. *Suppose that the pair $\{A, B_0\}$ is controllable and the pair $\{C, A\}$ is observable. Then differential linear repetitive processes of the form (7) with pass initial conditions defined by (8) are stable along the pass if, and only if,*

(a) $$r(D_0) < 1 \tag{49}$$

(b) $$|sI_n - A| \neq 0, \quad \mathrm{Re}\,(s) \geq 0 \tag{50}$$

(c) *all eigenvalues of the transfer function matrix*

$$G(s) = C\,(sI_n - A)^{-1}\,B_0 + D_0 \tag{51}$$

have modulus strictly less than unity $s = \imath\omega, \ \ \forall\,\omega \geq 0$.

All of these conditions can be tested by direct application of 1D linear systems tests. Note also that, in general, all three conditions must hold—the example of (42) only fails condition (c). Also they have well-defined physical interpretations which are discussed next.

Consider first condition (a) and consider, without loss of generality, the SISO case. Suppose also that zero state initial conditions and control inputs are applied, i.e. $d_{k+1} = 0,\ u_{k+1}(t) = 0,\ 0 \leq t \leq \alpha,\ k \geq 0$. Then the initial output on each pass is given by

$$y_k(0) = D_0^k y_0(0), \quad k \geq 0 \tag{52}$$

Hence in physical terms asymptotic stability of (7) requires that the sequence $\{y_k(0)\}_{k\geq 0}$ of initial pass profiles does not become unbounded (in a well-defined sense) as $k \to +\infty$.

Condition (b), in effect, demands that the contribution to the dynamics along the current pass from the input term is uniformly bounded (in a well-defined sense) with respect to the pass length. In the case of condition (c), apply the same conditions as in the discussion of condition (a). Then it can be shown (Smyth, 1992) that the process dynamics can be expressed in the form

$$y_k(\imath\omega) = G^k(\imath\omega) y_0(\imath\omega), \quad k \geq 0 \tag{53}$$

where $y(s)$ denotes the Laplace transform of $y(t)$. Hence this condition requires that each frequency component (as opposed to just the d.c. gain for asymptotic stability) of the initial profile is attenuated from pass to pass. Also it is easy to show that stability along the pass guarantees that the resulting limit profile is stable as a 1D linear time-invariant system.

Stability along the pass can, of course, be applied to the ILC problem considered earlier in this section. This aspect is extensively treated in (Amann, 1996) where the links to other stability theories for ILC schemes are also explored to great effect. In particular, the links with the long standing H_∞-based stability theory for ILC schemes are fully investigated.

In terms of the control of differential linear repetitive processes, it is clear that stability along the pass will, in general, be required. The work (Smyth, 1992) has also concluded that computable information on the following aspects of system performance would also be highly useful:

1. The rate of approach of the output sequence of pass profiles $\{y_k\}_{k\geq 1}$ to the resulting limit profile y_∞.
2. The magnitude of the error $y_k - y_\infty$ on any pass k.

Rogers and Owens (1992b; 1992c) have shown that easily computable information on these aspects is available at the expense of sufficient, but not necessary, conditions for stability along the pass. These tests are based on the assumed availability of

$$W(t) = C\int_0^t e^{A\tau}B_0\,\mathrm{d}\tau + D_0, \quad t \geq 0 \tag{54}$$

i.e. the step-response matrix of the standard 1D linear system with state-space description parameterised by the state space quadruple (A, B_0, C, D_0). The key task then is the computation of the total variation of each entry in $W(t)$—a task which can be achieved in a computer-aided analysis environment using numerically reliable algorithms.

Using this general approach, it can, for example, be shown for no extra computational cost that, if the control input sequence $\{u_k\}_{k\geq 1}$ is constant from pass to pass (for example, a unit step applied at $t = 0$ in one or more channels on each pass satisfies this condition), then

1. $\{y_k\}_{k\geq 1}$ approaches y_∞ at a geometric rate governed by a single computable scalar; and
2. in norm terms, the error $y_k - y_\infty$ on pass k lies in a computable band whose width decreases from pass to pass at a geometric rate governed by the same computable scalar.

Note that this 'worst case' error band can be replaced by a set of individual channel bands which are, in general, less conservative. The effective use of these computable measures in controller design is still very much an open research question. Also it is possible to give (by Parseval's theorem) a frequency domain interpretation of the convergence information of 1. above performance for both differential and discrete linear repetitive processes—see (Owens and Rogers, 1995) for the differential case, and (Rogers and Owens, 1996) for the discrete case. These are based on a so-called 1D Lyapunov equation characterisation of stability along the pass. In the differential case the central results are as follows.

Theorem 5. *The differential linear repetitive process (7) with pass initial conditions of the form of (8) is stable along the pass if, and only if,*

(a) *conditions (a) and (b) of Theorem 4 hold; and*

(b) $\exists$ *a rational polynomial matrix solution $P(s)$ of the Lyapunov equation*

$$G^T(-s)P(s)G(s) - P(s) = -I \tag{55}$$

bounded in an open neighbourhood of the imaginary axis with the properties that

(i) $$P(s) \equiv P^T(-s) \tag{56}$$

(ii) $$\beta_2^2 I \geq P(\imath\omega) = P^T(-\imath\omega) \geq \beta_1^2 I, \quad \forall\, \omega \geq 0 \tag{57}$$

for some choices of real scalars $\beta_i \geq 1,\ i = 1, 2$.

Note that the numbers β_i play no role in the stability analysis but, together with the solution matrix $P(s)$ of the so-called 1D Lyapunov equation (55), they are the key to obtaining bounds on expected system performance. This is developed next.

Suppose that (7)–(8) is stable along the pass. Then standard factorisation techniques enable $P(s)$ to be written as

$$P(s) = F^T(-s)F(s) \tag{58}$$

Also, without loss of generality, we take

$$\lim_{|s| \to +\infty} F(s) = P_\infty^{1/2} \tag{59}$$

where $\lim_{|\omega| \to +\infty} P(\imath\omega) = P_\infty$ and $P_\infty = P_\infty^T > 0$ is real and solves

$$D_0^T P_\infty D_0 - P_\infty = -I \tag{60}$$

It also follows that $F(s)$ is both stable and minimum phase and hence has a stable minimum phase inverse.

Given these facts, return to (7)–(8) in the case when the current pass input term is deleted. Then the process dynamics are described by

$$y_{k+1}(s) = G(s)y_k(s), \quad k \geq 0 \tag{61}$$

Also let

$$\hat{y}_k(s) = F(s)y_k(s), \quad k \geq 0 \tag{62}$$

denote 'filtered' (by the properties of $F(s)$) outputs. Then the following result gives (frequency domain) bounds on expected performance of (7)–(8).

Theorem 6. *Suppose that the differential linear repetitive process (7)–(8) is stable along the pass. Then* $\forall\, k \geq 0$

$$||\hat{y}_{k+1}||^2_{L_2^m(0,+\infty)} = ||\hat{y}||^2_{L_2^m(0,+\infty)} - ||y_k||^2_{L_2^m(0,+\infty)} \tag{63}$$

and hence the 'filtered' output sequence $\{||\hat{y}_k||_{L_2^m(0,+\infty)}\}_{k \geq 0}$ *is strictly monotonically decreasing to zero and satisfies, for* $k \geq 0$, *the inequality*

$$||\hat{y}_k||_{L_2^m(0,+\infty)} \leq \hat{\lambda}^k ||\hat{y}_0||_{L_2^m(0,+\infty)} \tag{64}$$

where

$$\hat{\lambda} := (1 - \beta_2^{-2})^{1/2} < 1 \tag{65}$$

Also the actual output sequence $\{||y_k||_{L_2^m(0,+\infty)}\}_{k\geq 0}$ *is bounded by*

$$||y_k||_{L_2^m(0,+\infty)} \leq M\hat{\lambda}^k ||y_0||_{L_2^m(0,+\infty)} \tag{66}$$

where

$$M := \beta_2\beta_1^{-1} \geq 1 \tag{67}$$

This result provides a frequency domain description of the convergence of the output sequence of the differential linear repetitive process (7)–(8) to the resulting limit profile under stability along the pass. The main features are as follows:

1. The sequence of 'filtered' outputs $\{\hat{y}_k\}_{k\geq 0}$ consists of monotone signals converging to zero (under zero control inputs) at a computable rate in $L_2^m(0,+\infty)$.
2. The actual output sequence $\{y\}_{k\geq 0}$ converges at the same geometric rate but is no longer monotonic. Further, the deviation from monotonicity is described by the parameter M computed from the solution of the frequency dependent (or 1D) Lyapunov equation of Theorem 5.

In computational terms, the results of Theorems 5 and 6 are not computationally attractive due to the need to examine the underlying Lyapunov equation (55) for all points on the imaginary axis of the complex plane. Recent work (Rogers and Owens, 1997) (building on some basic methods first used by (Agathoklis *et al.*, 1990) in the case of 2D linear systems described by the Roesser state-space model) has used the matrix Kronecker product to develop tests for stability along the pass via Theorem 5 which can be applied using generalised eigenvalue computations for constant matrices. Hence the existence (at least) of the convergence information of Theorem 6 is available for the same computational cost as the testing of the conditions of Theorem 5, and these may well prove to be computationally more attractive than the tests based on Theorem 4 in some cases of practical interest.

In contrast to 1D linear systems, more than one class of Lyapunov equations arise for linear repetitive processes/2D linear systems. As an alternative to the frequency domain (1D) Lyapunov equation of Theorems 5 and 6, it is possible to employ a constant coefficient, or 2D, Lyapunov equation to characterise stability along the pass of (7)–(8). This approach, see (Rogers and Owens, 1995) for a detailed treatment, is based on the so-called augmented plant matrix for (7) defined as

$$P = \begin{bmatrix} A & B_0 \\ C & D_0 \end{bmatrix} \tag{68}$$

Introduce $W = W_1 \oplus W_2$ and Q as real positive-definite symmetric matrices, where $\oplus$ denotes the direct sum of W_1 and W_2, i.e.

$$W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} \tag{69}$$

Then the candidate 2D Lyapunov equation for (7)–(8) is

$$P^T W^{1,0} + W^{1,0} P + P^T W^{0,1} P - W^{0,1} = -Q \tag{70}$$

where $W^{1,0} = W_1 \oplus 0_{m\times m}$ and $W^{0,1} = 0_{n\times n} \oplus W_2$. Then, using the results first developed in terms of the concept of a strictly continuous bounded real transfer function matrix (denoted by SCBR here) from circuit theory (see (Anderson and Vongpanithlerd, 1973) for the details), the following is the 2D Lyapunov equation stability result for (7)–(8).

Theorem 7. *Differential linear repetitive processes of the form (7)–(8) are stable along the pass if* $\exists$ *positive definite symmetric matrices* $W = W_1 \oplus W_2$ *and* Q *which satisfy the 2D Lyapunov equation (70).*

This result, i.e. that the SCBR property of $G(s)$ implies stability along the pass but not vice versa, parallels that previously reported for 2D linear systems described by the Roesser state-space model (Anderson *et al.*, 1986). In particular, the fact that this result is sufficient but not necessary for stability along the pass is established by straightforward (but numerous) modifications to the analysis of (Anderson *et al.*, 1986). There are, however, a number of relevant special cases where the theorem is both necessary and sufficient and by far the most important of these is if (7)–(8) is SISO. Also alternative characterisations of the SCBR property exist in terms of, for example, algebraic Riccati equations.

The development of efficient algorithms for solving (70) is still an open research area to some extent. The fact that Theorem 7 is, in general, a sufficient but not necessary condition for stability along the pass reduces the usefulness of the 2D Lyapunov equation in terms of stability tests. It still, however, has potentially a major role to play in certain other critical aspects of the control-related analysis of differential linear repetitive processes. One of these is to develop the first significant results on the wide-ranging problem of constructing 'informative' stability margins. This is still very much an open research area and is one to which much profitable research effort could be directed—see also the conclusions section of this paper.

A unique key feature of differential (and discrete) linear repetitive processes is the fact that the initial conditions alone can act as a destabilising mechanism. This (at first somewhat surprising) fact follows from developing necessary and sufficient conditions for asymptotic stability of (7) under the pass initial conditions of (9). Also a special case of (9) provides an as yet under-exploited link with a class of delay differential systems.

No loss of generality occurs here in considering the special case of (7) when $y_{k+1}(t) = x_{k+1}(t)$, i.e. $m = n$, $C = I_n$, $D = 0$, and $D_0 = 0$. In order to obtain necessary and sufficient conditions for asymptotic stability in this case, it is necessary to find the spectral values of the corresponding L_α. This can be achieved as follows.

First note that (19) in this case can be written in the differential form

$$\begin{aligned} \dot{w}(t) &= Aw(t) + B_0 z(t), \quad w(0) = \hat{Y} \\ \beta y(t) &= w(t) + z(t), \quad 0 \le t \le \alpha \end{aligned} \tag{71}$$

Consider first the case when $z(t) \equiv 0$ and $\beta \neq 0$. Then (71) can be written as

$$\beta y(t) = e^{(A+\beta^{-1}B_0)t}\hat{Y} \tag{72}$$

where

$$\Big\{\beta I_n - K_0 - \sum_{j=1}^{q} K_j e^{(A+\beta^{-1}B_0)t_j} - \int_0^\alpha K(\tau)e^{(A+\beta^{-1}B_0)\tau}\,\mathrm{d}\tau\Big\}\hat{Y} = 0 \tag{73}$$

It now follows immediately that if β is any non-zero solution of

$$\Big|\beta I_n - K_0 - \sum_{j=1}^{q} K_j e^{(A+\beta^{-1}B_0)t_j} - \int_o^\alpha K(\tau)e^{(A+\beta^{-1}B_0)\tau}\,\mathrm{d}\tau\Big| = 0 \tag{74}$$

then (73) has a non-zero solution vector $\hat{Y}$ generating a non-zero solution $y(t)$. This is impossible if $\beta I - L_\alpha$ is to be injective and hence we must have all solutions to (74) in the spectrum of L_α.

Suppose now that β is not a solution to (74) and that z is not necessarily zero. Then a similar analysis to that just given yields an equation of the form

$$\Big\{\beta I_n - K_0 - \sum_{j=1}^{q} K_j e^{(A+\beta^{-1}B_0)t_j} - \int_0^\alpha K(\tau)e^{(A+\beta^{-1}B_0)\tau}\,\mathrm{d}\tau\Big\}$$

$$= (\hat{G}z)(t), \qquad 0 \leq t \leq \alpha \tag{75}$$

where $\hat{G}$ is a bounded linear operator mapping E_α into itself. This equation leads to a unique solution for $\hat{Y}$ satisfying $||\hat{Y}|| \leq c_1||z||$, $\forall\, z \in E_\alpha$ and some real scalar $c_1 > 0$. Also the solution to (71) can be written in the form

$$y(t) = \beta^{-1}\Big\{z(t) + e^{(A+\beta^{-1}B_0)t}\hat{Y} + \int_0^\alpha e^{(A+\beta^{-1}B_0)(t-\tau)}\beta^{-1}B_0 z(\tau)\,\mathrm{d}\tau\Big\} \tag{76}$$

which implies the existence and boundedness of $(\beta I - L_\alpha)^{-1}$.

The above analysis shows that the spectrum of L_α in this case is simply the set of all solutions of (74). This result, in terms of asymptotic stability, is stated formally as follows.

Theorem 8. *Differential linear repetitive processes of the form (7) (with* $y_{k+1}(t) = x_{k+1}(t)$ *) and state initial conditions of the form (9) are asymptotically stable if, and only if, all solutions to (74) have modulus strictly less than unity.*

Proof. The analysis above had characterised the spectrum of L_α, and asymptotic stability holds provided that $r(L_\alpha) < 1$. Conversely, if $r(L_\alpha) < 1$, the continuity of the left-hand-side terms in (74) mean that the spectrum of L_α can have no cluster points on the boundary of the unit circle of the complex plane, i.e. $r(L_\alpha) < 1$. ∎

Note 1. The compactness of L_α could be used to provide an alternative characterisation of its spectrum.

Note 2. Theorem 8 and its consequence discussed below generalise in a natural manner to processes described by the full form of (7) and (9), i.e. without any restrictions on the presence and structure of the matrices in the output equation.

Theorem 8 immediately shows that asymptotic stability, and hence stability along the pass, of a differential linear repetitive process is critically dependent on the interaction between pass profiles and boundary conditions as expressed by $K_0, K_1, \cdots, K_q$ and $K(t)$ in (9). For example, if $K_0 = K_1 = \cdots K_q = 0$ and $K(t) \equiv 0$, the only solution of (74) is $\beta = 0$ and the process is asymptotically stable independent of the matrices A, B, and B_0. If, however, $K_1 = K_2 = \cdots = K_q = 0$ and $K(t) \equiv 0$, it follows immediately that the spectrum of L_α is simply the eigenvalues of K_0 and instability can be easily induced by choosing this matrix such that one of its eigenvalues is outside the unit circle in the complex plane.

This analysis clearly shows that it is the pass initial conditions which distinguish differential (and discrete) linear repetitive processes from other classes of linear dynamic systems. In particular, the pass initial conditions in a given application must be 'adequately modelled' in order to avoid erroneous results. This is not to say, however, that the structure of the pass initial conditions cannot be used, as discussed next, to show links with other classes of linear dynamic systems which may then be exploited to mutual benefit.

Consider the subclass of delay-differential systems in $\mathbb{R}^n$ defined by the state space equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_0 x(t-\alpha) + Bu(t), \quad t \geq 0 \\ x(t-\alpha) &:= x_0(t), \quad 0 \leq t \leq \alpha \end{aligned} \tag{77}$$

Here A, B_0, and B are constant $n \times n$, $n \times n$, and $n \times l$ matrices, respectively, and this subclass of delay-differential systems can be regarded as a special case of differential linear repetitive processes with a special case of the pass initial conditions (9). To show this, introduce the change of variables

$$\begin{aligned} u_{k+1}(t) &= u(k\alpha + t) \\ x_k(t) &= x((k-1)\alpha + t), \quad 0 \leq t \leq \alpha, \quad k \geq 0 \end{aligned} \tag{78}$$

Hence (77) can be treated as a special case of (7) with $n = m$, $C = I_n$, $D = 0$, $D_0 = 0$, and pass initial conditions

$$x_{k+1}(0) = x_k(\alpha), \quad k \geq 0 \tag{79}$$

i.e. a special case of (9).

Suppose, therefore, that the delay-differential system (77) is regarded as a differential linear repetitive process with state-space model

$$\begin{aligned} \dot{x}_{k+1}(t) &= Ax_{k+1}(t) + Bu_{k+1}(t) + B_0 x_k(t) \\ x_{k+1}(0) &= x_k(\alpha), \quad 0 \leq t \leq \alpha, \quad k \geq 0 \end{aligned} \tag{80}$$

Then Theorem 8 enables the following stability equivalence result to be stated.

Theorem 9. *The delay-differential system (77), when regarded as a differential linear repetitive process of the form (7), is asymptotically stable if, and only if,*

$$\left|\beta I_n - e^{(A+\beta^{-1}B_0)\alpha}\right| \neq 0, \quad \forall\, |z| \geq 1 \tag{81}$$

To write this last result in more familiar terms, write $\beta = e^{s\alpha}$ where s denotes the Laplace-transform variable. Then (81) reduces to

$$\left|sI_n - A - e^{-s\alpha}B_0\right| \neq 0, \quad \forall\, s:\ \mathrm{Re}\,(s) \geq 0 \tag{82}$$

which is just the 'characteristic equation' condition that would arise from directly applying the Laplace transform to (77). Hence in this case, the 'normal' and repetitive process concepts of asymptotic stability coincide.

This link to delay-differential systems is potentially very powerful and is, as yet, under-exploited—see also the conclusions section of this paper. Note also that (9) can also be used to establish links with other classes of linear systems—see (Owens and Rogers, 1997) for more details. It must be stressed again, however, that such links can only be exploited if the pass initial conditions are appropriately chosen. If this is not the case, incorrect results will be obtained. Also the variety possible here more than justifies the study of differential linear repetitive processes as a class of linear dynamic systems in their own right.

4. Systems Theory for Discrete Linear Repetitive Processes

The state-space model of a discrete linear repetitive process is given by

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + B_0 y_k(p) \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + D_0 y_k(p), \quad 0 \leq p \leq \alpha, \quad k \geq 0 \end{aligned} \tag{83}$$

Here on pass k, $x_{k+1}(p)$ is $n \times 1$ state vector, $y_k(p)$ is the $m \times 1$ vector pass profile, and $u_k(p)$ is the $l \times 1$ vector of control inputs. Note again that it is possible to specify two distinct types of pass initial conditions, write the resulting process state-space description in an abstract form, and apply the stability theory. The details are given in (Owens and Rogers, 1997; Rogers and Owens, 1992a). In the case of the simple pass state initial conditions, i.e. $x_{k+1}(0) = d_{k+1}$, (83) is asymptotically stable if, and only if, the matrix D_0 satisfies Theorem 1, and the following is the stability along the pass result.

Theorem 10. *Discrete linear repetitive processes described by (83) with pass state initial conditions of the form* $x_{k+1}(0) = d_{k+1}$, $k \geq 0$ *are stable along the pass if, and only if,*

(a) $$r(D_0) < 1, \quad r(A) < 1 \tag{84}$$

(b) *all eigenvalues of the transfer function matrix*

$$G(z) = C\left(zI_n - A\right)^{-1} B_0 + D_0 \tag{85}$$

have modulus strictly less than unity $\forall\, |z| = 1$.

These conditions can also be tested by direct application of 1D linear systems tests and all other stability results given in the previous section for differential linear processes extend in a natural manner to the discrete case—see, for example, (Rogers and Owens, 1993; 1996).

The dynamics of discrete linear repetitive processes clearly share some basic characteristics with 2D discrete linear systems recursive in the positive quadrant, i.e. systems which propagate information in two separate directions, often termed 'horizontal' and 'vertical' respectively, over the grid $Z^2 = \{(i,j) : i \geq 0, j \geq 0\}$. In particular, they basically propagate information in two separate directions, i.e. from pass to pass (k direction) and along a given pass (p direction). Hence one possible approach to the analysis of discrete linear repetitive processes is to treat them as 2D linear systems recursive over Z^2. A key difference, however, is the fact that the pass length of a repetitive process, which corresponds to the duration of one direction of information propagation in the 2D systems representations advocated here, is always finite by definition.

The modelling or representation of 2D linear discrete systems is somewhat more involved than the 1D case, see, for example, (Rocha, 1990) for a detailed treatment. At a basic level, the types of possible representations can be classified according to whether or not

1. an input/output structure is included; and
2. latent (auxiliary) variables are included in addition to the system variables.

As in the 1D case, state-space models are a very important class of internal representations of the system dynamics and several distinct versions exist, where the most commonly used are the Roesser model (Roesser, 1975) and the various forms of the Fornasini-Marchesini models (Fornasini and Marchesini, 1978).

The Roesser state-space model (omitting the output equation which has no role here) has the structure

$$\begin{aligned} x_h(h+1,v) &= A_1 x_h(h,v) + A_2 x_v(h,v) + B_1 u(h,v) \\ x_v(h,v+1) &= A_3 x_h(h,v) + A_4 x_v(h,v) + B_2 u(h,v) \end{aligned} \tag{86}$$

Here h and v are the (integer-valued) horizontal and vertical coefficients, x_h is the $n \times 1$ vector of horizontally transmitted information, x_v is the $m \times 1$ vector of vertically transmitted information, and u is the $l \times 1$ vector of control inputs.

In the Fornasini-Marchesini model structures, the state vector is not split into horizontal and vertical components. Again the output equation is not required in this work and with $z(i,j)$ denoting the appropriately dimensioned state vector the basic model of this type has the structure

$$z(h+1,v+1) = A_5 z(h+1,v) + A_6 z(h+1,v) + B_3 u(h+1,v) + B_4 u(h,v+1) \tag{87}$$

where, as in (86), u is the (appropriately dimensioned) vector of control inputs. Note also that every Fornasini Marchesini model can be transformed into a Roesser model and vice versa.

Another work (Rocha *et al.*, 1996) has argued that (83) is a Roesser (and hence Fornasini-Marchesini) model. In the case of the former, this claim is based on interpreting the state vector x as horizontally transmitted information and the pass profile y as vertically transmitted information. Using this 'equivalence', it is possible to show (Rocha *et al.*, 1996) that the same set of conditions result from the following operations:

1. Applying a set of necessary and sufficient conditions for stability along the pass to processes described by (83).

2. Applying a set of necessary and sufficient conditions for bounded-input/bounded-output (BIBO) stability (see, for example, the relevant references cited in (Rogers and Owens, 1992a) for the basic theory) of the Roesser model of (86) with the matrices A_1, A_2, A_3 and A_4 replaced by A, B_0, C, and D_0, respectively (or a Fornasini-Marchesini model equivalent).

This fact enables the interchange to great effect of stability tests between these two areas. It does not, however, show that (83) has the structure of a Roesser or Fornasini-Marchesini model which is what is required in order to use 2D linear systems theory to address key systems theoretic questions such as what controllability of discrete linear repetitive processes actually means.

The general problem of modelling the dynamics of (83) by 2D discrete linear state-space models has been extensively investigated recently (Gałkowski *et al.*, 1998). This work has shown that an appropriate place to start is the so-called augmented state vector for (83) which is defined as

$$Z(k,p) = \left[x_k^T(p), y_k^T(p)\right]^T \tag{88}$$

Then it follows immediately that the dynamics of discrete linear repetitive processes with state space model (83) can be written in the form

$$EZ(k+1,p+1) = A_8 Z(k+1,p) + A_9 Z(k,p) + B_5 u(k+1,p) \tag{89}$$

where

$$E = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad A_8 = \begin{bmatrix} A & 0 \\ C & -I_m \end{bmatrix}, \quad A_9 = \begin{bmatrix} 0 & B_0 \\ 0 & D_0 \end{bmatrix}, \quad B_5 = \begin{bmatrix} B \\ D \end{bmatrix} \tag{90}$$

and 0 denotes the zero matrix of appropriate dimensions.

This is a singular version of the model of (87) (with $A_5 = 0$, $B_3 = 0$). Note also that there is a considerable volume of literature on systems theory for 2D linear discrete systems described by singular versions of the Roesser or Fornasini-Marchesini models—see, for example, (Kaczorek, 1992; 1994). One key aspect of this work has been detailed investigations into the role and interpretation of singularity for these representations and the construction of nonsingular (sometimes termed regular) representations from their singular counterparts.

The work (Gałkowski *et al.*, 1998) has concluded that singularity is not an intrinsic feature (in a well-defined sense) of discrete linear repetitive processes. This, in turn, has led to the development of a 'transformation' theory for constructing nonsingular Roesser and Fornasini-Marchesini state space descriptions from their discrete counterparts. These nonsingular representations have then been used to establish a formal equivalence (unlike the work of (Rocha *et al.*, 1996)) between stability along the pass of processes described by (83) and BIBO stability of their 2D Roesser or Fornasini-Marchesini state space model interpretations which clearly must form the basis to investigate, for example, the potential role of 2D feedback control schemes for (83)—see also the conclusions section of this paper.

One key role for the Fornasini-Marchesini singular state-space model interpretation of the dynamics of (83) has been in the development of a transition matrix (also termed the fundamental matrix sequence) and hence an analytic expression for the trajectories generated by such processes in response to a given control input sequence and initial conditions. The following are the main results on this aspect with a full treatment in (Gałkowski *et al.*, 1998).

Consider first the transition matrix. Then this matrix, denoted by $T_{i,j}$, has the form

$$ET_{i,j} = \begin{cases} A_9 T_{-1,-1} + A_8 T_{0,-1} + I, & i = j = 0 \\ A_9 T_{i-1,j-1} + A_8 T_{i,j-1}, & i \neq 0 \text{ and/or } j \neq 0, \ \ i \geq -\mu_1, \ \ j \geq -\mu_2 \end{cases} \tag{91}$$

where (μ_1, μ_2) denotes the index of (89)–(90). To delineate this concept, first consider the Laurent expansion about infinity of the so-called characteristic matrix of (89)–(90), i.e.

$$E(z_1, z_2) = (z_1 z_2 E - z_1 A_8 - A_9)^{-1} \tag{92}$$

where z_1 and z_2 are shift operators in the p and k directions respectively, written formally in the following form:

$$E(z_1, z_2) = z_1^{-1} z_2^{-1} \sum_{i=-\mu_1}^{\infty} \sum_{j=-\mu_2}^{\infty} T_{i,j} z_1^{-i} z_2^{-j} \tag{93}$$

Lewis (1992) has shown that a sufficient condition for the existence of an expression of the form (93) with finite lower limits on the double summation for a two variable polynomial, say, $\Delta(z_1, z_2)$ is

$$\deg\big(\Delta(z_1, z_2)\big) = \deg_{z_1}\big(\Delta(z_1, z_2)\big) + \deg_{z_2}\big(\Delta(z_1, z_2)\big) \tag{94}$$

where the degree of the two-variable polynomial $\Delta(z_1, z_2)$ is defined as the degree in z of $\Delta(z, z)$ and $\deg_{z_h}(\Delta(z_1, z_2))$ denotes the degree in z_h. This means that the highest degree in z_1 and z_2 occurs in the same term and two-variable polynomials with this property are termed principal. It can be shown, using the transfer function description (for which (92) is the characteristic matrix), that the 2D linear systems models of the dynamics of (83) are also principal and it is this fact which is the key to showing that nonsingular 2D linear state-space models for these processes exist.

To evaluate (91), first partition $T_{i,j}$ as follows

$$T_{i,j} = \begin{bmatrix} T_{i,j}^{1,1} & T_{i,j}^{1,2} \\ T_{i,j}^{2,1} & T_{i,j}^{2,2} \end{bmatrix} \tag{95}$$

where $T_{i,j}^{1,1}$ is of dimension $n \times n$ and $T_{i,j}^{2,2}$ is of dimension $m \times m$. Then it follows immediately that the index (μ_1, μ_2) in this case is defined by $\mu_1 = 0$, $\mu_2 = 1$ and hence $T_{0,-1}$, i.e.

$$T_{0,-1} = \begin{bmatrix} 0 & 0 \\ 0 & I_m \end{bmatrix} \tag{96}$$

is the initial matrix in this case. The following result gives the formula for computing $T_{i,j}$, $i \geq 0, j \geq 1$.

Theorem 11. *The transition matrix for discrete linear repetitive processes described by (83) at point (i,j), $i \geq 0, j \geq 1$ is given by solving the following set of equations for $T_{i,j}$:*

$$T_{i,j}^{1,\lambda} = AT_{i,j-1}^{1\lambda} + B_0 T_{i-1,j-1}^{2\lambda}, \quad \lambda = 1, 2 \tag{97}$$

and the following set of equations for $T_{i,j+1}$:

$$T_{i,j}^{2\lambda} = CT_{i,j}^{1\lambda} + D_0 T_{i-1,j}^{2\lambda}, \quad \lambda = 1, 2 \tag{98}$$

Under pass state initial conditions of the form (89), the techniques of (Kaczorek, 1992) can be used to show that the model of (89)–(90) for a process (7)–(8) has a solution for any control input sequence if, and only if,

$$|z_1 z_2 E - z_1 A_8 - A_9| \not\equiv 0 \tag{99}$$

Here it is assumed without loss of generality that (89)–(90) has a unique solution, in which case the following result gives the general response formula for this for this model.

Theorem 12. *Consider the model (89)–(90) for the dynamics of discrete linear repetitive processes. Then the general response formula for this model is given by*

$$\begin{aligned} Z(k,p) = & \sum_{i=0}^{k-1} \sum_{j=-1}^{p-1} T_{i,j} B_5 u(k-i, p-j-1) \\ & + \sum_{i=0}^{k-1} T_{i,p-1} A_8 Z(k-i, 0) + \sum_{i=0}^{k-2} T_{i,p-1} A_9 Z(k-i-1, 0) \\ & + \sum_{j=-1}^{p-2} T_{k-1,j} A_9 Z(0, p-j-1) + T_{k-1,p-1} A_9 Z(0,0) \end{aligned} \tag{100}$$

In order to use this transition matrix to define and characterise the so-called local reachability and controllability of discrete linear repetitive processes, the following partial ordering of two-tuple integers will be used:

$$
\begin{aligned}
&(i,j) \leq (k,p) \ \text{ iff } \ i \leq p \ \text{ and } \ j \leq p \\
&(i,j) = (k,p) \ \text{ iff } \ i = p \ \text{ and } \ j = p \\
&(i,j) < (k,p) \ \text{ iff } \ (i,j) \leq (k,p) \ \text{ and } \ (i,j) \neq (k,p)
\end{aligned}
\tag{101}
$$

The dynamics of (83) evolve over

$$
D_e := \Big\{(k,p): \ k \geq 0, \ \ 0 \leq p \leq \alpha\Big\} \tag{102}
$$

but in practice only a finite number of passes K^* will actually be undertaken. Hence a natural definition of reachability/controllability for these processes is as follows: Given admissible boundary conditions and a sequence of control inputs, is it possible to achieve all vectors in the rectangle whose boundary in the pass to pass direction is defined by $0 \leq k \leq K^*$ and in the along the pass direction by $0 \leq p \leq \alpha$? The formalisation of this idea for local reachability follows next where for $(a,b),(c,d)$ the rectangle $[a,b],[c,d]$ is defined as

$$
\Big[(a,b),(c,d)\Big] := \Big\{(a,b) \leq (i,j) \leq (c,d)\Big\} \tag{103}
$$

In general, the properties of reachability and controllability for the 2D discrete linear systems/discrete linear repetitive processes under consideration here can be distinct. Here, for brevity, only the definition of local reachability is introduced—for a detailed treatment see (Kaczorek, 1992; 1994) for the 2D discrete linear systems case and (Gałkowski *et al.*, 1997) for discrete linear repetitive processes.

Definition 3. The dynamics of discrete linear repetitive processes modelled by (89)–(90) is said to be *locally reachable* in the rectangle $[(0,0),(h,f)]$, $0 \leq h \leq K^*$, $0 \leq f \leq \alpha$ if for admissible pass initial conditions (8) and every $z_t \in \mathbb{R}^{n+m} \exists$ a sequence of control input vectors $u(k,p)$ on $(0,0) \leq (k,p) < (h,f)$ such that $Z(h,f) = z_t$.

The properties of local reachability/controllability of discrete linear repetitive processes can be characterised by direct application of results for the the equivalent properties for 2D Roesser/Fornasini-Marchesini models due to Kaczorek (1992; 1994), resulting in matrix rank based conditions, see (Gałkowski *et al.*, 1998) for details. A currently open general research problem is the implications (if any) of local reachability/controllability properties on, for example, the structure and properties of current pass state feedback control laws with, for example, the structure $u_{k+1}(p) = Fx_{k+1}(p)$. Some initial work on this problem has also highlighted the need for the so-called simultaneous controllability which is defined as follows.

Definition 4. Let K^* be an arbitrarily chosen pass index for the discrete linear repetitive process state-space model of (83). Then processes described by this state-space model are said to be ***simultaneously***, or ***pass, controllable*** if for admissible pass initial conditions $\exists$ a sequence of control input vectors $u(k,p)$, $0 \leq p \leq \alpha$, $0 \leq k \leq K^*$ which will drive the system to a given set, say, $\{x^*(K^*,0),\ x^*(K^*,1),\dots,x^*(K^*,\alpha)\}$ on pass K^*.

One immediate approach to characterise this property is to use 2D systems models. In fact, however, it is not possible to completely characterise this property using the 2D systems interpretations of the process dynamics. The reason for this is explained below.

Pass reachability plays the same role for discrete linear repetitive processes as the so-called global reachability for 2D linear systems described by Roesser or Fornasini-Marchesini state-space models. In particular, global reachability in these cases is expressed in terms of the global state vector and hence in terms of an infinite collection of local state vectors with entries along the separation set. In the case of repetitive processes, however, this collection of local state vector entries is finite due to the fact that the pass length is finite by definition. The work (Gałkowski *et al.*, 1998) has attempted to characterise pass reachability using the 2D systems models which has shown that it is only possible to completely characterise the weaker concept of the so-called simultaneous local reachability.

Simultaneous local reachability means that the process is locally reachable in the rectangle $0 \leq k \leq K^*$, $0 \leq p \leq \alpha$ for each point p on this pass. This does not, however, guarantee pass reachability since two points (K^*,p^1) and (K^*,p^2) may be reached using two different input sequences which cannot be joined together to act simultaneously.

Given these facts, recent work has developed a 1D representation for the dynamics of discrete linear repetitive processes (Gałkowski *et al.*, 1997). In contrast to the 1D models of 2D linear systems developed in (Aravena *et al.*, 1990), this 1D representation has constant dimensions, and this fact itself makes it a potentially very powerful analysis base for discrete linear repetitive processes. The basic steps in its derivation are explained in more detail in (Gałkowski *et al.*, 1997).

Define the so-called global state and input vectors for (83) as

$$X(k) = \left[\hat{x}^T(k), \hat{y}^T(k)\right]^T \tag{104}$$

where

$$\hat{x}(k) = \left[x^T(k,1),\dots,x^T(k,\alpha)\right]^T$$
$$\hat{y}(k) = \left[y^T(k,0),\dots,y^T(k,\alpha-1)\right]^T \tag{105}$$

and

$$U(k) = \left[u^T(k,0),\dots,u^T(k,\alpha-1)\right]^T \tag{106}$$

Note that $x(k,0)$ does not belong to the global state vector and must therefore be treated as a part of the boundary conditions. Based on these definitions, the dynamics of (83) can equivalently be described by the 1D state-space model

$$X(k+1) = \hat{A}X(k) + \hat{B}U(k+1) + \epsilon(k) \tag{107}$$

where

$$\hat{A} = \begin{bmatrix} 0 & A_1 \\ 0 & A_2 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} \tag{108}$$

where

$$\begin{cases} A_1 = \begin{bmatrix} B_0 & 0 & \cdots & 0 \\ AB_0 & B_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{\alpha-1}B_0 & \cdots & AB_0 & B_0 \end{bmatrix} \\ A_2 = \begin{bmatrix} D_0 & 0 & \cdots & 0 \\ CB_0 & D_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{\alpha-2}B_0 & \cdots & CB_0 & B_0 \end{bmatrix} \end{cases} \tag{109}$$

$$\begin{cases} \hat{B}_1 = \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{\alpha-1}B & \cdots & AB & B \end{bmatrix} \\ \hat{B}_2 = \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{\alpha-2}B & \cdots & CB & D \end{bmatrix} \end{cases} \tag{110}$$

and

$$\epsilon(k) = \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} x(k+1,0) \tag{111}$$

where

$$\sigma_1 = \left[A^T, (A^2)^T, \ldots, (A^\alpha)^T\right]^T$$
$$\sigma_2 = \left[C^T, (CA)^T, \ldots, (CA^{\alpha-1})^T\right]^T \tag{112}$$

Note also that the input vector in (107) is updated in a manner directly related to the pass-by-pass nature of the underlying process dynamics.

Introducing the transformation

$$\hat{X}(k) = X(k) - \hat{B}U(k) \tag{113}$$

into (107) yields

$$\hat{X}(k+1) = \hat{A}\hat{X}(k) + \hat{A}\hat{B}U(k) + \epsilon(k) \tag{114}$$

where the matrix $\hat{A}\hat{B}$ is of the form

$$\hat{A}\hat{B} = \begin{bmatrix} R \\ S \end{bmatrix} \tag{115}$$

where

$$R = \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ r_2 & r_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ r_\alpha & r_{\alpha-1} & \cdots & r_1 \end{bmatrix} \tag{116}$$

$$S = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ s_2 & s_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ s_\alpha & s_{\alpha-1} & \cdots & s_1 \end{bmatrix} \tag{117}$$

The entries in R and S are given by

$$\begin{aligned} r_1 &= B_0 D \\ r_j &= A^{j-1}B_0D + \sum_{l=0}^{j-2} A^{j-l-2}B_0CA^lB, \quad j = 2, 3, \ldots, \alpha \end{aligned} \tag{118}$$

and

$$\begin{aligned} s_1 &= D_0D \\ s_2 &= CB_0D + D_0CB \\ s_j &= CA^{j-2}B_0D + D_0CA^{j-2}B + \sum_{l=0}^{j-3} CA^{j-l-3}B_0A^{j-2}B \\ j &= 3, 4, \ldots, \alpha \end{aligned} \tag{119}$$

Hence an equivalent 1D state-space model for the dynamics of (83) is

$$\begin{aligned} \hat{x}_1(k+1) &= A_1\hat{y}_1(k) + RU(k) + \sigma_1 x(k+1, 0) \\ \hat{y}_1(k+1) &= A_2\hat{y}_1(k) + SU(k) + \sigma_2 x(k+1, 0) \end{aligned} \tag{120}$$

where

$$\hat{X}(k) = \left[\hat{x}_1^T(k), \hat{y}^T(k)\right]^T \tag{121}$$

The 'free terms' $\sigma_l x(0, k+1)$, $l = 1, 2$ depend explicitly on the pass index k. Hence they must be interpreted as 'time-varying' in this sense. This feature is highly undesirable but, as shown below, these 'free' terms can be removed by employing further appropriate state transformations.

Another feature of this model is the 'non-standard' updating structure in the pass-to-pass direction. In particular, consider the sub-vectors $\hat{x}_1(k)$ and $\hat{y}_1(k)$ which form the global state vector $\hat{X}(k)$. Then the latter is computed recursively, i.e. $\hat{y}_1(k+1)$ is computed from $\hat{y}_1(k)$, but the former is not. Hence in terms of the structure of the 2D linear systems Fornasini-Marchesini state-space model (in all its various forms) the sub-vector $\hat{y}_1(k)$ can be interpreted as the state vector and the sub-vector $\hat{x}_1(k)$ as the output vector. Also in this context $x(0,p)$, $y(0,p)$, $0 \le p \le \alpha$ and $x(k,0)$, $k = 1, 2, \ldots$ are boundary conditions, but $y(k,0), k = 1, 2, \ldots$ are not boundary conditions since they can be uniquely determined from (113). The 'free' terms $\sigma_l x(k+1, 0)$, $l = 1, 2$ can be regarded as 'permitted' inputs.

To remove the time-varying terms from (120), introduce the transformations

$$\begin{aligned} \omega(k) &= \hat{y}(k) + \pi(k) \\ \zeta(k) &= \hat{x}_1(k) + \upsilon(k) \end{aligned} \tag{122}$$

Then it follows immediately that $\pi(k)$ and $\upsilon(k)$ must satisfy the following pair of equations:

$$\begin{aligned} \pi(k+1) &= A_2 \pi(k) - \sigma_2 x(k+1, 0) \\ \upsilon(k+1) &= A_1 \pi(k) - \sigma_1 x(k+1, 0) \end{aligned} \tag{123}$$

Also substituting these transformations into the model of (120) gives a 1D discrete linear system equivalent to the discrete linear repetitive process state-space model of (83). This result is stated formally as follows.

Theorem 13. *The dynamics of discrete linear repetitive processes can be equivalently described by the 1D linear time invariant state space model*

$$\begin{aligned} \omega(k+1) &= A_2 \omega(k) + SU(k) \\ \zeta(k+1) &= A_1 \omega(k) + RU(k) \end{aligned} \tag{124}$$

See (Gałkowski *et al.*, 1997) for a more detailed discussion of the structure of this representation, including a transfer function matrix representation. Note also the existence of alternative derivations of this 1D representation (Xu, 1997).

Since this model is nonsingular, there is no difference between the properties of reachability and controllability (unlike the case of 2D linear systems described by Roesser or Fornasini-Marchesini state-space models (see, for example, (Kaczorek, 1994)). It is also a routine to conclude that discrete linear repetitive processes are pass-controllable in the sense of Definition 4 if, and only if, the sub-vector $\omega(k)$ of their equivalent 1D state space representation is (state) controllable. This result is stated formally as follows.

Theorem 14. *Discrete linear repetitive processes are pass-controllable in the sense of Definition 4 if, and only if, the so-called pass controllability matrix*

$$\Omega_p := [S, A_2S, \dots, A^{m-1}S] \tag{125}$$

has rank equal to αm.

Again see (Gałkowski *et al.*, 1997) for some further developments of Theorem 14 in the form of important special cases where a further simplification is possible.

5. Conclusions and Open Research Problems

Early approaches to the analysis and control of linear repetitive processes were based on first 'transforming' them into equivalent 1D linear systems. This approach was based on using the concept of the total distance traversed to, in effect, remove the dependence of the system variables on the pass index. Subsequently it was shown that this approach is incorrect except in a few very restrictive special cases.

This fact led, in turn, to the development of a rigorous stability theory for linear repetitive processes of constant pass length. Also this theory is based on an abstract model of the process in a Banach-space setting which includes all such processes as special cases. Hence it is of sufficient generality to form the basis for the development of a rigorous control theory and, in due course, attendant controller design algorithms/software for sub-classes of particular interest.

In the special cases of differential and discrete linear repetitive processes, the resulting stability conditions can be tested by, in effect, direct application of 1D linear systems tests. Also if the process under consideration is stable then, after a 'sufficiently large' number of passes, its dynamics can be replaced by those of a stable 1D linear system (differential or discrete as appropriate). The implications of this key result in the formulation and solutions of control/regulation schemes for these processes has not been investigated yet beyond a very superficial coverage of a number of special cases.

Extensive work has been undertaken on the development of performance bounds or measures for differential and discrete linear repetitive processes. This has yielded computable bounds or bands in the time domain and convergence rate information in the frequency domain via a 1D Lyapunov equation formulation of stability. As yet, little progress has been made on the definition and characterisation of useful/physically meaningful stability margins for these processes. One possible approach here is a further development of the so-called 2D Lyapunov equation interpretation of stability where the work to-date appears to provide a potential starting point.

Differential and discrete linear repetitive processes have clear structural links with, in particular, a subclass of delay differential systems in the case of the former and 2D discrete linear systems recursive over the positive quadrant in the latter. This leads to the assertion that these processes can simply be analysed by direct application/modification of the existing systems theory. In fact, however, differential and discrete linear systems can be totally distinct from other classes of linear systems. The essential mechanism for this is the pass initial conditions which alone can, as demonstrated in this paper, destabilise these processes. Hence the pass initial conditions must be 'adequately modelled' in any given case to avoid the probability of incorrect results at the most basic systems analysis level.

Provided the pass initial conditions are appropriately selected/modelled, it is possible to establish structural links with other classes of linear systems which can then be used to mutual benefit. In the differential case, the links with a subclass of 1D linear systems with a delay in the state vector have not been fully exploited yet. Given the wealth of results on delay differential systems, both in terms of the basic theory and its applications, it is expected that further research in this general area will prove highly profitable. Possible topics for short to medium term work include further development of the 2D Lyapunov equation based approach to stability analysis and controllability/observability.

Under 'simple' pass initial conditions, the dynamics of discrete linear repetitive processes can be represented by well-known state-space models for 2D linear systems recursive over the positive quadrant. In this case (given the currently available results) it is possible to interchange analysis tools to a great effect between these two classes of linear systems. The work to-date, in addition to stability tests, has focused on the construction of a transition matrix and hence on to the definitions and characterisations of systems theoretic properties such as local reachability/controllability. This work has shown that large elements of the relevant systems theory for 2D discrete linear systems recursive over the positive quadrant can be directly applied to the corresponding problems for discrete linear repetitive processes.

Unlike the 1D case, there is more than one truly distinct concept of reachability/controllability (and related/dual properties such as observability) for 2D linear systems. This is also true for discrete linear repetitive processes where, for example, it is so-called simultaneous, or pass, controllability which is the relevant property in some cases. Also it is known that (at best) only necessary conditions for this property can be obtained using the 2D linear systems state-space model interpretations of the process dynamics.

This last fact has led to the recent development of a formal (i.e. the essential process dynamics are preserved) 1D linear systems representations for the dynamics of discrete linear repetitive processes. Unlike the 1D models for 2D linear systems developed previously (Aravena *et al.*, 1990), this 1D representation is characterised by state-space matrices of both constant dimensions and entries. Also it leads immediately to a simple matrix rank characterisation of the pass controllability property.

The reason that this 1D representation is not 'time-varying' in the sense of (Aravena *et al.*, 1990) is due to the fact that the pass length of a linear repetitive process (which corresponds to the duration of one direction of information propagation in a 2D linear systems interpretation of the process dynamics is finite by definition. This property alone makes this 1D representation a (potentially) very powerful basis for the solution of currently open research problems for discrete linear repetitive processes, e.g. state and output feedback stabilising control laws (see below), observers, optimal control, etc. Work is underway on a number of these aspects and will be reported in due course.

It is also possible to develop 2D transfer function matrix descriptions for the dynamics of differential and discrete linear repetitive processes (Rogers and Owens, 1992a). One immediate use for these descriptions is that they give simple block diagram representations of the process dynamics which clearly highlight the crucial interpass interaction. The more general question of the use of these descriptions to characterise key systems theoretic properties, and the specification and design of appropriate control schemes, has received some attention—for example, see (Johnson *et al.*, 1996) which gives some interesting results on the extension of Rosenbrock's system matrix theory to differential linear repetitive processes. Again, however, much work remains to be done in this general area.

To-date, relatively little research effort has been directed towards the specification of control schemes for linear repetitive processes and the development of algorithms/tools for designing the resulting controllers. A key basic problem here is the specification of physically meaningful general objectives for any such control scheme. Clearly any such objectives must include stability and, in particular, stability along the pass as the basic requirement. Hence the stabilisation problem for differential and discrete linear repetitive processes can be stated as 'the specification and design of compensator structures which when applied to the process ensure stability along the pass.'

Recall that if a differential or discrete linear repetitive process is stable along the pass then, after a 'sufficiently large' number of passes, its dynamics can be replaced by those of a stable 1D linear system—the limit profile (differential or discrete as appropriate). This fact leads to the following as one possible general control objective for these processes:

> Drive the process to a limit profile with 'acceptable' along the pass dynamics in a 'reasonable' number of passes and, simultaneously, maintain a 'tolerable' error $y_k - y_\infty$ on pass k.

The terms in quotation marks here are features whose exact interpretation will be a function of the particular application under consideration. To-date, little work has been done on refining this general specification into forms suitable for in-depth control scheme specification, design, and evaluation studies.

Given that differential and discrete linear repetitive processes have clear structural links with 1D linear systems, appropriately structured feedback control laws/schemes are a natural approach to solving control problems for these processes. As in the 1D case, state and output feedback based control schemes can be defined. In the case of differential processes, a state feedback law takes the following form:

$$u_{k+1}(t) = Fx_{k+1}(t), \quad 0 \leq t \leq \alpha, \ k \geq 0 \tag{126}$$

where F is an $l \times n$ matrix to be selected.

Applying this law to (7) yields a closed-loop state-space model which is closed in the sense that it has an identical structure to (7). This, in turn, means that the stability theory can be applied to give conditions for closed-loop asymptotic stability and stability along the pass. Also the following facts immediately arise:

1. Asymptotic stability, and hence stability along the pass, is invariant under this state feedback action unless there is 'direct feedthrough' from the current pass input to the current pass profile, i.e. $D \neq 0$.

2. Implementation of this state feedback control law requires that each element in $x_{k+1}(t)$ must be available for measurement. If not then an observer theory must be developed or state feedback control laws must be abandoned.

3. Although cases exist where (126) can be designed to ensure closed-loop stability along the pass, in general this class of control laws are (potentially) weak. In particular, they have no influence on the B_0 matrix, i.e. the contribution from the previous pass profile to the current pass profile or, alternatively, the instability mechanism which is unique to differential or discrete linear repetitive processes when compared with 1D linear systems.

The general problem arising under this last point is returned to below after the role of output feedback control schemes has been briefly discussed.

Consider a 'point' t on pass k of a differential linear repetitive process. Then at this point the information in the following set is causal and can therefore be used for output feedback pass control purposes:

$$Y := \left\{ y_k(\tau) : 0 \leq \tau \leq t \right\} \cup \left\{ y_h(t) : 0 \leq t \leq \alpha,\ 0 \leq h \leq k-1 \right\} \qquad (127)$$

Clearly, the most appealing schemes from an implementation standpoint would be the so-called local or instantaneously activated schemes, i.e. those which only explicitly use information at point (t,k). Again it is possible to find cases where the resulting design problem can be solved in one step and, equally, cases where such a control scheme, e.g. the natural generalisation of dynamic unity negative feedback control, cannot even ensure stability along the pass closed loop. In such cases, the obvious next step is to employ control schemes which explicitly employ information from previous pass profiles.

Aside from special cases, little work has been done yet in the general area of the control of linear repetitive processes—except for the ILC application area. The state and output feedback control schemes discussed briefly above are, in many ways, a natural generalisation of their 1D linear systems counterparts. Hence they can be expected to inherit many (if not all) of the strengths and weaknesses of such schemes. Note also that the work to date on ILC schemes has shown the potential of 2D 'predictive control' type schemes, i.e. a combination of feedback action based on the current pass state or output error and feedforward action from the previous pass profiles/inputs/errors. Also there has been some highly promising recent work (Wood *et al.*, 1997) on studying the stabilisation problem for differential or discrete linear repetitive processes in terms of systems theory over a ring.

Overall substantial progress has been made on key elements of a basic systems theory for differential and discrete linear repetitive processes. Much work remains to be done on refining/expanding this theory and, in particular, on the control aspects. Aside from the theoretical interest, the need for such work is motivated by the ever increasing and diverse range of applications for such theory and its consequences.

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