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A FORMAL APPROACH TO DISCRETE SYSTEMS THEORY

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Based on earlier results, an abstract multi-dimensional discrete systems theory is formulated here for further discussion. We believe that a more general framework including variable-parameter systems and nonlinear systems together with some discrete systems which cannot be treated on a common rectangular grid, may widen the scope of possible applications of n-D discrete systems theory. Some general results are summarized and examples are given.

1. Introduction

In the last two decades a considerable effort in systems theory has been devoted to multi-dimensional systems. The acronym n-D systems described somewhat better the situation involving several independent (discrete) variables rather than more input or output functions. Primarily the interest was arisen by applications in variable parameter systems, and it soon became clear that discretization efforts in classical systems theory cannot stop on the threshold of continuous functions of several variables also because of the existence of a well-developed area of numerical solution to partial differential equations (finite-difference methods). Another important source of application came from digital signal processing. Purely mathematical considerations demanded to fill up the neglected area of difference equations with several independent variables as a natural generalization of the long established theory of finite differences and the corresponding equations, the more so because a far developed theory of discrete systems showed many excellent and important results.

The attempts to develop a 2-D discrete systems theory as a generalization of the corresponding 1-D theory were widely successful. But soon it was realized that such generalizations cannot characterize and describe all specific properties of these systems, even more, many of these generalizations are responsible for the errors and mistakes which had to later corrected. Still there are different opinions as for the ultimate cause of these errors and for the main distinction between 1-D and n-D systems. We hope that the subsequent (perhaps rather abstract) analysis will show that, at least in the case of linear discrete systems, the origin of difficulties has to be placed neither in the non-existence of isolated poles of the transfer function, nor in the 'multidimensional time', nor in the lack of a factorization theorem for polynomials. All these facts, important for some special systems (often described by the acronym

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LSI systems), are consequences of the structure of the solution space for these systems. The main distinction between 1-D and n-D linear systems is the dimension of the space of solutions. Linear systems with finite-dimensional spaces of solutions are comparatively simple, but multi-dimensional systems fall into this category only in some exceptional cases.

We believe that conceptual problems in n-D systems theory can be solved by a 'top-to-bottom' approach, i.e. we have to start with an abstract systems theory. Such an approach gave valuable results also in the 1-D theory, e.g. in the works of Zadeh and Desoer (1968), Mesarovic and Takahara (1975; 1989), in a series of papers by Willems (1991) and many others (see the cited works for further bibliography). Unfortunately, n-D discrete systems are not fully covered in these works, although remarkable general results for linear systems have been obtained using Willems' behavioural approach (Fornasini *et al.*, 1993; Rocha, 1990) or using Gröbner bases (Oberst, 1990). On the other hand, 2-D and also n-D discrete linear systems in a special setting (e.g. with some or all independent variables restricted to nonnegative values) are well-understood and described in several monographs (Bose, 1982). Some of the unsolved problems are difficult because the general theory is still not satisfactory. The most striking example of such difficulties are efforts to solve some of these problems in the framework of functional transforms without attempts to justify the application of these methods.

An attempt of a general approach to n-D discrete systems theory is presented here for a further discussion and development. As in (Mesarovic and Takahara, 1975; 1989), we want to set up an abstract theory for a special type of systems, not fully covered in these books. Our approach is close to the behavioural concept of Willems and widens in a special direction the results of (Mesarovic and Takahara, 1989). The concepts introduced here have to remain as general as possible with the use of a minimal number of axioms, i.e. the most general mathematical structures. Such an approach, besides its theoretical significance, may widen the scope of applicability of n-D systems theory. The proofs of theorems, when published elsewhere, are not reproduced here.

The structure of the paper is as follows. Section 2 introduces the basic concepts. Section 3 deals with locally N-linear systems and state space considerations. Section 4 outlines some concepts of stability and Section 5 provides appropriate examples. It might be recommended to start reading with Section 5 so as to see the motivation behind the abstract approach presented in the paper.

2. Basic Concepts

We have to start with the basic notation and general results.

Definition 1. Let X and Y be sets of mappings into linear spaces $L_i, i = X, Y$ over the field $\mathcal{T}$. Then $S \subset X \times Y$ is called a *system*.

Theorem 1. *For any system there exists a set G and a mapping $\rho : G \times X \to Y$ such that*

$$(x, y) \in S \ \textit{ iff } \ \exists g \in G \ \textit{ such that } \ y = \rho(g, x) \tag{1}$$

For a proof, see (Mesarovic and Takahara, 1975). The function ρ and the element $g \in G$ in this theorem are called the *response function* and the *state object* of the system S, respectively. Note that the nontriviality of this result is hidden in its 'only if' part: There are no such elements $g \in G$ for which $(x, \rho(g, x)) \notin S$. The elements of G can be viewed as a parametrization of the system S.

Definition 2. The system S is called *linear*, iff

$$\forall (x_i, y_i) \in S, \quad i = 1, 2 \quad \text{and} \quad \forall c \in \mathcal{T}$$

there is

$$(cx_1, cy_1) \in S, \quad (x_1 + x_2, y_1 + y_2) \in S$$

For linear systems, the following basic result has been proven in (Mesarovic and Takahara, 1975):

Theorem 2. *Let X and Y be linear spaces over the same field $\mathcal{T}$. Then $S \subset X \times Y$ is a linear system if and only if there exists a linear space G over the field $\mathcal{T}$ and a function $R : G \times X \to Y$ such that:*

1. *$(x, R(g, x)) \in S$ for every $x \in X$*
2. *There exists a pair of linear mappings*

   $$R_1 : G \to Y \quad \textit{and} \quad R_2 : X \to Y$$

   such that for all $(g, x) \in G \times X$ there is

   $$R(g, x) = R_1(g) + R_2(x)$$

Definition 3. A linear discrete system S is called *N-linear* if the linear space $G = \{y : (0, y) \in S\}$ is of finite dimension N.

Definition 4. Let A and B be at most countable sets, and

$$X = \{x : A \longrightarrow L_X\}, \qquad Y = \{y : B \longrightarrow L_Y\}$$

Then X, Y is called a *discrete system.*

Definition 5. Let Φ_A and Φ_B be one-to-one mappings respectively from the sets A and B into $\mathbb{Z}^n$ (the set of n-dimensional vectors with integer coordinates). Then the system S with

$$X = \{x : \Phi_A \longrightarrow L_X\}, \qquad Y = \{x : \Phi_B \longrightarrow L_Y\}$$

is called a *(free) n-D (n-dimensional) system.*

The well-known diagonalization procedure mapping $\mathbb{Z}^2$ into $\mathbb{Z}$ can be viewed as an example of such a mapping.

Since for an at most countable set $A \subset \mathbb{Z}^m$ there exists a mapping $\Phi : A \to A^*$ with $A^* \subset \mathbb{Z}^n$, any discrete system can be viewed as an n-D discrete system with n arbitrarily fixed.

Example 1. We may consider a specific way in which any sequence $h : A \to \mathbb{C}$, $A \subset \mathbb{Z}^n$, with some additional assumptions, defines two systems. Keeping the symbols x and y for the input and output, respectively, these systems can be defined by convolution, namely $y = h * x$ and $x = h * y$. Here, $*$ denotes convolution and the assumptions must guarantee the existence of $h * (\cdot)$. (Note that a distinction between the input and output is taken into account, in contrast to the behavioural approach of (Fornasini *et al.*, 1993; Rocha, 1990; Willems, 1991).)

Both these systems are linear and discrete. While the first system is simple (in (Mesarovic and Takahara, 1975) it is called the 'functional'), the other (defining the so-called deconvolution problem) is in principle and computationally rather complicated. ♦

In some applications such as multiple-pass processes, models of learning, and nonstationary or almost periodic 1-D signals, a kind of segmentation technique can be applied as follows. A given 1-D signal $h : \mathbb{N}^+ \to \mathbb{C}$ can be subdivided in 'parts' of equal length τ and a two-dimensional signal $h^{[2]} : T \times \mathbb{N}^+ \to \mathbb{C}$ can be defined as follows: $h(n) = h^{[2]}(i,j)$, $i = n \,(\mathrm{mod}\, \tau)$, $j = n \,(\mathrm{div})\, \tau$, where div stands for integer division. In the above sense h and $h^{[2]}$ define a 1-D and a 2-D system, respectively. Similar constructions can widely be generalized and the 'dimensionality' of systems arbitrarily changed. It depends on their actual use whether this kind of arbitrariness of dimensionality can yield useful results. Its main disadvantage is that the algebraic structure of the domain of a signal is distorted: while here for h it was an ordered semigroup, this is no longer true for $h^{[2]}$.

Conversely, for some basic 2-D systems (see below), preferably those with final support of the involved signals, a 1-D model of the signal created by writing the rows of the matrix subsequently can be considered. Here again the original algebraic structure is distorted. For example, in image processing applications this approach destroys the correlation between some neighbouring pixels.

Convolutional systems will be reconsidered below.

Disapproval with the arbitrariness of the value of n in the above example opens the question of the algebraic structure of a signal domain. Another way of reasoning may start with the following definition.

Definition 6. Let A and B be at most countable sets with an associative and commutative operation called the addition. If there exists an element $\mu \in A$ (and similarly a ν for the set B) such that the sets $A + \mu$ and $B + \nu$ are semigroups, then the corresponding system as in Definition 1 is called the *(semigroup structured) discrete system.*

If these semigroups have m generators, the semigroup-structured discrete system is called the *m-D structured system.*

It has been shown (Gregor, 1991) that this concept of structured systems may and should be slightly generalized so as to be adapted to invariance considerations.

The algebraic structure of the sets A and B is introduced first of all because we want to study different types of invariances. Invariances with respect to some types of shift operators are of principal importance. It seems reasonable to define different types of shifts.

Definition 7. Let $\bar{A}$ be an additive and commutative group and let $A \subset \bar{A}$ be such that there exists a $\mu \in \bar{A}$ for which the set $A + \mu = \{t : t = \nu + \mu,\ \nu \in A\}$ is a semigroup. Then

1. The operator s^σ defined for all $x : A \to L_X$ and for all $\sigma \in \bar{A}$ by $s^\sigma x = \tilde{x}$,

$$\tilde{x}(t) = x(t - \sigma) \ \ \forall\, t \in A + \sigma$$

   is called the *shift operator.* Hence $\tilde{x} : A + \sigma \to L_X$.

2. The operator r^σ defined for all $x : A \to L_X$ and for all $\sigma \in \bar{A}$ by $r^\sigma x = \hat{x}$,

$$\hat{x}(t) = \begin{cases} x(t - \sigma) & \text{for } \ t \in (A + \sigma) \cap A \\ 0 & \text{for } \ t \in A \setminus (A + \sigma) \end{cases}$$

   is called the *translation operator*. Hence $\hat{x} : A \to L$.

3. A modified translation operator r^σ is called *completed with* ξ when a mapping $\xi : (t \in A \setminus (A + \sigma)) \to L_X$ is given and $r^\sigma\, x = \xi$ on this set.

The operators s and r are analogous, but not identical to the operators σ and λ in (Mesarovic and Takahara, 1975; 1989). They generalize the notion of forward and backward shifts as they are known in 1-D Laplace or $\mathcal{Z}$ transforms. It is important to note the domains of the shifted mappings.

Various invariance properties can now be defined. Although the value σ in the previous definitions is considered as a constant, the invariance properties of systems mostly refer to situations where the elements of X and Y are invariant with respect to all shifts σ of a certain set. Also note that the shift operators are meaningful for both free and structured systems.

Definition 8. A (free or structured) discrete system is called *s-invariant* iff

$$\forall\, \sigma \in A\ \forall\, (x, y) \in S \quad \text{there is} \quad (s^\sigma x, s^\sigma y) \in S$$

A discrete (free or structured) system S is called *r-invariant* iff

$$\forall\, \sigma \in A\ \forall\, (x, y) \in S,\ y = \rho(g, x) \quad \text{there is} \quad \rho(g, r^\sigma\, x) = r^\sigma\, \rho(g, x)$$

Definition 9. Let the sets A and B in Definition 4 be endowed with a linear ordering $\leq$ which coincides for the elements of $A \cap B$. Such an ordering will subsequently be called the *(abstract) time*. A discrete structured or free system with linear ordering will be called a *dynamic system.*

A dynamic system is called *one-sided (two-sided)* if the sets under consideration are well-ordered (not well-ordered).

In order to deal with dynamic systems, we shall use the following notation. Let the set A be ordered and $t \in A$. Then

$$A_t = \{t' : t' \geq t\}, \qquad A^t = \{t' : \; t' < t\}, \qquad \bar{A}^t = A^t \cup \{t\}$$

and for restrictions

$$x_t = x|_{A_t}, \qquad x^t = x|_{A^t}, \qquad \text{etc.}$$

For $S \subset X \times Y$ we have $S_t \subset X_t \times Y_t$ with $X_t = \{x_t : x|_{A_t}, x \in X\}$ and similarly for S^t. For $s < t$ we set $x_s^t = (x|_{A_t})|_{A^s}$. With these notations we shall use the concatenations $\hat{x}$ as follows:

$$\hat{x}(t) = \begin{cases} x(t) & \text{for } t < \tau \\ x^*(t) & \text{for } t \geq \tau \end{cases}$$

Note that for x, $x^* \in X$ and for some $\tau \in A$ it may be $\hat{x} \notin X$. If two different orderings have to be discussed simultaneously, we shall put the indexes and exponents in brackets.

Dynamic systems might possess a property commonly called causality. This concept rose some doubts about n-D systems theory mainly due to the lack of clear concepts of the past and future. To keep its basic content, we may formalize that special feature of some systems, which is characterized by a certain definiteness of its future when its history is known. This can easily be based on the concept of abstract time.

The concept of past-determinacy has been heuristically described as a system property, which allows us to foresee in all details the future system behaviour, provided the input-output pairs have been observed for a time interval long enough. A formal definition of past-determinacy can be given as follows:

Definition 10. A dynamic system $S \subset X \times Y$ is called *past-determined from* τ if there exists a τ such that

1. $\forall\,(x,y),(x',y') \in S,\;\; (\bar{x}^\tau, \bar{y}^\tau) = (\bar{x}'^\tau, \bar{y}'^\tau),\;\; \forall\, t \geq \tau : \bar{x}^t = \bar{x}'^t \Longrightarrow \bar{y}_t = \bar{y}'_t$
2. $\forall\,(x^\tau, y^\tau) \in S^\tau \;\; \forall x'_\tau \;\; \exists y'_\tau$ such that $(x^\tau.x'_\tau, y^\tau.y'_\tau) \in S$

The system is called *strongly past-determined* if in the first condition it suffices to suppose $x^t = x'^t$.

Example 2. Let

$$\begin{aligned} z(n+1) &= F\,z(n) + G\,x(n) \\ y(n) &= Hz(n), \qquad n \geq 0 \end{aligned}$$

where F, G, H are constant square matrices of dimension N. Then

$$y(k) = H\,F^k\,z(0) + H\sum_{i=0}^{k-1} F^{k-1-i}G\,x(i)$$

If

$$y'(k) = H\,F^k\,z'(0) + H\sum_{i=0}^{k-1} F^{k-1-i}G\,x'(i)$$

and for $\tau = n$ we have $(x^\tau, y^\tau) = (x'^\tau, y'^\tau)$, then

$$H\,F^q\,z(0) = H\,F^q\,z'(0), \qquad q = 0, 1, \ldots, n-1$$

If $z, z' \in E_n$, from the Cayley-Hamilton theorem it follows that also $H\,F^n\,z(0) = H\,F^n\,z'(0)$ and therefore if $x^t = x'^t \forall\, t \geq \tau = n$ then also $\bar{y}^t = \bar{y}'^t$ and this statement remains true if $x^t = x'^t\ \forall\, t > \tau = n$. Hence the system is strongly past-determined from n. Note that with the right-hand side of the first equation in the form $G\,x(n+1)$ the statement is no longer valid. Recursively computable n-D systems do not need to be past-determined when the output requires 'future' inputs. We shall see later that such a situation may arise when the implication (3) with the output mask M replaced by the input mask E becomes false. ♦

2.1. Basic n-Dimensional Discrete Systems

An important special case of n-dimensional discrete systems is formed by those mappings x, y which map subsets of $\mathbb{Z}^n$ into the field of complex numbers. In what follows, we shall use capitals to denote arbitrary subsets of $\mathbb{Z}^n$. Addition is defined as in $\mathbb{Z}^n$ (component-wise), $A+B = \{\gamma = \alpha+\beta,\ \alpha \in A,\ \beta \in B\}$, while the $\cup$, $\cap$, $\setminus$ etc. retain their set-theoretical meaning. Subsets may have additional structures, such as semigroups, different types of ordering, etc. Capitals will also denote matrices, but this will not cause misunderstandings.

We shall discuss equations of the form

$$F(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{|E|}, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{|M|},) = 0 \tag{2}$$

where $E = \{\varepsilon_i,\ i = 1, 2, \ldots, |E|\}$, $M = \{\mu_i,\ i = 1, 2, \ldots, |M|\}$ and

$$\bar{x}_i = x(\alpha + \varepsilon_i), \qquad \bar{y}_i = y(\alpha + \mu_i), \qquad \alpha \in A \subset \mathbb{Z}^n$$

and $x : A + E \to \mathbb{C}$, $y : A + M \to \mathbb{C}$. We shall also refer to (2) in a shorter form $F(\underline{x}, \underline{y}) = 0$. Since e.g. x_i and other arguments of the function F can be expressed as linear combinations of backward and forward partial differences of the mappings x, y, (recall a similar situation in ordinary difference equations) we may call (2) a partial difference equation. Clearly, any partial difference equation defines a mathematical model of a discrete system S. The mappings x and y are often called the input and output, respectively, and the finite sets E, M are the input and output masks. Although most of what follows will deal with the case of an infinite set A in (2), the case of a finite set A is not excluded. On the other hand, the set S might be empty: such a trivial case will be excluded from further investigations. The system (2) will henceforth be called the *basic n-D discrete system*, and we want to apply the foregoing (abstract) concepts to such systems.

Theorem 3. *The basic n-D discrete system (2) is linear iff there exists a linear mapping G from $\mathbb{C}^q$, $q = |M| + |E|$, into $\mathbb{C}$ such that*

$$F(\underline{x}, \underline{y}) = 0 \Longleftrightarrow G(\underline{x}, \underline{y}) = 0$$

This theorem follows easily e.g. from the fact that any linear functional can be represented as a scalar product. The equivalence ensures that systems may be linear also in the cases when their linearity is not made explicitly manifested.

2.2. n-D Dynamic Systems

The following theorem paves the way for including basic n-D discrete systems into formal systems theory:

Theorem 4. *Let A, M be subsets of $\mathbb{Z}^n$, M finite with at least two elements. Then there exists a mapping $\mu : A \to M$ and a linear ordering $\prec$ of A such that A is well-ordered with respect to $\prec$ and*

$$\alpha' + \mu(\alpha') \in \alpha + M \Longrightarrow \alpha' \prec \alpha \quad \text{for all} \quad \alpha, \alpha' \in A \tag{3}$$

The proof of this statement was published in (Bosák and Gregor, 1987). To motivate this statement and to describe its corollaries, some heuristic explanation seems useful. Taking the set $\alpha + M$ as a 'shifted output mask', the ordering introduced here includes a choice of one of the elements $\mu(\alpha)$ of the mask with the following property: if the values of the output at the points $\alpha' + \mu(\alpha')$ are known for all $\alpha' \prec \alpha$, then for this particular value of α there is one and only one component of the vector $\underline{y}$ in (2), with an unknown value. Given an input $\underline{x}$, this value can be computed from (2). These heuristic considerations will be made more precise later. Here we would like to show that the ordering introduced above will serve for the purpose of possible recursive computation of the output. From the proof it also follows that the ordering which satisfies the condition (3) is not unique.

Definition 11. Any ordering which satisfies the conditions in Theorem 4 will be called the *system-time* of the system defined by (2).

While this definition introduces a concept different from the concept of the stationary time set (Mesarovic and Takahara, 1975), for reasons explained below we want to formulate some of its properties and, subsequently, to introduce other time concepts.

The previous remark on the heuristics of Theorem 4 can further be completed by the following convention: the component of y of the form $y(\alpha+\mu(\alpha))$, with μ as in Theorem 4 will be called the *leading term* of (2).

Theorem 5. *Every equation (2) with an ordering which satisfies condition (3) defines a dynamic one-sided system provided the output mask has at least two elements, i.e.* $|M|\geq 2$ *and (2) has a unique solution* $y(\alpha+\mu(\alpha))$ *for every* $\alpha\in A$.

Example 3. Consider the equation

$$y^3(i+1,k)+a\,y(i+1,k)+f\big(y(i,k),\ y(i,k+1)\big)=x\,(i,\,k)$$

with $a>0,\ f(\cdot,\cdot)>0,\ x(\cdot,\cdot)<0$ for $i\geq 0,\ 0\leq k\leq i$. Assume that $y(i,k)=1$ for all $i=k$ and $y(i,k)=0$ for all $i=k-1$.

The set A is here triangularly-shaped part of the first quadrant, the output mask consists of three elements and the solution could be a transformation of a matrix x of coefficients for a system of polynomials of degree i into another system of normalized polynomials with coefficients y.

With $\mu(i,k)=(1,0)$ we may take the ordering $\prec$ to be as follows:

$$(i,k)\prec(p,q)\iff\Big((i<p\ \text{ and }\ k=q)\quad\text{or}\quad(i=p\ \text{ and }\ k>q)\Big)$$

The leading term becomes $y(i+1,k)$. Consider now the equation

$$\eta^3+a\eta+b=0$$

It is easy to find that for positive values a and b it has exactly one real root. Therefore our nonlinear difference equation satisfies all the assumptions of Theorem 5, and it defines a one-sided dynamic system. Moreover, the values of the output can be recursively calculated. ♦

Many practical applications in image and signal processing (e.g. the so-called FIR filters), some numerical computations and other fields of interest deal with the systems where $|M|=1$. With the commonly-accepted assumption of a finite input mask ($|E|<\infty$) they are comparatively simple from a system theoretic point of view, but they may be (and with $|E|=\infty$ they certainly are) computationally rather complex and difficult to handle.

Definition 12. The system S defined by (2) with $|M|=1$ will be called the *basic non-dynamic n-D discrete system.*

The discussion of non-dynamic n-D systems will be postponed, but some common properties of both dynamic and non-dynamic systems may be formulated now.

As a corollary of Theorem 3, we will consider basic linear n-D discrete systems in the following form:

$$\sum_{\mu\in M}a_\mu(\alpha)\,y(\alpha+\mu)=\sum_{\varepsilon\in E}d_\varepsilon(\alpha)\,x(\alpha+\varepsilon),\qquad\alpha\in A\subset\mathbb{Z}^n \tag{4}$$

Recall that eqn. (4) is commonly called the linear partial difference equation with variable coefficients. In accordance with Theorem 5, eqn. (4) together with a fixed ordering (see Theorem 4) defines a dynamic system. A more general class of systems, where some concepts and methods derived from linear systems theory can be applied, can also be defined.

Definition 13. A basic n-D discrete system (2) is called *quasi-linear* if for every 'leading term' $\mu(\alpha)$ there exists a function Φ_k such that

$$y(\alpha + \mu(\alpha)) = \Phi_k(\underline{x}, \underline{y}^0), \qquad \alpha \in A \subset \mathbb{Z}^n \tag{5}$$

where $\underline{y}^0$ is the vector $\underline{y}$ with the coordinate $y(\alpha + \mu(\alpha))$ dropped.

This definition (involving a special class of nonlinear partial-difference equations) also applies in an evident manner to non-dynamic systems. From Theorem 4 it follows that the number of functions Φ_k is finite.

Theorem 6. (Gregor, 1991) *A basic linear n-D system (2) is s-invariant iff all its coefficients a_μ and d_ϵ are constant (independent of α).*

While the concept of s-invariance essentially coincides with stationarity as in (Mesarovic and Takahara, 1989), here it is introduced without reference to state-space concepts. It also coincides with the commonly-used abbreviation 'LSI systems'. As will be seen below (see Theorem 8 and Example 5), the specialization of s-invariance to the concept of r-invariance is important in systems theory (Gregor, 1991; Pondělíček, 1982). In fact, it is a prerequisite of the application of z-transform to the solution to (6).

In what follows, we want to further analyse basic n-D systems in terms of general systems theory, namely their invariance properties, their states and behaviour. Although this framework seems to be sufficiently broad, three additional remarks are in order:

1. We are starting with a single equation, defining the system under consideration. All subsequent (and also previous) results can be formulated for mappings like $y : A \to \mathbb{C}^m$, $A \subset \mathbb{Z}^n$, dealing with vectors of n-D sequences, equations of the type (4) with square matrix coefficients, etc. Except for rather complicated symbolics, conceptually such an approach could not produce essentially different results except for one particular situation: Matrix multiplication has nontrivial divisors of zero and therefore in (4) with matrix coefficients a_μ special methods are necessary if these coefficients are singular matrices. Such so-called singular systems are rather difficult to handle; important results have been published on this problem, but they are out of the scope of this paper.
2. The existence of an inverse of at least some coefficients of linear equations of the type (4) is essential for uniqueness considerations. Therefore equations with nonsquare matrix coefficients are not included. Linear systems of this type are treated as AR models in the behavioural approach in (Fornasini *et al.*, 1993; Rocha, 1990.

3. In the 'real word' of mathematics and applications there exist n-D discrete systems which cannot be embedded into the framework of basic systems defined here, or such inclusion demands some additional construction (see examples below). For example, functionals defined on grids or other subsets not belonging to $\mathbb{Z}^n$ may give rise to such systems (recall here hexagonal grids, imagery of echocardiography and other medical applications). Many of these problems deserve special attention, but we feel that basic methods (if not the results) of the analysis of basic n-D systems will strongly support such investigations.

2.3. Canonical Forms of Quasilinear Systems

Applying Theorem 1 to linear and also quasilinear basic n-D dynamic systems, their explicit parametrization (see Theorem 1) can be given, i.e. with the ordering as in Theorem 4 some set G can be defined such that any mapping $g : D \to \mathbb{C}$ defines uniquely, for any input x, an output y such that $(x, y) \in S$. A pair (G, g) with $G = \{g : D \to \mathbb{C}\}$ will henceforth be called an initial state of the system S for reasons described later.

The following theorem has been proved in (Bosák and Gregor, 1987):

Theorem 7. *For a quasilinear system S, let the following set D of points be defined:*

$$D = (A + M) \setminus \bigcup_{\alpha \in A} \{\alpha + \mu(\alpha)\} \tag{6}$$

and let a set of mappings $G = \{g : D \to \mathbb{C}\}$ be considered. Then there exists a mapping $\rho : G \times X \to Y$ such that $\forall\, x \in X$, $\forall\, g \in G$ there is $(x, \rho(g, x)) \in S$.

Corollary 1. *For a linear dynamic system (3) any mapping $g \in G$ defines uniquely a function $\rho : G \times X \to Y$ such that for any x there is $(x, \rho(g, x)) \in S$, if $a_\mu(\alpha) \neq 0$ for all 'leading terms' $\mu(\alpha)$.*

The canonical forms of dynamic systems together with Theorem 4 enable us to construct the values of the output for any input by recursion. Due to such recursion not only qualitative properties of the output, as in (Veit, 1995), but also structural properties, e.g. 'state space' considerations can be derived.

For linear systems, the mappings $g : D \to \mathbb{C}$ form a linear space. Recall Theorem 2 with the function $R_1 : G \to Y$ introduced here. Denote by δ_γ the delta sequence belonging to the point $\gamma \in D$ and $y_\gamma = R_1(\delta_\gamma)$, which means $y_\gamma|_D = \delta_\gamma$. Then the function $R_1(g)$ in Theorem 2 can be written down (at least formally) as follows:

$$R_1(g) = \sum_{\gamma \in D} y_\gamma$$

Example 4. From the above considerations it follows evidently that any n-D basic linear dynamic system can be N-linear iff the set D is finite. As an example of such a simple system, the difference equation

$$y(i+1, k) + a\, y(i, k+1) + b\, y(i, k) = 0, \qquad ab \neq 0$$

for $0 \leq i \leq N-2$, $k \geq 0$ can be considered. Indeed, for given values $y(i,0)$, $i = 0,1,\ldots N-1$ this equation has a unique solution. If we denote by $y_q(i,k)$ such a solution, which assumes the value one at $(q,0)$ and the value zero at all other points $(l,0)$, $l = 0,1,\ldots,N-1$, then the linear combinations of these N solutions form a general solution of the difference equation. Similar examples can easily be constructed. ♦

Trying to formulate a similar result for the function R_2 in Theorem 2, we are led to the concept of a weight function or an abstract transfer function, as it is called in (Mesarovic and Takahara, 1975). It has been shown that in order to formalize this concept, some additional structures have to be introduced. Namely, the sets X and Y have to belong to a convolutional algebra K which is a commutative ring with the convolution operation as its multiplication.

White taking the sum $\sum_\beta a(\alpha-\beta)b(\beta)$ as a template for defining a convolution at the point α, two ways of its use can be distinguished. Either the sum has for each α only a finite number of nonzero summands, or some type of convergence has to be applied. While the first condition imposes some restrictions on the domains and their algebraic structure, the other demands restrictions of the allowed inputs and outputs (e.g. they may belong to l^p spaces).

Definition 14. Let K be a convolutional algebra in a linear system $S \subset X \times Y$, $X, Y \subset K$. Then S is called *convolutional* if there exists an element $h \in K$ such that $\rho(0,x) = h * x$.

Since applications of functional-transform techniques are widely-used in linear systems theory, this theorem shows the 'widest' frame of their use. Simple examples of eqns. (3) can be shown, which define non-r-invariant systems, and attempts to 'solve' such systems by z-transform techniques must fail (Gregor, 1988).

The following theorem has been proved in (Gregor, 1991):

Theorem 8. *A system over a convolutional algebra K is convolutional if and only if it is r-invariant.*

While systems defined by difference equations with constant coefficients are s-invariant, it is easy to give examples (Gregor, 1991) of s-invariant systems (also defined by difference equations with constant coefficients) which are not r-invariant. For these systems the concept of transfer functions cannot be meaningfully defined.

Example 5. The 2-D difference equation

$$a\,y(i-1,k+1) + y(i,k) = x(i,k), \qquad a \neq 0, \qquad i \geq 0,\ k \geq 0$$

with $y(-1,k) = 0$ for all $k > 0$ has evidently a unique solution. For $x = \delta$ the solution $y = \delta$. If this system 'obeyed the common rules', we would have $y = \delta * x = x$ which is evidently wrong. The system defined by this equation is not r-invariant, it has no meaningfull 'impulse response', it cannot be solved by z-transform, etc. ♦

3. Locally N-Linear Systems and State-Space Description

For n-D discrete systems with $n > 1$ the introduced abstract time has to be examined more closely. The output at any point of the time set depends not only on the 'previous' input, but also on the initial values which were not yet included. While in one-dimensional systems the influence of all the initial values 'gets included' after a sufficiently long time interval, the cardinality of the set D implies that this cannot be the case for any n-D system. Moreover, a time interval, constructed e.g. as the intersection $I(t, t') = \{t \prec \tau\} \cap \{t' \prec \tau\}$, may consist of an infinite number of points. Intuitively, state-space concepts are closely connected to past-determinacy. The introduction of any type of state-space description requires another type of time set to be introduced. The time set considered so far has been a linearly well-ordered set, which implies transfinite steps. A question could be posed as follows: Which of the above convenient properties has to be dropped so as to eliminate the transfinite steps and to construct a time set which could be 'locally' finite, i.e. for any of its element the number of predecessors would be finite. In this way we arrive at a new abstract time concept.

In what follows, we shall show that the construction of such 'state time' follows easily from the system time as introduced in Definition 11 and Theorem 4.

Construction. Let A and M be subsets of $\mathbb{Z}^n$, M finite with at least two elements. We know that there exists a mapping $\mu : A \to M$ and a (partial) ordering $\prec$ of A such that

$$\alpha' + \mu(\alpha') \in \alpha + M \Longrightarrow \alpha' \prec \alpha \quad \text{for all} \quad \alpha, \alpha' \in A$$

i.e. the mask M can be positioned such that it covers exactly one point where the value $y(\alpha)$ has not been calculated yet. Write

$$\begin{aligned} \Lambda_0 &= D \\ \Lambda_{q+1} &= \Lambda_q \bigcup \left\{ \cup_{\alpha \in A} (\alpha + M) : \left|(\alpha + M) \setminus \Lambda_q\right| = 1 \right\} \end{aligned}$$

i.e. in order to form Λ_{q+1}, we adjoin to Λ_q all such points of $A + M$, where the values of output can be calculated from (1) using it exactly once.

Clearly,

$$\Lambda_0 \subset \Lambda_1 \subset \cdots$$

From the proof of Theorem 4, it follows that

$$\bigcup_{k=0}^{\infty} \Lambda_k = A + M$$

which is nontrivial. Figuratively speaking, the Λ_k's are slices of the set $A + M$.

Let the sets $A^{(q)}$ be defined as follows:

$$A^{(q)} = A \setminus \Lambda_{(q)}, \qquad q = 0, 1, \ldots$$

Consider (4) or (5) with A replaced by $A^{(q)}$ and otherwise unchanged. We obtain a countable set of systems S^q which all have the same mask M and a common ordering—they all evolve in the same abstract time, defined as the system time. To each of these systems belongs an initial set, which we denote by $D^{(q)}$. The index q and the definition of Λ introduce a (partial) ordering in the set $A+M$.

Definition 15. For any $\gamma \in A+M$ denote by $k(\gamma)$ the smallest integer such that $\gamma \in \Lambda_k$. The ordering defined by ($_p$ for partial)

$$\gamma \prec_p \gamma' \Longleftrightarrow k(\gamma) < k(\gamma')$$

is called the *state time* of the system S.

To explain this construction in common terms, we may consider the slices of the set $A+M$. Each of them consists of all the points which can be calculated in one step from (4) or (5). In this way, Λ_q is enlarged to Λ_{q+1} and all the immediate successors of the elements of Λ_q are constructed. This succession (ordering) of the slices is called the state time.

Note that all the values of output at a fixed instant of the state time can be obtained independently, i.e. by parallel computation.

The following conclusion on interrelation of the state and system time can be formulated:

Theorem 9. *For any $\gamma, \gamma' \in A+M$ the inequality $\gamma \prec \gamma'$ in the ordering called system time implies $\gamma \prec_p \gamma'$ in the (partial) ordering called state time.*

The proof follows from the construction of $\prec_p$.

Since we identified time with a certain ordering of the set $A+M$, we may speak of any point of this set as being attached to a time instant, say $q \in N$. Conversely, for any such integer value q there exists a subset U_q subdividing $A+M$ into 'past' and 'future'. The cardinality of the 'past' here is always finite. In particular, for any 'time instant' q of the state time, the 'past' contains at most a finite subset of the initial values on D. Note that the state time satisfies all the requirements formulated above. In particular, at any point of this state time the number of elapsed time instants is finite. The first important corollary of this fact can be formulated as follows:

Theorem 10. *Let nonnegative integer q be a fixed instant of the state time of a linear discrete dynamic system S. Then any element of the set $(0, y^q) \in S^q$ belongs to a linear space of finite dimension.*

Proof. The interval $[0, q]$ contains $q+1$ elements (time instants) and in each of them the value of y is found by recursion, i.e. as a linear combination of a finite number of initial values and those calculated previously. The set of all initial conditions with a finite support forms a finite-dimensional linear space. ∎

Figuratively speaking, this theorem says that any linear discrete dynamic system is locally N-linear.

The properties of a discrete n-D system with respect to state time can be more transparent than those considered in system time.

The pair (G, g) has been called an (initial) state object of the system S (see Theorem 1). For basic linear and quasilinear n-D systems we have, according to the definition of state-time, a sequence of systems S^q and their corresponding (initial) state objects. Note the different meaning of S^q and $S^{(q)}$: while the former refers to a restriction of the system S up to a point q of system time, in the latter q denotes an instant of the state time.

Definition 16. The set $\Sigma = \{(G^q, g_q),\ ; q = 0, 1, \dots\}$ is called the *consistent set of initial objects* for systems defined by (4) or (5) if

$$\forall q\ \forall g_q \in G^q\ \forall x^{(q)}\quad \rho_{(q)}\big(g_q, x^{(q)}\big) \in S^q$$
$$\Longrightarrow \Big(\exists\, g \in G,\ \exists x \text{ and } \rho(g, x)^q = \rho_{(q)}\big(g_q, x^{(q)}\big)\Big)$$

The meaning and significance of this definition can be best understood by recalling that, in general, the initial state objects of the systems $S^{(q)}$ might belong to several different dynamic systems.

Definition 17. A consistent set of initial objects is called the *state space* of system S. The linearity of the state space is understood as follows:

$$\Big((G^q, g_q^i) \in \Sigma,\ i = 1, 2\Big) \Longrightarrow (G^q, a g_q^1 + b g_q^2) \in \Sigma$$

Definition 18. The mapping $\Phi : \Sigma \to \Sigma$ is called the *state-transition rule* of the system S if $\Phi(G^q, g_q) = (G^{q+1}, g_{q+1})$, $q = 0, 1, \dots$, where Σ is the state space of the system S.

In simple cases the state-transition rule is the first of the two equations usualy describing the state space model.

Example 6. We shall consider eqn. (4) with constant coefficients and with the following specifications: $n = 2$, $\alpha = (i, k)$, the domain A will be either the first quadrant $i, k \geq 0$, or the half-plane $i + k \geq 0$. The mask M will be supposed to contain the point 0, and $-M \subset A$, $i_m = \min_M(i)$, $k_m = \min_M(k)$. The system time, i.e. the ordering of the set A will be either a lexicographic ordering or a reverse lexicographic ordering. We consider therefore four different systems. According to Theorem 7, the set D will consist either of the points $\{(i, k) : \min(i_m, k_m) \leq i + k \leq 0\}$ for the half-plane, or $\{(i, k) : i_m \leq i \leq 0, k_m \leq k \leq 0\}$.

The reason why these four systems can be handled at a time is in the construction of the state-time. The slices Λ_q in all the cases are the sets $\{(i, k) : i + k = q$ and the state-transition rule has to map the set G^q into G^{q+1}. If the mask M had only three points, the simple rule would be eqn. (4) itself. If $|M| > 3$, additional 'state variables' have to be introduced so as to 'cover' all the state-time instants belonging to the mask. Clearly, no more than $i_m\ k_m$ such additional variables are necessary and their (much more interesting) minimal number depends on the shape of the mask.

This is why a vector g of the so-called state variables is introduced and the equations of the type

$$g(i+1,k+1) = Ag(i+1,k) + Bg(i,k+1)$$

are generally described as state equations (with an additional term for the input). In our case we arrived at the so-called Fornasini-Marchesini state-space model. It is worth noticing that the introduction of a number of state variables resembles the common procedure of transforming a single differential equation of order n into a system of first-order equations. In the case of an n-D system, a similar approach may be made easier when applying some results from algebra on varietes and ideals of polynomial rings.

Along the lines described in this example, other state-space models, or state-transition rules as we called them, can be constructed. Restrictions to shift invariant systems (i.e. systems described by equations with constant coefficients) seem unnecessary. ♦

4. Stability

Various concepts of stability form the core of system theory. Stability of a system is often felt as a kind of continuity: 'small changes' of the environment cause 'small changes' in the system response. Since stability is rather a property of the system and not the property of a specific input-output pair, the 'small changes' have to be specified by introducing a metric (or at least a topology) in the 'environment' and in the set of the corresponding system responses.

Recalling Theorem 1, we may introduce two concepts of stability, corresponding to two types of the 'environment' causing changes of the system response. First, various elements of the set G with a fixed input $x \in X$ cause different outputs $y \in Y$, and second, with a fixed state object $g \in G$, changes of the input $x \in X$ lead to changes of the ouput.

Hence two types of stability can be defined when introducing some type of topology in the sets G, Y or in the sets X, Y. General concepts of stability were introduced such as the Lyapunov-type stability and input-output stability in (Mesarovic and Takahara, 1989), and for n-D discrete systems no changes are necessary. We may turn directly to more practical specializations of these concepts for n-D dynamic systems considered in metric spaces. We shall write for any mappings u, v such that $u, v : A \to L_U$, $A \subset \mathbb{Z}^n$ as $d[u,v] = \sup_A |u(\cdot) - v(\cdot)|$ or $d_p^p[u,v] = \sum_A |u(\cdot) - v(\cdot)|^p$, $p > 1$ provided that the last sum exists.

Definition 19. The output $y = \rho_x(g^*)$ of an n-D dynamic system S is called *stable* at g^* for a fixed x in the sense of Lyapunov if

$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ \Big(\forall g \in G : d[g^*, g] < \delta\Big) \Rightarrow d\big[\rho_x(g), \rho_x(g^*)\big] < \varepsilon$$

Definition 20. The output $y = \rho_g(x^*)$ of an n-D dynamic system S is called *input-output stable* at x^* for a fixed g if

$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ \Big(\forall x \in X : d[x^*, x] < \delta\Big) \Rightarrow d\big[\rho_g(x), \rho_g(x^*)\big] < \varepsilon$$

The commonly-used concept of BIBO stability is an extremely specialized example of the input-output stability defined here. When basic linear systems with constant coefficients are considered, then under some assumptions the conditions of this type of stability are well-known and will not be repeated here.

5. Some Examples

The examples in this section aim at showing that widening the scope of n-D systems theory, perhaps with the above generalizations, could widen the scope of its applicability.

Example 7. In nonlinear mechanics, in order to solve the Korteweg-de Vries equation an approximation of the following form can be considered (Baumann, 1993):

$$u_t - 6\,u\,u_x + u_{xxx} = 0$$

where the subscripts denote partial derivatives with respect to the indicated variables. Numerical solution can start with the discretization of this equation as follows. The unknown function $u(x,t)$ is considered for $x = mh$, $m = 0, 1, \ldots, M$ and $t = nk$, $n = 0, 1, \ldots$, and the partial derivatives are replaced by differences. With the brief notation $u(mh, nk) = u_m^n$ and with some minor changes we arrive at

$$\begin{aligned} u_m^{n+1} &= u_m^{n-1} + \frac{6k}{3h}\left(u_{m+1}^n + u_m^n + u_{m-1}^n\right)\left(u_{m+1}^n - u_{m-1}^n\right) \\ &\quad - \frac{k}{h^3}\left(u_{m+2}^n - 2u_{m+1}^n + 2u_{m-1}^n - u_{m-2}^n\right) \end{aligned}$$

This is a canonical quasilinear system with $n = 2$, $A = [0, M] \times N_0$, $|B| = 7$ except that the set G cannot be derived solely from the initial values of the original Korteweg-de Vries equation as given in the 'continuous' formulation of the problem. From these initial data we obtain the values u only on the set $\{(m,n) : m = 0\} \subset G$ and additional considerations become necessary so as to guarantee a unique output. ♦

Example 8. An image reconstruction model can be based on the assumption that the photographic density, luminance or some other physical parameter in the digital representation of an image has been measured at any point as some kind of average value of these quantities for the original in the neighbourhood of this point. Taking this 'averaging mechanism' as a system Σ with the original values as its input and measured values as its output, we want to study some 'inverse' system, i.e. a method of reconstructing the original from the measured data. Assuming that the digital representation of the image is a mapping $y : A \to \mathbb{R}$, $A \subset \mathbb{Z}^2$ and defining the neighbourhood as the five points of a cross-like mask, we may give the inverse system in the basic form as follows. Denoting the measured and original values at a point (i, k) by $x(i, k)$ and $y(i, k)$, respectively, the original image y can be reconstructed from the measured data x when solving a PDE of the form

$$\mathbf{aver}\Big(y(i,k), y(i+1,k), y(i-1,k), y(i,k+1), y(i,k-1)\Big) = x(i,k)$$

where $\mathbf{aver}(\cdot)$ is some kind of average value of its arguments, $0 \le i \le M$, $0 \le k \le N$. Such an equation defines a system in $\mathbb{Z}^2$.

Considering $\mathbf{aver}(\cdot) = Ar(\cdot)$ as the weighted arithmetic mean, we obtain a linear system. It can be considered as a dynamic system and a procedure of recursively calculating its output (the original image) can be given along the lines described above.

When $\mathbf{aver}(\cdot) = Ge(\cdot)$ (i.e. a weighted geometric mean), then taking the logarithm we arrive again at a linear system, but now the existence of a suitable solution of this linear equation is not so evident. On the other hand, the solution of the (non-linear) PDE with $Ge(\cdot)$ as its left-hand side seems to cause no difficulties, since it is easy to give it a quasilinear form.

If $\mathbf{aver}(\cdot) = Hr(\cdot)$ with Hr for the harmonic mean, the equation could have the following form:

$$\frac{8}{x(i,k)} = \frac{1}{y(i+1,k)} + \frac{1}{y(i-1,k)} + \frac{1}{y(i,k+1)} + \frac{1}{y(i,k-1)} + \frac{4}{y(i,k)}$$

and the existence of its solution and its uniqueness are not obvious. Again a quasi-linear formulation can give the result.

When $\mathbf{aver}(\cdot)$ denotes the median of its arguments, no obvious way of solution seems to be at hand. ♦

Example 9. Let the starting point of the previous example be revisited. Clearly, it depends on the concept of a 'discrete neighbourhood'. In our model each point had four other points in its neighbourhood, since the plane had been subdivided by equal rectangles. Taking regular hexagons or equilateral triangles instead, this number will change to three or six, respectively. Since the vertices form a countable set, we may speak of a discrete system (see Definition 4) and there exists a one-to-one mapping into $\mathbb{Z}^2$, it is a free 2-D system (see Definition 5). Although the corresponding system can again be handled as a linear (or quasilinear) system, it is not in basic form any more.

It has been shown (Veit, 1995) that after some renumbering of the countable set of vertices, the system could be described as a structured system in basic form. Since the set of vertices is not endowed with a semigroup structure, the original system is not structured (in the sense of Definition 6) and its basic form cannot be s-invariant.

Comparison of the asymptotic behaviour of such systems with a Dirac impulse at its input could yield some interesting distinctions between the spread of an excitation through nets of various structure. ♦

Example 10. Fix two real values x, y and consider the equations

$$b(x+n+1, y+m) + b(x+n, y+m+1) = b(x+n, y+m)$$

and

$$y+m\ b(x+n, y+m) = (x+n+y+m)b(x+n, y+m+1)$$

for $x+n > 0$, $y+m > 0$. Together with the conditions $b(x+n,\ 1) = 1/(x+n)$, $b(1, y+m) = 1/(y+m)$ they are satisfied by the (specialized) Beta function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t$$

Does one of these (or both) determine the Beta function uniquely for all positive values of x and y? Note that the sequence $f(n,m) = \lambda^n(1-\lambda)^m$ satisfies the first of these equations for any real λ. We may ask whether there exist two sequences λ_k and μ_k such that $b(n,m) = \sum_k \mu_k \lambda_k^n (1-\lambda_k)^m$. ♦

Example 11. Consider the set $C_k = \{(m,n), m, n \in \mathbb{Z},\ 0 \le n < k\}$ with the following rule of addition:

$$(m_1, n_1) + (m_2, n_2) = \Big(m_1 + m_2, (n_1 + n_2) \mod k\Big)$$

An equation of the form (4) can be defined with $\alpha \in C_k$. This equation then defines a linear discrete structured system which is not in basic form. In mathematics we would call eqn. (4) in this case a partial-difference equation on a cylinder. The construction of an ordering, i.e. the definition of a time concept, is not included in the previous reasoning. The existence and uniqueness of solution of (4) is close to analogous questions for periodic n-D systems. ♦

6. Some Conclusions

The above constructions are formulated as a framework for further discussions. They are generalizations of models applied so far and, similarly to other general approaches, they have to be completed by further new concepts as they appear in actual applications.

As it has been noted earlier, systems of linear difference equations with non-square matrix coefficients are not included into this framework. It seems that here the behavioural approach or geometric approaches may give better results. We also paid less attention to transform methods, such as the z-transform, and also to the state-space representation of n-D systems. In our treatment both these concepts are secondary as they may appear at a certain stage of system analysis. The application of z-transform is justified only in a very special case of n-D systems, although it is formally used in many papers on n-D systems theory. Conditions of its application have already been described elsewhere (Gregor, 1988). Here they would make the structure of our reasoning less transparent. The state-space description is often studied as the starting point of n-D systems theory for no evident reason. We did not want to restrict our attention here to overspecialized concepts.

In our opinion, the discussion of general concepts and methods could find new impulses for further development of both theory and applications of n-D discrete systems.

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