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CONTROLLABILITY OF 2-D SYSTEMS: A SURVEY

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In this paper, a survey of recent results concerning controllability of 2-D systems is presented. Various types of linear and nonlinear 2-D systems with constant coefficients are discussed. Several controllability conditions for various kinds of 2-D systems are formulated without proofs. Moreover, many supplementary remarks and bibliographical comments are given. The relationships between different concepts of controllability are also explained.

1. Introduction

Controllability is one of the fundamental concepts in modern mathematical control theory. Many dynamical systems are such that the control does not affect the complete state of the dynamical system but only some of its components. Therefore it is important to determine whether or not a complete system control is possible. Roughly speaking, controllability generally means that it is possible to steer the dynamical system from an arbitrary initial state to an arbitrary final state using a given set of admissible controls. In the literature there are many different definitions of controllability which depend on the type of dynamical system. The extensive list of publications concerning various controllability problems, containing more than 500 positions can be found in the monograph (Klamka, 1991a). Moreover, a survey of recent results and the current state of controllability theory for different types of dynamical systems can be found in the paper (Klamka, 1993b).

A growing interest has been observed over the past few years in problems involving signals and systems that depend on more than one independent variable. The motivations for studying 2-D systems have been well justified in several papers and monographs (Fornasini and Marchesini, 1979; Kaczorek, 1985; Klamka, 1991a; 1993a; Kurek and Zaremba, 1993; Roesser, 1975). Most of the major results concerning the multidimensional signals and systems are developed for two-dimensional cases. Discrete dynamical systems with two independent variables, called the 2-D systems, are important in image processing, multivariable network realizability, and in multidimensional digital filters (Fornasini and Marchesini, 1979; Kaczorek, 1985; Kurek, 1985; Roesser, 1975).

During the last two decades controllability of 2-D systems has been considered in many papers and books (Fornasini and Marchesini, 1979; Kaczorek, 1985; Klamka, 1991a; 1993a; Roesser, 1975). The main purpose of this paper is to present a compact

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review over the existing controllability results for 2-D systems. The majority of the results in this area concerns linear 2-D systems with constant coefficients.

The paper is organized as follows. Section 2 contains systems descriptions and fundamental results concerning unconstrained controllability for the most popular linear 2-D models with constant coefficients. In Section 3 unconstrained controllability of linear singular 2-D systems with constant coefficients is discussed. Section 4 is devoted to a study of constrained controllability of linear 2-D systems. Special attention is paid to the so-called positive controllability. Section 5 presents results on positive controllability for linear positive 2-D systems. In Section 6 controllability of the so-called continuous-discrete linear 2-D systems is investigated. Local controllability of nonlinear 2-D systems with constrained controls is considered in Section 7. Finally, in Section 8 concluding remarks and comments concerning possible extensions are presented. Since the paper should be limited to a reasonable size, it is impossible to give a comprehensive lecture on the subject. In consequence, only selected fundamental results without proofs are presented. Sections consist of a few major results, some additional bibliographical comments and supplementary remarks.

2. Unconstrained Controllability

In the theory of 2-D systems several different models are considered (Kaczorek, 1985; Klamka 1991a). The most popular and the most frequently used are the Fornasini-Marchesini model (Fornasini and Marchesini, 1979) and the Roesser model (Roesser,1975).

First, let us consider the Fornasini-Marchesini model of a linear 2-D system with constant coefficients given by the following difference equation (Fornasini and Marchesini, 1979):

$$x(i+1,j+1) = A_0 x(i,j) + A_1 x(i+1,j) + A_2 x(i,j+1) + Bu(i,j) \quad (1)$$

where $i,j \in \mathbb{Z}_+ = \{0,1,2,3,\dots\}$, $x(i,j) \in \mathbb{R}^n$ is a local state vector, $u(i,j) \in U \subset \mathbb{R}^m$ denotes an input vector, U is a given set, A_0, A_1, A_2, and B are real matrices of appropriate dimensions. Boundary conditions for eqn. (1) are given by the following equalities:

$$\begin{cases} x(i,0) = x_{i0} \in \mathbb{R}^n & \text{for } i \in \mathbb{Z}_+ \\ x(0,j) = x_{0j} \in \mathbb{R}^n & \text{for } j \in \mathbb{Z}_+ \end{cases} \quad (2)$$

In order to present the general response formula for eqn. (1) in a convenient compact form, it is necessary to introduce the $(n \times n)$-dimensional state transition matrix $A^{i,j}$ defined as follows (Kaczorek, 1986a):

(i) $A^{0,0} = I$,

(ii) $A^{-i,j} = A^{i,-j} = A^{-i,-j} = 0$ for $i,j > 0$,

(iii) $A^{i,j} = A_0 A^{i-1,j-1} + A_1 A^{i,j-1} + A_2 A^{i-1,j} = A^{i-1,j-1} A_0 + A^{i,j-1} A_1 + A^{i-1,j} A_2$ for $i,j > 0$,

where I is the identity matrix.

Therefore, the general response formula for eqn. (1) with boundary conditions (2) and a given sequence of admissible controls has the following compact form (Kaczorek, 1986a; Kaczorek and Kurek, 1985):

$$\begin{aligned} x(i,j) &= A^{i-1,j-1}A_0x_{00} + \sum_{p=0}^{p=i}(A^{i-p,j-1}A_1 + A^{i-p-1,j-1}A_0)x_{p,0} \\ &\quad + \sum_{q=0}^{q=j}(A^{i-1,j-q}A_2 + A^{i-1,j-q-1}A_0)x_{0,q} \\ &\quad + \sum_{p=0}^{p=i-1}\sum_{q=0}^{q=j-1} A^{i-p-1,j-q-1}Bu(p,q) \end{aligned} \tag{3}$$

It is well-known that for 2-D systems it is possible to introduce several different notions of controllability. For example, we may consider global controllability of 2-D systems (Gaishun and Goryachkin, 1988; 1991; Gaishun and Hoang Van Kuang, 1992; 1993) or the so-called straight-line controllability of 2-D systems (Kaczorek, 1987b; 1987c; 1987d).

Now, let us recall the most popular and frequently-used fundamental definition of unconstrained controllability in a given rectangle $[(0,0),(r,s)]$ for linear 2-D systems with constant coefficients (Fornasisni and Marchesini, 1982; Kaczorek, 1985; 1992a; 1994b; 1994e; 1996e; Klamka, 1984a; 1991a).

Definition 1. System (1) with boundary conditions (2) is said to be *controllable in a given rectangle* $[(0,0),(r,s)]$ if for all boundary conditions (2) and every vector $x_{rs} \in \mathbb{R}^n$, there exists a sequence of controls $u(i,j) \in \mathbb{R}^m$, $(0,0) \le (i,j) < (r,s)$ such that $x(r,s) = x_{rs}$.

From the general formula (3) it follows immediately that for zero boundary conditions $x(i,0) = x_{i0} = x(0,j) = x_{0j} = 0$, for $i,j \in \mathbb{Z}_+$, the solution $x(i,j)$ of eqn. (1) is simply given by the equality

$$x(i,j) = W_{ij}u_{ij}$$

where $W_{ij} = [A^{i-1,j-1}B,\ A^{i-2,j-1}B, \dots, A^{0,j-1}B,\ A^{i-1,j-2}B, \dots, A^{1,0}B,\ B]$ is an $(n \times ij)$-dimensional matrix with constant coefficients and the sequence of admissible controls u_{ij} is given by

$$\begin{aligned} u_{ij} = \big[&u^T(0,0), u^T(1,0), \dots, u^T(i-1,0), u^T(0,1), \\ &\dots, u^T(i-2,j-1), u^T(i-1,j-1)\big]^T \end{aligned}$$

Theorem 1. (Kaczorek, 1985) *System (1) is controllable in a given rectangle* $[(0,0),(r,s)]$ *with unconstrained controls if and only if*

$$\operatorname{rank} W_{rs} = n$$

Corollary 1. (Kaczorek, 1985) *System (1) is controllable in a given rectangle* $[(0,0),(r,s)]$ *if and only if the* $(n \times n)$*-dimensional symmetric matrix* $W_{rs}W_{rs}^T$ *is nonsingular.*

Now, let us consider the Roesser model of a 2-D linear system given by the set of two difference equation (Roesser, 1975; Kaczorek, 1985; 1989b)

$$\begin{vmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{vmatrix} = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \begin{vmatrix} x^h(i,j) \\ x^v(i,j) \end{vmatrix} + \begin{vmatrix} B_1 \\ B_2 \end{vmatrix} u(i,j) \tag{4}$$

where $x^h(i,j) \in \mathbb{R}^{n_1}$ is the horizontal state vector, $x^v(i,j) \in \mathbb{R}^{n_2}$ is the vertical state vector, A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2 are real matrices of appropriate dimensions. Boundary conditions are given by

$$\begin{cases} x^h(0,j) = x^h_{0j} \in \mathbb{R}^{n_1} & \text{for } j = \mathbb{Z}_+ \\ x^v(i,0) = x^v_{i0} \in \mathbb{R}^{n_2} & \text{for } j = \mathbb{Z}_+ \end{cases} \tag{5}$$

Let $n = n_1 + n_2$ and let us introduce the n-dimensional vector $x' \in \mathbb{R}^n$, the $(n \times n)$-dimensional matrix A' and the $(n \times m)$-dimensional matrix B' defined as follows:

$$x' = \begin{vmatrix} x^h(i,j) \\ x^v(i,j) \end{vmatrix}, \qquad A' = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}, \qquad B' = \begin{vmatrix} B_1 \\ B_2 \end{vmatrix}$$

The state transition matrix $A'^{i,j}$ for the Roesser 2-D system is defined as follows (Kaczorek, 1985; 1989b; Klamka, 1984b; Roesser 1975):

(i) $A'^{0,0} = I$,

(ii) $A'^{1,0} = \begin{vmatrix} A_{11} & A_{12} \\ 0 & 0 \end{vmatrix}, \qquad A'^{0,1} = \begin{vmatrix} 0 & 0 \\ A_{21} & A_{22} \end{vmatrix},$

(iii) $A'^{i,j} = A'^{1,0}A'^{i-1,j} + A'^{0,1}A'^{i,j-1}$ for $i,j = 1,2,3,\dots,$

(iv) $A'^{i,j} = 0$ for $i < 0$ and/or $j < 0$.

Using the state transition matrix $A'^{i,j}$ it is possible to express the solution of the Roesser model in a convenient, compact form which is quite similar to those given by formula (3) for the Fornasini-Marchesini model (Kaczorek, 1985; 1989b; Roesser 1975).

Definition 2. (Eising, 1979; Kaczorek, 1985; 1989b; Roesser, 1975) The Roesser model of the 2-D system (4) is said to be *controllable in a given rectangle* $[(0,0),(r,s)]$ if for all boundary conditions (5) and every vector $x'_{rs} \in \mathbb{R}^n$, there exists a sequence of controls $u(i,j) \in \mathbb{R}^m$, $(0,0) \le (i,j) < (r,s)$ such that $x'(r,s) = x'_{rs}$.

Using the state transition matrix $A'^{i,j}$ and the formula for the solution it is possible to formulate the following necessary and sufficient condition for controllability of the Roesser model in a given rectangle $[(0,0),(r,s)]$.

Theorem 2. (Eising, 1979; Kaczorek, 1985; Roesser 1975) *The Roesser model (4) is controllable in a given rectangle* $[(0,0),(r,s)]$ *if and only if*

$$\operatorname{rank} W'_{rs} = n$$

where $W'_{rs} = [M'(0,1), M'(1,0), \dots, M'(i,j), \dots, M'(r,s)]$ *and the* $(n \times m)$*-matrices* $M'(i,j)$ *are defined as follows:*

$$M'(i,j) = A'^{i-1,j} B'^{1,0} + A'^{i,j-1} B'^{0,1}$$

Corollary 2. (Eising, 1979; Kaczorek, 1985; Roesser 1975) *The Roesser model (4) is controllable in a given rectangle* $[(0,0),(r,s)]$ *if and only if the* $(n \times n)$*-dimensional symmetric matrix* $W'_{rs} W'^{T}_{rs}$ *is nonsingular.*

It should be pointed out that a more general concept of controllability, namely the so-called straight-line controllability has been considered in (Kaczorek 1987c; 1987d). Moreover, using the concept of the attainable set, it is possible to introduce the notion of reachability for 2-D systems (Kaczorek, 1993b). The relationships between the concepts of controllability and reachability are explained in the papers (Kaczorek, 1994b; 1994e).

3. Singular Systems

Recently the classical regular 2-D systems have been extended to singular 2-D systems also known as implicit, descriptor or generalized 2-D systems (Kaczorek, 1988a; 1988b; 1988c; 1989c; 1990; 1991; 1992b; 1994c; 1995d; 1996d; Klamka, 1989; 1991b; 1991c; 1995a; Lewis, 1992). The motivations for studying singular 2-D systems have been well justified in several papers (Kaczorek, 1994c; Kocięcki, 1993; Lewis, 1992) where the reader may find many examples of possible practical applications of singular 2-D systems in network theory, chemical engineering, robotics or economics. Two-dimensional singular systems, unlike the familiar state-space 2-D systems, do not require any notion of causality or recursibility. Instead, they require a milder notion of regularity which is required for existence and uniqueness of solutions. Thus the singular 2-D systems are more natural and suited for description of naturally occurring 2-D systems, e.g. for applications in image processing. It is also possible to consider linear singular 2-D systems with delays (Kaczorek, 1992b) and infinite-dimensional singular 2-D systems (Klamka, 1995a). A survey of recent results and the current state of the theory of 2-D singular systems can be found e.g. in (Kaczorek, 1991; Lewis, 1992).

Let us consider a linear singular 2-D system with constant coefficients described by the following difference equation (Kaczorek, 1988a; Lewis, 1992):

$$Ex(i+1,j+1) = A_0 x(i,j) + A_1 x(i+1,j) + A_2 x(i,j+1) + Bu(i,j) \qquad (6)$$

where A_0, A_1, A_2 are the same matrices as in eqn. (1) and E is an $(n \times n)$-dimensional (possibly singular) matrix.

It is well-known that a given singular 2-D system has a unique solution only for admissible boundary conditions (Kaczorek, 1988a; 1988b; 1988c; 1991; 1992c). The linear subspace of admissible boundary conditions depends on the matrices E, A_0, A_1, A_2 and the input sequence. However, the following lemma holds.

Lemma 1. (Kaczorek, 1992c) *All boundary conditions are admissible in any rectangle and for any input sequence if and only if*

$$\operatorname{rank}\left[z_1 z_2 E - A_0 - z_1 A_1 - z_2 A_2\right] = \operatorname{rank}\left[z_1 z_2 E - A_0 - z_1 A_1 - z_2 A_2 \mid B\right]$$

for some $z_1, z_2 \in \mathbb{C}$.

The solution for admissible boundary conditions and a given admissible control sequence has the following form (Kaczorek, 1988a; 1988b; 1988c; 1989c; 1990; 1991; 1992b; Lewis, 1992):

$$\begin{aligned} Ex(i,j) &= A^{i-1,j-1}A_0 x(0,0) + \sum_{p=0}^{p=i}(A^{i-p,j-1}A_1 + A^{i-p-1,j-1}A_0)x(p,0) \\ &\quad + \sum_{q=0}^{q=j}(A^{i-1,j-q}A_2 + A^{i-1,j-q-1}A_0)x(0,q) \\ &\quad + \sum_{p=0}^{p=i-1}\sum_{q=0}^{q=j-1} A^{i-p-1,j-q-1}Bu(p,q) \end{aligned} \tag{7}$$

Theorem 3. (Kaczorek, 1989a) *The singular system (5) is locally controllable in the rectangle* $[(0,0),(r,s)]$ *if and only if*

$$\operatorname{rank} M_{rs} = n$$

where

$$\begin{aligned} M_{rs} = \big[&M(0,0), M(1,0), \dots, M(r,0), M(0,1), \dots, M(0,s), M(1,1), \\ &\dots, M(r-1,s), M(r,1), \dots, M(r,s-1)\big] \end{aligned}$$

and

$$M(i,j) = A^{r-i-1,s-j-1}B \quad \text{for} \quad i = 1,2,3,\dots,r, \quad j = 1,2,3,\dots,s$$

Corollary 3. (Kaczorek, 1989a) *System (6) is controllable in a given rectangle* $[(0,0),(r,s)]$ *if and only if the* $(n \times n)$*-dimensional symmetric matrix* $M_{rs}M_{rs}^T$ *is nonsingular.*

Similar controllability conditions can be derived for the Roesser model of linear singular 2-D systems (Kaczorek, 1989c).

Finally, let us observe, that it is possible to consider a more general linear singular 2-D system with a rectangular matrix E (Kaczorek, 1990; 1991).

4. Constrained Controllability

In recent years controllability problems for various kinds of 2-D systems have been considered in many publications. So far most works in this direction have been concerned, however, with the so-called unconstrained controllability problems (Eising, 1979; Fornasini and Marchesini, 1982; Kaczorek, 1985; Klamka, 1991a; Lin et al., 1987; Roesser, 1975; Sebek et al., 1988). Only a few papers deal with the so-called constrained controllability problems, i.e. with the case when the control functions are restricted to take their values in a prescribed admissible set (Kaczorek, 1995a; 1996f; Klamka, 1988a; 1994b; 1996a). Moreover, it should also be stressed that up to now constrained controllability problems for abstract retarded dynamical systems defined in infinite-dimensional Hilbert spaces have not been considered in the literature.

In this section we shall present some necessary and sufficient conditions for controllability of the linear 2-D system (1) with constrained controls.

Let $U \subset \mathbb{R}^m$ be a given arbitrary set in $\mathbb{R}^m$. The sequence of controls $u = \{u(i,j);\ (0,0) \leq (i,j),\ u(i,j) \in U\}$ is called an admissible sequence of controls. The set of all such admissible sequences of controls forms the so-called admissible set of controls. In the sequel, for the nonlinear 2-D systems we shall also use the following notations: $\Omega^0 \subset \mathbb{R}^m$ is a neighbourhood of zero, $U^c \subset \mathbb{R}^m$ is a closed convex cone with vertex at zero and $U^{c0} = U^c \cap \Omega^0$.

Let $\mathbb{R}_+ = [0,\infty)$. We will denote by $\mathbb{R}^n_+$ the set of n-tuples of nonnegative real numbers, i.e. $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_k \geq 0\ ,\ \text{for}\ k = 1,2,\dots,n\}$. Let the sequence of admissible controls be given by

$$u_{ij} = \big[u^T(0,0), u^T(1,0), \dots, u^T(i-1,0), u^T(0,1), \\ \dots, u^T(i-2,j-1), u^T(i-1,j-1)\big]^T \in U_{ij}$$

where T denotes the transposition, and $U_{ij} = U \times U \times \cdots \times U$ is the ij-times Cartesian product of the set U.

Consider the special case when $U = \mathbb{R}^m_+$ is a closed convex cone of m-dimensional vectors with nonnegative elements and write $U^+_{ij} = \mathbb{R}^m_+ \times \mathbb{R}^m_+ \times \cdots \times \mathbb{R}^m_+$.

Following (Klamka, 1994c), let $V_{rs} \in \mathbb{R}^n$ denote the reachable cone at the point (r,s) from zero boundary conditions and for nonnegative controls, i.e. the image of U^+_{rs} under a linear mapping W_{rs}. Hence

$$V_{rs} = \Big\{x_{rs} \in \mathbb{R}^n : x(r,s) = W_{rs}u_{rs},\ \ u_{rs} \in U_{rs}\Big\}$$

V_{rs} is a closed convex cone with vertex at zero, generated by the columns of the matrix W_{rs}. Therefore it is possible to define the so-called polar cone V^*_{rs} as follows:

$$V^*_{rs} = \Big\{x^* \in \mathbb{R}^n : \langle x^*, x\rangle \leq 0,\quad \text{for all}\ x \in V_{rs}\Big\}$$

It should be pointed out that V_{rs} is contained in V_{hk} for every $(r,s) \leq (h,k)$.

Let us observe that for constrained controls, in contrast to the classical unconstrained case, the first term in the formula (3) containing boundary conditions (2) drags the reachable cones V_{ij} around in the positive orthant $\mathbb{R}^n_+$. Hence the 2-D system (1) with constrained controls and nonzero boundary conditions (2) may be controllable in the rectangle $[(0,0),(r,s)]$, even if is not controllable in the rectangle $[(0,0),(h,k)]$ for $(h,k) > (r,s)$. This was impossible in linear discrete systems for unconstrained controls. Therefore, in the sequel for simplicity of considerations, we generally assume zero boundary conditions for nonnegative controls.

Definition 3. (Klamka, 1994c) System (1) is said to be $\mathbb{R}^m_+$*-controllable in a given rectangle* $[(0,0),(r,s,)]$ if, for each vector $x_{rs} \in \mathbb{R}^n$, there exists a sequence of nonnegative admissible controls u_{rs} such that $x(r,s) = x_{rs}$, or equivalently $V_{rs} = \mathbb{R}^n$.

It should be mentioned that a quite similar definition is valid for U^c-controllability, when U^c is an arbitrary cone in $\mathbb{R}^m$ and, more generally, for U-controllability if U is an arbitrary set in $\mathbb{R}^m$.

Theorem 4. (Klamka, 1994c) *System (1) is* $\mathbb{R}^m_+$*-controllable in a given rectangle* $[(0,0),(r,s)]$ *if and only if*

(i) $\operatorname{rank} W_{rs} = n$,

(ii) $V^*_{rs} = \{0\}$,

where V^*_{rs} *denotes a polar cone.*

Let us observe that (i) is a necessary and sufficient condition for unconstrained controllability of the system (1) in a given rectangle (Fornasini and Marchesini, 1979). Therefore, constrained $\mathbb{R}^m_+$-controllability always implies unconstrained controllability in the same rectangle.

In the present section, various constrained controllability problems for linear 2-D systems with constant coefficients have been discussed. The result of this section may be used in controllability considerations for nonlinear 2-D systems.

5. Positive Systems

Positive systems, in which the state is constrained to lie in the positive orthant $\mathbb{R}^n_+$, are common in signal processing, image processing, digital communications, economic and social sciences, biological and chemical applications (Schanbacher, 1989). The state variables may represent populations, image description, quantities of good, or masses of chemical species. In the past, the underlying positivity of these systems has often been ignored or accommodated in an *ad-hoc* fashion, in order to take advantage of the well-developed theory of linear systems, which assumes that the states are drawn from a vector space.

Recently, there have been a number of attempts to address system issues directly in the context of discrete positive systems (Coxson and Shapiro, 1987; Kaczorek, 1996c). These research efforts have answered some questions and have raised many new ones. They have revealed important qualitative differences from the corresponding theories for unconstrained systems. Until now, scarce attention has been paid to the important case where the control of systems is realizable in only one direction. Our main concern is to ask which positive states can be reached for positivity preserving systems if the controls are taken to be positive. It is important to note here that there are two constraints involved: the positivity constraint on the state and the corresponding positive constraint on the controls.

In this section we examine the issue of controllability for positive linear 2-D systems. We look at the connections between reachability, reachability from zero boundary conditions, and the rank criterion for controllability for systems in which both the state and controls are constrained by practical considerations to lie in the positive orthant. While these concepts are equivalent for unconstrained systems, there are significant changes in the positive case. We analyze these differences and show how they affect the control strategy. Using the methods of linear algebra we formulate and prove the necessary and sufficient conditions for constrained controllability of positive 2-D systems. Several examples which are representative of a large class of industrial applications motivate and illustrate our results.

We may consider system (1) with nonnegative coefficients $A_0 \in \mathbb{R}_+^{n\times n}$, $A_1 \in \mathbb{R}_+^{n\times n}$, $A_2 \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times m}$, nonnegative boundary conditions (2), $x_{i0} \in \mathbb{R}_+^n$, $x_{0j} \in \mathbb{R}_+^n$, and nonnegative admissible controls, i.e. $u(i,j) \in \mathbb{R}_+^m$, for $(i,j) \in \mathbb{Z}_+ \times Z_+$. Such a system is called the positive linear 2-D system. For a positive linear 2-D system (1) the (r,s)-reachable cone $V_{rs} \subset \mathbb{R}_+^n$.

For positive 2-D systems the matrix W_{ij} has nonnegative coefficients (Kaczorek, 1996c). Let us observe that matrices which are nonnegative and have nonnegative inverses can be expressed as the product of a nonsingular diagonal matrix and a permutation matrix. The product of a nonsingular diagonal matrix (not necessarily nonnegative) and a permutation matrix is called the monomial. It has one nonzero entry in each row and column.

Since for positive 2-D systems the trajectory always remains in the positive orthant in the n-dimensional state space, the positive 2-D systems are never controllable in any rectangle $[(0,0),(r,s)]$. Therefore, for positive 2-D systems it is convenient to introduce a weaker concept of controllability, namely the so-called positive controllability in a given rectangle $[(0,0),(r,s)]$.

Definition 4. (Kaczorek, 1996c) A positive 2-D system (1) is said to be *positive controllable in a given rectangle* $[(0,0),(r,s)]$ if, for each nonnegative vector $x_{rs} \in \mathbb{R}_+^n$, there exists a sequence of nonnegative admissible controls u_{rs} such that $x(r,s) = x_{rs}$, or equivalently $V_{rs} = \mathbb{R}_+^n$.

Let us observe that the positive 2-D system (1) is never positive controllable in any rectangle $[(0,0),(r,s)]$ for nonnegative boundary conditions (2). This statement follows immediately from the positivity of the state transition matrix $A^{i,j}$ for every (i,j) and the form of the solution (3). Therefore, in the next section we shall concentrate on positive controllability for zero boundary conditions (2).

Now, under the assumption that A is not a nilpotent matrix, we shall formulate a necessary and sufficient condition for positive controllability of the positive 2-D system (1).

Theorem 5. (Kaczorek, 1996c) *The positive system (1) is positive controllable in a given rectangle* $[(0,0),(r,s)]$ *if and only if the matrix* W_{rs} *has an* $(n \times n)$*-dimensional monomial submatrix.*

In the next theorem, we shall present a simple sufficient condition for positive controllability of positive 2-D systems.

Theorem 6. (Kaczorek, 1996c) *The positive system (1) is positive controllable in a given rectangle* $[(0,0),(r,s)]$ *if*

(i) $\operatorname{rank} W_{rs} = n$,

(ii) *there exists at least one matrix* $W_{rs}^{-1} \in \mathbb{R}_{+}^{rs \times n}$,

where W_{rs}^{-1} *denote the right inverse of the matrix* W_{rs}.

In this section different kinds of constrained controllability in a given rectangle for linear positive 2-D systems have been considered. Using pure algebraic methods, some conditions for positive controllability of positive 2-D systems in a given rectangle have been formulated.

6. Continuous-Discrete Systems

2-D continuous-discrete linear dynamical systems, i.e. dynamical systems described by an indexed set of linear ordinary differential equations with appropriate initial and boundary conditions (Kaczorek, 1994d; 1994g; Kurek and Zaremba, 1993).

Let us consider a linear time-invariant 2-D continuous-discrete dynamical system described by the following set of ordinary differential equations (Kaczorek, 1994a; 1994c; 1994d; 1994f; 1994g):

$$\dot{x}_{k+1}(t) = A_0 x_k(t) + A_1 x_{k+1}(t) + B u_k(t) \tag{8}$$

where $t \in \mathbb{R}^+$ and $k \in \mathbb{Z}^+$, $\mathbb{R}^+$ stands for the set of nonnegative real numbers, $\mathbb{Z}^+$ denotes the set of nonnegative integer numbers, $x_k(t) \in \mathbb{R}^n$ is the local state vector, $u_k(t) \in \mathbb{R}^m$ is the control function, A_0, A_1, B are constant matrices of appropriate dimensions.

Let us observe that in order to solve the differential eqn. (1), it is necessary to give boundary and initial conditions of the following form:

$$x_0(t) = p(t) \quad \text{for} \quad t \in \mathbb{R}^+, \qquad x_k(0) = q(k) \quad \text{for} \quad k \in \mathbb{Z}^+$$

where $p(t)$ and $q(k)$ are given functions such that $p(0) = q(0)$.

Without loss of generality, for simplicity of considerations, let us assume that $p(t) = 0$ and $q(k) = 0$, i.e. we have homogeneous boundary and initial conditions.

Moreover, it is assumed that the admissible controls are square-integrable functions of time, i.e. $u_k(t) \in L_2([0,\infty),\mathbb{R}^m)$.

Let us introduce the following notation:

$$x(t,k) = \Big(x_0(t), x_1(t), \dots, x_k(t)\Big), \qquad k = 0,1,2,\dots$$

$$u(t,k) = \Big(u_0(t), u_1(t), \dots, u_k(t)\Big), \qquad k = 0,1,2,\dots$$

Definition 5. (Kaczorek, 1994d; 1994f; 1994g) The continuous-discrete system (8) is said to be *controllable in a given rectangle* $[(0,0),(T,K)]$ if, for each vector $x_{TK} \in \mathbb{R}^n$, there exists a sequence of admissible controls $u_k(t)$, $k = 0,1,2,\dots,(K-1)$, $t \in [0,T]$ such that $x(T,K) = x_{TK}$.

Write

$$F = \begin{vmatrix} A_0 & A_1 & 0 & 0 & \cdots \\ 0 & A_0 & A_1 & 0 & \cdots \\ 0 & 0 & A_0 & A_1 & \cdots \\ 0 & 0 & 0 & A_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}, \qquad G = \begin{vmatrix} B & 0 & 0 & 0 & \cdots \\ 0 & B & 0 & 0 & \cdots \\ 0 & 0 & B & 0 & \cdots \\ 0 & 0 & 0 & B & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

Therefore eqn. (8) with homogeneous initial and boundary conditions can be expressed equivalently as the linear difference equation

$$x(t,k+1) = Fx(t,k) + Gu(t,k), \qquad k \in \mathbb{Z}^+ \tag{9}$$

with zero initial condition $x(t,0) = 0$.

Now, using the results given in (Kaczorek, 1994a; 1994d; 1994g) it is possible to write down the general response formula for the continuous-discrete dynamical system (8) with homogeneous initial and boundary conditions:

$$x(t,k) = \sum_{i=0}^{i=k-1} F^{k-i-1}Gu(i)$$

Let us introduce the symmetric matrix $W(k)$ defined as follows:

$$W(k) = \sum_{i=0}^{i=k-1} F^i G G^T F^{Ti}, \qquad k = 1,2,3,\dots$$

Theorem 7. (Kaczorek, 1994d) *The continuous-discrete dynamical system (8) is controllable in k steps if and only if the matrix $W(k)$ is nonsingular.*

Let us note that in this framework we can easily study a quite general class of continuous-discrete dynamical systems defined in infinite-dimensional Banach or Hilbert spaces (Klamka, 1995a; 1995b). Finally, it should be stressed that it is also possible to consider singular continuous-discrete systems. The idea of these systems has been presented in (Kaczorek, 1994c; 1994d; 1996b; 1996d).

7. Nonlinear Systems

Up to now the problem of controllability in continuous and discrete time for linear dynamic systems has been extensively investigated in many monographs and papers (Kaczorek, 1985; 1993a; Klamka, 1991a; 1993a). However, this is not true for nonlinear dynamic systems, especially with constrained controls. Only a few papers concern constrained controllability problems for 2-D nonlinear systems (Klamka, 1992; 1996d; 1997a). The paper (Klamka, 1992) contains some results concerning local controllability of nonlinear 2-D systems without differentiability assumptions. Finally, the papers (Klamka, 1996d; 1997a) deal with local controllability of general nonlinear 2-D systems with constant coefficients. Moreover, it should be mentioned that a formula for the solution and controllability results concerning 2-D bilinear systems can be found in (Kaczorek, 1995c).

In the present section, local constrained controllability problems for nonlinear discrete 2-D systems with constant coefficients are formulated and discussed. Using some nonlinear mapping theorems taken from functional analysis and linear approximation methods (Klamka, 1992; 1996c) sufficient conditions for constrained controllability are derived and proved. The present section reports the main results given in (Klamka, 1988a; 1992; 1996d; 1997a).

Let us consider a general nonlinear discrete 2-D system with constant coefficients described by the following difference equation (Kaczorek, 1987a; Klamka, 1992; 1994a; 1996c):

$$x(i+1,j+1) = f\big(x(i,j), x(i+1,j), x(i,j+1), u(i,j)\big) \tag{10}$$

where $(i,j) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, $\mathbb{Z}^+$ is the set of nonnegative integers numbers, $x(i,j) \in \mathbb{R}^n$ is the state vector at the point (i,j), $u(i,j) \in \mathbb{R}^m$ is the control vector at the point (i,j), $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ stands for a given function.

The boundary conditions for the nonlinear difference eqn. (10) are given by

$$x(i,0) = x_{i0} \in \mathbb{R}^n \quad \text{and} \quad x(0,j) = x_{0j} \in \mathbb{R}^n \quad \text{for} \quad (i,j) \in \mathbb{Z}^+ \times \mathbb{Z}^+$$

where x_{i0} and x_{0j} are known vectors.

Instead of the nonlinear 2-D system (10), we shall also consider an associated linear discrete 2-D system with constant coefficients described by the following difference equation (Klamka, 1988a):

$$x(i+1,j+1) = A_0 x(i,j) + A_1 x(i+1,j) + A_2 x(i,j+1) + Bu(i,j) \tag{11}$$

defined for $(i,j) \geq (0,0)$, where A_0, A_1, A_2 are constant $(n \times n)$-dimensional matrices and B is an $(n \times m)$-dimensional constant matrix.

For nonlinear discrete 2-D systems, it is possible to define many different concepts of controllability, analogously to linear 2-D systems (Klamka, 1988a; 1992; 1996d; 1997a).

Let $U \subset \mathbb{R}^m$ be a given set of control values. In the sequel, we shall concentrate on local and global U-controllability in a given rectangle $[(0,0),(p,q)]$ for nonlinear 2-D systems.

Definition 6. (Klamka, 1992; 1994a; 1996c) System (10) is said to be *globally U-controllable in a given rectangle* $[(0,0),(p,q)]$ if for zero boundary conditions $x_{i0}=0,\ i=0,1,2,\dots,p,\ x_{0j}=0,\ j=0,1,2,\dots,q$ and every vector $x'\in\mathbb{R}^n$, there exists an admissible sequence of controls $\{u(i,j)\in U;\ (0,0)\le(i,j)<(p,q)\}$ such that the corresponding solution to (10) satisfies the condition $x(p,q)=x'$.

Definition 7. (Klamka, 1992; 1994a; 1996c) System (10) is said to be *locally U-controllable in a given rectangle* $[(0,0),(p,q)]$ if for zero boundary conditions $x_{i0}=0,\ i=0,1,2,\dots,p,\ x_{0j}=0,\ j=0,1,2,\dots,q$, there exists a neighbourhood of zero $D\subset\mathbb{R}^n$ such that for every point $x'\in D$ there exists an admissible sequence of controls $\{u(i,j)\in U;\ (0,0)\le(i,j)<(p,q)\}$ such that the corresponding solution to (10) satisfies the condition $x(p,q)=x'$.

It is well-known (Klamka, 1988a) that for the sets U containing zero as an interior point, local constrained controllability in the rectangle $[(0,0),(p,q)]$ of a linear 2-D system is equivalent to global unconstrained controllability in the same rectangle. Hence the following result is valid.

Lemma 2. (Klamka, 1988a) *Linear system (11) is locally Ω^0-controllable in the rectangle $[(0,0),(p,q)]$ if and only if it is globally $\mathbb{R}^m$-controllable in the rectangle $[(0,0),(p,q)]$.*

In the next part of this section, we shall formulate and prove sufficient conditions of local U-controllability in a given rectangle $[(0,0),(p,q)]$ for the general nonlinear 2-D system (10) with constant coefficients. It is generally assumed that:

1. $f(0,0,0,0)=0$,
2. the function $f(x,y,z,u)$ is continuously differentiable with respect to all its arguments in some neighbourhood of zero in the product space $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m$.

Taking into account Assumption 2, let us introduce the following notation for partial derivatives of the function $f(x,y,z,u)$:

$$A_0=f'_x(0,0,0,0),\quad A_1=f'_y(0,0,0,0),\quad A_2=f'_z(0,0,0,0),\quad B=f'_u(0,0,0,0) \tag{12}$$

where A_0, A_1, A_2 are $(n\times n)$-dimensional constant matrices and B is an $(n\times m)$-dimensional constant matrix.

Therefore, using standard methods (Klamka, 1992), it is possible to construct a linear approximation (11) of the nonlinear 2-D system (10). This linear approximation is valid in some neighbourhood of zero in the product space $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m$, and is given by the linear difference equation (11) with matrices A_0, A_1, A_2, B defined above. The proofs of these results are based on some lemmas taken from functional analysis and concerning the so-called nonlinear covering operators (Klamka, 1996c).

Now, we are in a position to formulate and prove the main result on local U-controllability in the rectangle $[(0,0),(p,q)]$ for the nonlinear 2-D system (10) with constant coefficients.

Theorem 8. (Klamka, 1996d; 1997a) *Suppose that $U^c \subset \mathbb{R}^m$ is a closed convex cone with vertex at zero. Then the nonlinear discrete 2-D system (10) is locally U^{c0}-controllable in the rectangle $[(0,0),(p,q)]$ if its linear approximation near the origin given by the difference equation (11) is globally U^c-controllable in the same rectangle $[(0,0),(p,q)]$.*

Using the results stated in Section 4 and Theorem 8, we can formulate the following corollary:

Corollary 4. (Klamka, 1996d; 1997a) *Under the assumptions stated in Theorem 8, the nonlinear system (10) is locally U^{c0}-controllable in the rectangle $[(0,0),(p,q)]$ if*

(i) $\operatorname{rank} W_{rs} = n$,

(ii) $V_{rs}^* = \{0\}$.

In the case when U contains zero as an interior point, we have the following sufficient condition for local constrained controllability of nonlinear 2-D systems (10) with constant coefficents.

Corollary 5. (Klamka, 1988a; 1992; 1996d; 1997a) *Let $0 \in \operatorname{int}(U)$. Then the nonlinear system (10) is locally U-controllable in the rectangle $[(0,0),(p,q)]$ if its linear approximation near the origin given by the difference equation (11) is locally U-controllable in the same rectangle $[(0,0),(p,q)]$.*

Let us consider a special case of nonlinear 2-D system, namely the system described by the following nonlinear difference equation (Klamka, 1997a):

$$\begin{aligned} x(i+1,j+1) &= A_0\big(x(i,j)\big)x(i,j) + A_1\big(x(i+1,j)\big)x(i+1,j) \\ &\quad + A_2\big(x(i,j+1)\big)x(i,j+1) + B\big(x(i,j)\big)u(i,j) \end{aligned} \tag{13}$$

where $A_0(x(i,j))$, $A_1(x(i+1,j))$, $A_2(x(i,j+1))$, $B(x(i,j))$ are nonlinear differentiable matrices of suitable dimensions.

In this special case, the associated linear discrete 2-D system with constant coefficients is described by the following difference equation (Klamka, 1997a):

$$x(i+1,j+1) = A_0'x(i,j) + A_1'x(i+1,j) + A_2'x(i,j+1) + B'u(i,j) \tag{14}$$

where

$$A_0' = A_0(0), \quad A_1' = A_1(0), \quad A_2' = A_2(0), \quad B' = B(0)$$

Hence, applying Theorem 8 we obtain immediately the following sufficient condition for local controllability of the nonlinear 2-D system (13):

Theorem 9. (Klamka, 1997a) *Suppose that $U^c \in \mathbb{R}^m$ is a closed convex cone with vertex at zero. Then the nonlinear discrete 2-D system (13) is locally U^{c0}-controllable in the rectangle $[(0,0),(p,q)]$ if its linear approximation near the origin given by the difference equation (14) is globally U^c-controllable in the same rectangle $[(0,0),(p,q)]$.*

In the present section only one simple model of nonlinear 2-D systems with constant coefficients has been considered. The results can be extended in many directions.

For example, it is possible to formulate sufficient local controllability conditions for other models of nonlinear 2-D systems with constant coefficients, for nonlinear 2-D systems with variable coefficients, and finally, for M-D nonlinear systems with constant or variable coefficients.

8. Conclusions

The paper contains fundamental theorems concerning unconstrained and constrained controllability problems for both linear and nonlinear 2-D systems with constant coefficients. In the literature there are many other controllability results, derived for more general 2-D systems. For example, controllability of the following multidimensional discrete systems has been considered recently:

- linear 2-D systems with variable coefficients (Kaczorek, 1987a; Klamka, 1988b),
- linear 2-D systems defined in infinite-dimensional linear spaces, e.g. Hilbert spaces or Banach spaces (Klamka, 1983b; 1988b; 1993b; 1995a; 1995b; 1996b),
- linear 2-D systems with delays (Kaczorek, 1992b; 1995f),
- linear M-D systems i.e. discrete systems with M independent variables (Gałkowski, 1991; 1992; 1993a; 1993b; 1994; Kaczorek, 1986b; Kaczorek and Klamka, 1987; Klamka, 1983a; 1983c; 1988b; 1991a; 1997b; Kurek, 1987),
- nonlinear 2-D systems with variable coefficients (Kaczorek, 1994h; Klamka, 1992).

For different classes of the above multidimensional discrete systems it is necessary to introduce different types of controllability. For example, for infinite-dimensional systems it is necessary to introduce two fundamental notions of controllability namely, approximate (weak) controllability and exact (strong) controllability (Klamka, 1983b; 1988b; 1995a; 1995b).

Controllability of dynamical systems is strongly connected with the so-called minimum energy control problems. The minimum-energy problem for various kinds of 2-D systems has been considered in many publications (Dzieliński, 1993; Kaczorek, 1989a; 1990; 1994a; 1994f; 1995b; 1995e; 1996a; Kaczorek and Klamka, 1986; 1987; 1988; Klamka, 1983a; 1983b; 1984a; 1990; 1991a; 1991c; 1993b).

Moreover, it should be pointed out that in much the same way as in classical linear dynamical systems, there are relationships between controllability and spectrum assignability for 2-D systems (Kaczorek, 1985; 1986c; 1993b; 1996b).

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