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LINEAR QUADRATIC CONTROL PROBLEM WITH FIXED FINAL STATE FOR DISCRETE-TIME DISTRIBUTED SYSTEMS

LOTFI CHRAÏBI*, JAMILA KARRAKCHOU*, ABDELATIF OUANSAFI**, MOSTAFA RACHIK***

The problem considered is that of minimizing a quadratic cost functional for a discrete distributed system with fixed initial and final states. It is shown that under suitable controllability assumptions, there is a close relationship between this problem and that of exact controllability with minimization of a time-varying energy criterion. The HUM technique is then extended to treat the exact controllability problem in the time-varying case and applied to provide an explicit form for the optimal control and the optimal cost.

Keywords: exact controllability, feedback law, open loop, optimal control, quadratic control

1. Introduction

The present work deals with the linear quadratic control problem for a discrete-time distributed parameter system with fixed final state. The system considered is described by the difference equation

$$\begin{cases} x_{i+1} = \Phi x_i + D u_i, \quad i \in \{0, \dots, N-1\}, \\ x_0 \in X, \end{cases} \tag{1}$$

where $x_i \in X$, $\Phi : X \longrightarrow X$ and $D : U \longrightarrow X$ are bounded linear operators, X and U being Hilbert spaces. For a desired state $d \in X$, consider the set of admissible controls

$$\Sigma_P = \left\{ u = (u_0, \dots, u_{N-1}) \in U^N : \ x_N^u = d \right\}, \tag{2}$$

where x_N^u is the state of system (1) at stage N corresponding to a control u.

* Ecole Mohammadia d'Ingénieurs, B.P. 765, Rabat, Morocco, e-mail: j.karrak@emi.ac.ma; lotfichraibi@wanadoo.net.ma

** Faculté des Sciences, Départament de Mathématiques et d'Informatique, B.P. 1014, Rabat, Morocco, e-mail: ouansafi@emi.ac.ma

*** Faculté des Sciences, Benmsik Casa II, Casablanca, Morocco

The control problem with a fixed final state, denoted by (P), is the following: Find $u_P = (u_0^P, u_1^P, \dots, u_{N-1}^P)$ in Σ_P minimizing the cost functional

$$J(u) = \sum_{i=0}^{N-1} \{\langle Qx_i, x_i\rangle + \langle Ru_i, u_i\rangle\}, \tag{3}$$

where Q and R are self-adjoint and non-negative with $\langle Ru, u\rangle \geq \delta\|u\|^2$, $\delta > 0$.

The unconstrained case of this problem is well documented in the literature. It was initially considered in (Lee *et al.*, 1972) where the optimal control was given in a feedback form using a discrete Riccati equation. Recently, the same problem has also been studied in several papers for discrete systems with delays, using the HUM approach (Hilbert Uniqueness Method). The optimal control is obtained there efficiently in an open loop (Karrakchou and Rachik, 1995; Karrakchou *et al.*, 1998).

The linear quadratic control problem with a final condition was examined in (Curtain, 1984) for systems governed by evolution operators. In that paper, a relationship between the problem considered and a minimum-energy control problem was established, and then the optimal control was expressed, in the case of approximately controllable systems, as the weak limit of a sequence of approximating controls. These results were complemented in (Emirsajłow, 1989) where both open loop and feedback descriptions of the optimal control were presented under a sufficient controllability condition.

Inspired by the idea developed in (Curtain, 1984), we show in this paper that a solution to the problem (P) can be obtained by solving a time-varying exact controllability problem. This will require perturbation of the state equation by the feedback law appearing in (Lee *et al.*, 1972) and therefore investigation of a possible equivalence between the controllability assumptions about the original system and those about the perturbed one. Then the HUM techniques developed in (Karrakchou *et al.*, 1995; Chraïbi *et al.*, 1998) are extended to the time-varying case and applied to provide an explicit expression for the solution. As an illustrative example, the numerical simulation of a production control system is presented.

Notation:

- σ_α^β stands for the set of indices defined by $\sigma_\alpha^\beta = \{\alpha, \alpha+1, \dots, \beta-1, \beta\}$.
- $l^2(\sigma_\alpha^\beta, Y) = \{(x_i)_{i\in\sigma_\alpha^\beta},\ x_i \in Y;\ Y$ is a Hilbert space$\}$ is a Hilbert space with usual addition, scalar multiplication and with the inner product

$$\langle x, y\rangle_{l^2(\sigma_\alpha^\beta, Y)} = \sum_{i\in\sigma_\alpha^\beta} \langle x_i, y_i\rangle. \tag{4}$$

- For a family of operators $\{L_i\colon\ Y \to Y,\ i \in \sigma_0^{N-1},\ N \geq 1\}$, the operator $\overline{L}$ is defined by

$$\overline{L}\colon\ l^2(\sigma_0^{N-1}, Y) \to l^2(\sigma_0^{N-1}, Y)\colon\ z \mapsto (L_0(z_0), \dots, L_{N-1}(z_{N-1})).$$

- For a given finite sequence of operators $(A_k)_{k\in I}$ on a Hilbert space, the operator $\prod_{k=i}^{j} A_k$ is given by

$$\prod_{k=i}^{j} A_k = \begin{cases} A_i A_{i+1} \cdots A_{j-1} A_j & \text{if} \quad i \leq j, \\ I & \text{if} \quad j = i-1, \\ 0 & \text{if} \quad j < i-1. \end{cases}$$

2. Perturbed System

2.1. Preliminary Results

Let $H_P : l^2(\sigma_0^{N-1}, U) \longrightarrow X$ be the controllability operator of system (1) defined by

$$H_P(u) = \sum_{k=0}^{N-1} \Phi^{N-k-1} D u_k. \tag{5}$$

If V_P denotes the set of reachable states,

$$V_P = \operatorname{range} H_P,$$

then we have the following result on the existence and uniqueness of a solution to the problem (P):

Proposition 1. *The problem (P) possesses a unique solution if and only if*

$$d - \Phi^N x_0 \in V_P.$$

Proof. That $d - \Phi^N x_0 \in V_P$ amounts to the existence of a control u in $l^2(\sigma_0^{N-1}, U)$ such that

$$d - \Phi^N x_0 = H_P(u) = \sum_{k=0}^{N-1} \Phi^{N-k-1} D u_k. \tag{6}$$

Therefore

$$d = \Phi^N x_0 + \sum_{k=0}^{N-1} \Phi^{N-k-1} D u_k = x_N^u. \tag{7}$$

Hence $u \in \sum_P$ and $\sum_P \neq \emptyset$. Moreover, $\sum_P$ is closed and convex, so the functional J attains a unique minimum $u = (u_0^P, \ldots, u_{N-1}^P) \in \sum_P$ (Lions, 1968). Conversely, if (P) admits a unique solution u, the corresponding response at the step N is d, which forces

$$x_N^u = d = \Phi^N x_0 + H_P(u). \tag{8}$$

Hence $d - \Phi^N x_0 \in \operatorname{range} H_P = V_p$. ■

Let us now recall some results concerning the unconstrained problem (1)–(3) presented in (Lee *et al.*, 1972).

Theorem 1. *The optimal solution to the problem (1)–(3) with a free-terminal state has feedback form and is given by*

$$u_i = -R^{-1}D^*K_{i+1}\big(I + DR^{-1}D^*K_{i+1}\big)^{-1}\Phi x_i, \quad i \in \sigma_0^{N-1}, \tag{9}$$

where $(x_i)_{i\in\sigma_0^{N-1}}$ *is the optimal state and* $\{K_i,\ i \in \sigma_0^{N-1}\}$ *is a family of bounded self-adjoint non-negative operators which constitute the unique solution to the discrete Riccati equation*

$$\begin{cases} K_i = \Phi^*K_{i+1}\big(I + DR^{-1}D^*K_{i+1}\big)^{-1}\Phi + Q, \quad i = N-1, N-2, \ldots, 0, \\ K_N = 0. \end{cases} \tag{10}$$

Moreover, the optimal cost is given by

$$J^*(u) = \big\langle K_0x_0, x_0\big\rangle.$$

2.2. Perturbation by a Feedback Law

Now consider the control system (1) starting at time i with an initial state $h \in X$,

$$\begin{cases} x_{j+1} = \Phi x_j + Du_j, \quad j \in \sigma_i^{N-1}, \\ x_i = h. \end{cases} \tag{11}$$

For an arbitrary control $v = (v_i, \ldots, v_{N-1}) \in l^2(\sigma_i^{N-1}; U)$, suppose that the system so obtained is excited by the control

$$u_j = v_j - R^{-1}D^*K_{j+1}\big(I + DR^{-1}D^*K_{j+1}\big)^{-1}\Phi x_j, \quad j \in \sigma_i^{N-1}. \tag{12}$$

The difference equation (11) then becomes

$$x_{j+1} = \Big[I - DR^{-1}D^*K_{j+1}\big(I + DR^{-1}D^*K_{j+1}\big)^{-1}\Big]\Phi x_j + Dv_j, \quad j \in \sigma_i^{N-1},$$

and hence

$$x_{j+1} = \big(I + DR^{-1}D^*K_{j+1}\big)^{-1}\Phi x_j + Dv_j.$$

Taking $B_j = (I + DR^{-1}D^*K_{j+1})^{-1}\Phi$, for $j \in \sigma_0^{N-1}$, the difference equation

$$\begin{cases} z_{j+1} = B_jz_j + Dv_j, \quad j \in \sigma_i^{N-1}, \\ z_i = h \end{cases} \tag{13}$$

can be considered as a perturbed system of eqn. (11) by the family of operators

$$\big\{-DR^{-1}D^*K_{j+1}B_j, \quad j \in \sigma_i^{N-1}\big\}.$$

Remark 1. It is clear that if u and v are two controls in $l^2(\sigma_i^{N-1}; U)$ satisfying (12), we have

$$x_j^u = z_j^v, \quad j \in \sigma_i^{N-1},$$

where x^u (resp. z^v) is the solution to (11) (resp. (13)) corresponding to the control u (resp. v).

Proposition 2. *Assume that the system (11) is excited by applying the control law (12) with* $v = (v_j)_{j\in\sigma_i^{N-1}} \in l^2(\sigma_i^{N-1}, U)$. *The following equality holds for* $i \in \sigma_0^{N-1}$*:*

$$\langle K_i h, h\rangle = \sum_{j=i}^{N-1} \langle x_j, \left(Q + B_j^* K_{j+1} D R^{-1} D^* K_{j+1} B_j\right) x_j\rangle - 2\sum_{j=i}^{N-1} \langle D^* K_{j+1} B_j x_j, v_j\rangle - \sum_{j=i}^{N-1} \langle K_{j+1} D v_j, D v_j\rangle . \tag{14}$$

Proof. Using (12), the right-hand side (RHS) of the identity (14) can be written down as

$$\text{RHS} = \sum_{j=i}^{N-1} \langle x_j, Q x_j\rangle + \sum_{j=i}^{N-1} \langle B_j^* K_{j+1} D \left(v_j - u_j\right), x_j\rangle - 2\sum_{j=i}^{N-1} \langle D^* K_{j+1} B_j x_j, v_j\rangle - \sum_{j=i}^{N-1} \langle K_{j+1} D v_j, D v_j\rangle ,$$

which implies

$$\text{RHS} = \sum_{j=i}^{N-1} \langle x_j, Q x_j\rangle - \sum_{j=i}^{N-1} \langle B_j^* K_{j+1} D v_j, x_j\rangle - \sum_{j=i}^{N-1} \langle B_j^* K_{j+1} D u_j, x_j\rangle - \sum_{j=i}^{N-1} \langle K_{j+1} D v_j, D v_j\rangle . \tag{15}$$

Since

$$D u_j = x_{j+1} - \Phi x_j, \quad D v_j = z_{j+1} - B_j z_j = x_{j+1} - B_j x_j,$$

eqn. (15) becomes

$$\text{RHS} = \sum_{j=i}^{N-1} \langle x_j, Q x_j\rangle - 2\sum_{j=i}^{N-1} \langle B_j x_j, K_{j+1} x_{j+1}\rangle + \sum_{j=i}^{N-1} \langle K_{j+1} B_j x_j, B_j x_j\rangle + \sum_{j=i}^{N-1} \langle x_j, \Phi^* K_{j+1} B_j x_j\rangle - \sum_{j=i}^{N-1} \langle K_{j+1} D v_j, D v_j\rangle . \tag{16}$$

Moreover,

$$\sum_{j=i}^{N-1}\langle K_{j+1}Dv_j, Dv_j\rangle = \sum_{j=i}^{N-1}\langle K_{j+1}x_{j+1}, x_{j+1}\rangle - 2\sum_{j=i}^{N-1}\langle K_{j+1}x_{j+1}, B_jx_j\rangle + \sum_{j=i}^{N-1}\langle K_{j+1}B_jx_j, B_jx_j\rangle,$$

which yields

$$\mathrm{RHS} = \sum_{j=i}^{N-1}\langle x_j, (Q+\Phi^*K_{j+1}B_j)\,x_j\rangle - \sum_{j=i}^{N-1}\langle K_{j+1}x_{j+1}, x_{j+1}\rangle .$$

Then, from (10), we obtain

$$\begin{aligned}\mathrm{RHS} &= \sum_{j=i}^{N-1}\langle x_j, K_jx_j\rangle - \sum_{j=i}^{N-1}\langle K_{j+1}x_{j+1}, x_{j+1}\rangle \\ &= \sum_{j=i}^{N-1}\langle x_j, K_jx_j\rangle - \sum_{j=i+1}^{N-1}\langle K_jx_j, x_j\rangle \\ &= \langle K_ix_i, x_i\rangle = \langle K_ih, h\rangle.\end{aligned}$$ ■

Consider now the relation (12) and the controlled system (13) for $i = 0$ and $h = x_0$, i.e.

$$\begin{cases} z_{i+1} = B_iz_i + Dv_i, & i \in \sigma_0^{N-1}, \\ z_0 = x_0 \in X. \end{cases} \tag{17}$$

The equation (17) can be regarded as a perturbed system of (1) by $\{-DR^{-1}D^*K_{j+1}B_j\}_{j\in\sigma_0^{N-1}}$. The final state z_N^v of this system is

$$z_N^v = \left(\prod_{j=0}^{N-1} B_{N-j-1}\right)x_0 + \sum_{j=0}^{N-1}\left(\prod_{k=0}^{N-j-2} B_{N-k-1}\right)Dv_j.$$

Consequently, the corresponding controllability operator $H_E : l^2(\sigma_0^{N-1}, U) \longrightarrow X$ is defined by

$$H_E(v) = \sum_{j=0}^{N-1}\left(\prod_{k=0}^{N-j-2} B_{N-k-1}\right)Dv_j. \tag{18}$$

It is now reasonable to examine whether the original and perturbed systems have the same controllability properties. We shall generally deal with this question in the next subsection.

2.3. Controllability Result

It is known that the controllability of a system is characterized by the range of its controllability operator. We state this classical result for the system (1).

Proposition 3. *We have the following characterizations:*

(i) The system (1) is exactly controllable if and only if

$$\operatorname{range} H_P = X.$$

(ii) The system (1) is approximately controllable if and only if

$$\overline{\operatorname{range} H_P} = X,$$

which is equivalent to

$$\ker H_P^* = \{0\}.$$

Proof. See (Curtain and Pritchard, 1978). ∎

Suppose that the system (1) is excited by a control $u = (u_0, \dots, u_{N-1}) \in l^2(\sigma_0^{N-1}; U)$, given by

$$u_i = v_i - F_i x_i, \quad i \in \sigma_0^{N-1},$$

where $\{F_i : X \to U;\ i \in \sigma_0^{N-1}\}$ is a family of operators. Then the perturbed system is as follows:

$$\begin{cases} z_{i+1} = (\Phi - DF_i)z_i + Dv_i, & i \in \sigma_0^{N-1}, \\ z_0 = x_0 \in X. \end{cases} \tag{19}$$

Theorem 2. *We have*

$$\operatorname{range} H_P = \operatorname{range} H_V,$$

where $H_V : l^2(\sigma_0^{N-1}, U) \longrightarrow X$ *is the controllability operator for the system (19), defined by*

$$H_V(v) = \sum_{p=0}^{N-1} \left[\prod_{k=0}^{N-p-2} (\Phi - DF_{N-k-1}) \right] Dv_p.$$

For the proof of this proposition, we need the following technical lemma.

Lemma 1. *Let* $\{\Phi\} \cup \{G_i,\ i \in \sigma_0^{N-1}\}$ $(N \geq 1)$ *be a family of operators defined on a space* X. *For* $r \in \sigma_1^{N-1}$, *we have*

$$\prod_{p=0}^{r-1} (\Phi - G_{N-p-1}) = \Phi^r - \sum_{k=N-r}^{N-1} \Phi^{N-k-1} G_k \left[\prod_{p=0}^{k+r-N-1} (\Phi - G_{k-p-1}) \right]. \tag{20}$$

Proof. See the Appendix. ■

Proof of Theorem 2. Given $y \in \operatorname{range} H_V$, there is a control $v \in l^2(\sigma_0^{N-1}, U)$ such that

$$y = H_V(v) = \sum_{s=0}^{N-1} \left[\prod_{k=0}^{N-s-2} \left(\Phi - DF_{N-k-1} \right) \right] Dv_s.$$

Applying (20) for $G_k = DF_k$, $k \in \sigma_0^{N-1}$ and $r = N - s - 1$, we obtain

$$\begin{aligned} y &= \sum_{s=0}^{N-1} \left\{ \Phi^{N-s-1} - \sum_{k=s+1}^{N-1} \Phi^{N-k-1} DF_k \left[\prod_{p=0}^{k-s-2} \left(\Phi - DF_{k-p-1} \right) \right] \right\} Dv_s \\ &= \sum_{s=0}^{N-1} \Phi^{N-s-1} Dv_s \\ &\quad - \sum_{s=0}^{N-1} \sum_{k=s}^{N-1} \Phi^{N-k-1} DF_k \left[\prod_{p=0}^{k-s-2} \left(\Phi - DF_{k-p-1} \right) \right] Dv_s. \end{aligned}$$

Hence

$$\begin{aligned} y &= \sum_{k=0}^{N-1} \Phi^{N-k-1} Dv_k \\ &\quad - \sum_{k=0}^{N-1} \Phi^{N-k-1} DF_k \sum_{s=0}^{k} \left[\prod_{p=0}^{k-s-2} \left(\Phi - DF_{k-p-1} \right) \right] Dv_s, \\ &= \sum_{k=0}^{N-1} \Phi^{N-k-1} D \left\{ v_k - F_k \sum_{s=0}^{k-1} \left[\prod_{p=0}^{k-s-2} \left(\Phi - DF_{k-p-1} \right) \right] Dv_s \right\}, \end{aligned}$$

which yields

$$y = H_V(w),$$

where $w = (w_0, \ldots, w_{N-1})$ is given by

$$w_k = v_k - F_k \sum_{s=0}^{k-1} \left[\prod_{p=0}^{k-s-2} \left(\Phi - DF_{k-p-1} \right) \right] Dv_s, \quad k \in \sigma_0^{N-1}.$$

Thus

$$\operatorname{range} H_V \subset \operatorname{range} H_P.$$

Conversely, the control system (1) is the perturbation of (19) by $\{DF_k\}_{k \in \sigma_0^{N-1}}$. Consequently, the reverse inclusion is established. ■

The above theorem shows that the system (1) preserves the controllability properties when it is subject to a feedback perturbation. It is a simple matter to deduce the following result.

Corollary 1. *We have*

$$\operatorname{range} H_P = \operatorname{range} H_E.$$

Proof. We have already seen that eqn. (17) is the perturbed system of eqn. (1) by $\{-DR^{-1}D^*K_{j+1}B_j\}_{j\in\sigma_0^{N-1}}$. Hence the result is proved by applying Theorem 2 for $F_i = R^{-1}D^*K_{i+1}B_i$, $i \in \sigma_0^{N-1}$. ∎

It is now obvious that the system (1) is exactly controllable (resp. approximately controllable) if and only if so is the system (17).

3. Relationship between the Problem (P) and a Problem of Exact Controllability

Corollary 1, when combined with Proposition 2, enables us to establish a relationship between the problem (P) and a minimum-energy control problem.

3.1. Statement of a Minimum-Energy Control Problem

Define the set of admissible controls

$$\Sigma_E = \left\{ v \in l^2(\sigma_0^{N-1}, U) : \ z_N^v = d \right\}, \tag{21}$$

where z_N^v is the final state corresponding to the control v and satisfying the perturbed equation (17). Consider the control problem (E) stated as follows: Find $v_E = (v_0^E, \dots, v_{N-1}^E) \in \Sigma_E$ minimizing the criterion

$$W(v) = \sum_{i=0}^{N-1} \left\langle (R + D^*K_{i+1}D)v_i, v_i \right\rangle. \tag{22}$$

Proposition 4. *The problem (E) possesses a unique solution if and only if*

$$d - \left(\prod_{j=0}^{N-1} B_{N-j-1} \right) x_0 \in \operatorname{range} H_E = V_P. \tag{23}$$

Moreover,

$$d - \Phi^N x_0 \in V_P \Longleftrightarrow d - \left(\prod_{j=0}^{N-1} B_{N-j-1} \right) x_0 \in V_P. \tag{24}$$

Proof. Arguments similar to those used in Proposition 1 lead to the first part of our assertion. For the second part, note that $d - \Phi^N x_0 \in V_P \Longleftrightarrow \exists u \in l^2(\sigma_0^{N-1}, U)$ such that

$$d = \Phi^N x_0 + H_P(u) = x_N^u = z_N^v,$$

where u and v are related by (12). Therefore

$$d - \Phi^N x_0 \in V_P \Longleftrightarrow z_N^v = \left(\prod_{j=0}^{N-1} B_{N-j-1}\right) x_0 + H_E(v) = d$$

$$\Longleftrightarrow d - \left(\prod_{j=0}^{N-1} B_{N-j-1}\right) x_0 \in \operatorname{range} H_E = V_P.$$

This completes the proof. ■

We can now formulate our main result in this section.

Theorem 3. *If $d - \Phi^N x_0 \in V_P$, then the optimal control for the problem (P) is given by*

$$u_i^P = v_i^E - R^{-1} D^* K_{i+1} B_i z_i^E, \quad i \in \sigma_0^{N-1}, \tag{25}$$

where $v_E = (v_0^E, \dots, v_{N-1}^E)$ is the optimal control for the problem (E) and z_E is the corresponding optimal response. Moreover, the optimal cost is given by

$$J(u_p) = W(v_E) + \langle K_0 x_0, x_0 \rangle . \tag{26}$$

Proof. Since $d - \Phi^N x_0 \in V_P$, the last proposition assures the existence of a unique v_E. Furthermore, for an arbitrary control $v \in \sum_E$ consider the control given by (12) and apply the identity (14) for $i = 0$ and $h = x_0$, which yields

$$\langle K_0 x_0, x_0 \rangle = \sum_{j=0}^{N-1} \langle x_j, \left(Q + B_j^* K_{j+1} D R^{-1} D^* K_{j+1} B_j\right) x_j \rangle$$

$$- 2 \sum_{j=0}^{N-1} \langle D^* K_{j+1} B_j x_j, v_j \rangle - \sum_{j=i}^{N-1} \langle K_{j+1} D v_j, D v_j \rangle .$$

This implies

$$\langle K_0 x_0, x_0 \rangle = \sum_{j=0}^{N-1} \langle Q x_j, x_j \rangle$$

$$+ \sum_{j=0}^{N-1} \langle R(R^{-1} D^* K_{j+1} B_j x_j), R^{-1} D^* K_{j+1} B_j x_j \rangle$$

$$-2 \sum_{j=0}^{N-1} \langle R(R^{-1} D^* K_{j+1} B_j x_j), v_j \rangle - \sum_{j=0}^{N-1} \langle D^* K_{j+1} D v_j, v_j \rangle .$$

Using (12), the last equality becomes

$$\begin{aligned}\langle K_0x_0, x_0\rangle = &\sum_{j=0}^{N-1}\langle Qx_j, x_j\rangle\\ &+\sum_{j=0}^{N-1}\Big\{\langle R(u_j-v_j), u_j)\rangle + \langle R(u_j-v_j), v_j\rangle\Big\}\\ &-\sum_{j=0}^{N-1}\langle D^*K_{j+1}Dv_j, v_j\rangle.\end{aligned}$$

Hence

$$\begin{aligned}\langle K_0x_0, x_0\rangle = &\sum_{j=0}^{N-1}\Big\{\langle Qx_j, x_j\rangle + \langle Ru_j, u_j\rangle\Big\}\\ &-\sum_{j=0}^{N-1}\langle (R+D^*K_{j+1}D)v_j, v_j\rangle,\end{aligned}$$

which means

$$\langle K_0x_0, x_0\rangle = J(u) - W(v), \quad \forall\, v\in\Sigma_E^d.$$

Since this equality holds true for $v = v_E$, the optimal solution to (E), we obtain

$$J(u) - J(u_p) = W(v) - W(v_E) \geq 0,$$

which completes the proof. ■

4. Solution to the Problem (P)

The aim of this section is to give an explicit solution to the problem (P). As a consequence of Theorem 3, it suffices to solve the exact controllability problem (E) to find the control u_P.

4.1. Minimum-Energy Control Problem

Note that the problem (E) is an exact controllability one with a time-varying dynamic and energy criterion. This motivates us to devote the present subsection to an independent analysis of the exact controllability problem with time-varying operators. The HUM approach is adopted here and the results obtained in (Karrakchou *et al.*, 1995) are generalized to this case.

Consider the difference equation

$$\begin{cases} x_{i+1} = \Phi_i x_i + D_i v_i, & i\in\sigma_0^{N-1},\\ x_0\in X, \end{cases} \tag{27}$$

where $\{\Phi_i : X \longrightarrow X;\ i \in \sigma_0^{N-1}\}$ and $\{D_i : U \longrightarrow X;\ i \in \sigma_0^{N-1}\}$ are linear and bounded. The control problem is to find $v^* = (v_0, \ldots, v_{N-1}) \in \Sigma = \{v \in l^2(\sigma_0^{N-1}, U)\colon x_N^v = d\}$ minimizing the energy criterion

$$P(v) = \sum_{i=0}^{N-1} \langle R_i v_i, v_i \rangle, \tag{28}$$

where $x = (x_i)_{i\in\sigma_0^N}$ is the solution to the difference equation (27), and $R_i\colon U \longrightarrow U$, $i \in \sigma_0^{N-1}$ are self-adjoint and non-negative with $\langle R_i u, u\rangle \geq \delta \|u\|^2, \delta > 0$. It is well-known that the controllability operator $H\colon l^2(\sigma_0^{N-1}, U) \longrightarrow X$ for the system (27) is given by

$$H(v) = \sum_{j=0}^{N-1} \left(\prod_{k=0}^{N-j-2} \Phi_{N-k-1} \right) D_j v_j,$$

and its adjoint $H^*\colon X \longrightarrow l^2(\sigma_0^{N-1}, U)$ is

$$H^* y = \left((H^* y)_0, \ldots, (H^* y)_{N-1} \right),$$

where

$$(H^* y)_j = D_j^* \left(\prod_{k=j+1}^{N-1} \Phi_k^* \right) y, \quad j \in \sigma_0^{N-1}.$$

Define the scalar product

$$\langle x, y \rangle_{X_E} = \langle H_R^* x, H_R^* y \rangle_{l^2(\sigma_0^{N-1}, U)}. \tag{29}$$

on the space X, where H_R is the control-energy operator given by

$$H_R = H \bar{R}^{-1/2}.$$

Assume that the system (27) is approximately controllable. Then the semi-norm $\|\cdot\|_{X_E}$, obtained from the scalar product (29) and given by

$$\|y\|_{X_E} = \left\{ \sum_{j=0}^{N-1} \left\| R_j^{-1/2} D_j^* \left(\prod_{k=j+1}^{N-1} \Phi_k^* \right) y \right\|^2 \right\}^{1/2},$$

defines a norm on X. Therefore X is identified with a dense subspace of X_E, where X_E is the completion of X for the norm $\|\cdot\|_{X_E}$.

Let $\Lambda_R\colon X \to X$ be the bounded self-adjoint operator defined by

$$\Lambda_R(x) = H_R H_R^* x, \tag{30}$$

so Λ_R has an extension to an isomorphism, also denoted by Λ_R, defined from X_E

to its dual X_E'. Moreover, the following inclusion takes place:

Lemma 2. *We have*

$$\operatorname{range} H \subset X_E',$$

i.e. the elements of range H *are linear continuous forms on* X_E.

Proof. See the Appendix. ∎

Now all the necessary tools are available to state the following main result:

Proposition 5. *Suppose that the control system (27) is approximately controllable. If* $d - (\prod_{j=0}^{N-1} \Phi_{N-j-1})x_0 \in X_E'$, *then the optimal control of the problem (27)–(28) is given by*

$$v_j = R_j^{-1} D_j^* \left(\prod_{k=j+1}^{N-1} \Phi_k^* \right) g, \quad j \in \sigma_0^{N-1}, \tag{31}$$

where g *is the solution to the algebraic equation*

$$\Lambda_R g = d - \left(\prod_{j=0}^{N-1} \Phi_{N-j-1} \right) x_0. \tag{32}$$

The minimum energy is

$$P(v) = \|g\|_{X_E}^2. \tag{33}$$

Proof. Observe that the control given by (31) can be rewritten as

$$v^* = \bar{R}^{-1} H^*(g). \tag{34}$$

Therefore

$$\begin{aligned} x_N^{v^*} &= \left(\prod_{j=0}^{N-1} \Phi_{N-j-1} \right) x_0 + H\big(\bar{R}^{-1} H^*(g)\big) \\ &= \left(\prod_{j=0}^{N-1} \Phi_{N-j-1} \right) x_0 + H \bar{R}^{-1/2} \bar{R}^{-1/2} H^*(g) \\ &= \left(\prod_{j=0}^{N-1} \Phi_{N-j-1} \right) x_0 + H_R H_R^*(g) \\ &= \left(\prod_{j=0}^{N-1} \Phi_{N-j-1} \right) x_0 + \Lambda_R(g). \end{aligned}$$

Since g is the solution to (32), it follows that $x_N^{v^*} = d$. On the other hand, we have

$$x_N^{v^*} = x_N^v, \quad \forall\, v \in \Sigma^d.$$

Consequently, we have

$$H(v - v^*) = 0, \quad \forall\, v \in \Sigma^d,$$

which implies

$$\langle H(v - v^*), g \rangle = 0, \quad \forall\, v \in \Sigma^d.$$

Hence from (34) it follows that

$$\begin{aligned}\langle H(v - v^*), g \rangle &= \sum_{j=0}^{N-1} \langle R_j \left(v_j - v_j^*\right), v_j^* \rangle = 0, \\ &= \langle \bar{R}\left(v - v^*\right), v^* \rangle = 0, \quad \forall\, v \in \Sigma^d.\end{aligned}$$

This yields

$$\langle \bar{R}v, v \rangle - \langle \bar{R}v^*, v^* \rangle = \langle \bar{R}\left(v - v^*\right), v - v^* \rangle \geq 0,$$

which completes the proof. ■

4.2. Optimal Control for the Problem (*P*)

Based on Proposition 5, an explicit solution to the problem (P) can be derived.

Theorem 4. *Suppose that (1) is approximately controllable and $d - \Phi^N x_0 \in V_P$. Then the optimal control for the problem (P) is given by*

$$u_i^P = \left(R + D^* K_{i+1} D\right)^{-1} D^* \left(\prod_{k=i+1}^{N-1} B_k^*\right) g - R^{-1} D^* K_{i+1} B_i z_i^E, \quad i \in \sigma_0^{N-1}, \tag{35}$$

where g is the solution to the equation

$$\Lambda_R(g) = d - \left(\prod_{j=0}^{N-1} B_{N-j-1}\right) x_0, \tag{36}$$

with $\Lambda_E = H_E \bar{R}^{-1} H_E^*$. Moreover, the optimal cost is

$$J(u_p) = \|g\|_{X_E}^2 + \langle K_0 x_0, x_0 \rangle. \tag{37}$$

Proof. From Corollary 1, it is clear that if (1) is approximately controllable, so is the system (17). Since $d - \Phi^N x_0 \in V_P$, from Proposition 4 we deduce that

$$d - \left(\prod_{j=0}^{N-1} B_{N-j-1}\right) x_0 \in V_P,$$

which, by Lemma 2, implies

$$d - \left(\prod_{j=0}^{N-1} B_{N-j-1}\right) x_0 \in X'_E.$$

Hence the problem (E) satisfies the hypothesis of Proposition 5 for $\Phi_j = B_j$, $D = D_j$ and $R_j = R + D^* K_{j+1} D$, $j \in \sigma_0^{N-1}$, and consequently v_E can be expressed by

$$v_j^E = \left(R + D^* K_{j+1} D\right)^{-1} D^* \left(\prod_{k=j+1}^{N-1} B_k^*\right) g, \tag{38}$$

where g is the solution to the equation

$$\left(H_E \bar{R}^{-1} H_E^*\right)(g) = d - \left(\prod_{j=0}^{N-1} B_{N-j-1}\right) x_0.$$

By substitution in (25) (cf. Theorem 1), the expression (35) is obtained. ■

5. Application

For illustration, consider the production control problem described in (Faradzhev *et al.*, 1986):

$$\begin{cases} x_{i+1} = x_i + \alpha(u_i - x_i), & i \in \sigma_0^{N-1}, \\ x_0 \in \mathbb{R}, \end{cases} \tag{39}$$

where x_i is the production volume and u_i is the production control, $i \in \sigma_0^{N-1}$, α signifies the coefficient adjusting the production volume to the control, $0 < \alpha \leq 1$.

The purpose of this application is to find an optimal control $u = (u_0, \ldots, u_{N-1})$ allowing the system (39) to reach a desired production volume d with a minimum cost J such that

$$J(u) = r \left\{ \sum_{i=0}^{N-1} x_i^2 + \sum_{i=0}^{N-1} u_i^2 \right\}, \tag{40}$$

where r is a shipment coefficient, $0 < r \leq 1$.

By applying Theorem 1 for $\Phi = 1 - \alpha$, $D = \alpha$ and $Q = R = r$, the corresponding discrete Riccati equation is given by

$$\begin{cases} k_i = \dfrac{(1-\alpha)^2 r k_{i+1}}{r + \alpha^2 k_{i+1}} + r, & i = N-1, N-2, \ldots, 1, 0, \\ k_N = 0. \end{cases} \tag{41}$$

Then the perturbed system is described by

$$\begin{cases} z_i = b_i z_i + \alpha v_i, & i \in \sigma_0^{N-1}, \\ z_0 = x_0, \end{cases} \tag{42}$$

such that $b_i = r(1-\alpha)/(r+\alpha^2 k_{i+1})$, $i \in \sigma_0^{N-1}$.

The corresponding energy control problem (E) is to minimize

$$W(v) = \sum_{i=0}^{N-1} (r+\alpha^2 k_{i+1}) v_i^2, \tag{43}$$

over

$$\Sigma_E = \Big\{ v = (v_0, \dots, v_{N-1}) \in \mathbb{R}^N : \; z_N^v = d \Big\}. \tag{44}$$

Consequently, the minimum-energy control solution v_E is computed as

$$v_i^E = \frac{\alpha \left(\prod_{k=i+1}^{N-1} b_k \right) g}{\left(r + \alpha^2 k_{i+1} \right)}, \quad i \in \sigma_0^{N-1}, \tag{45}$$

with g defined by

$$g = \frac{1}{\Lambda} \left[d - \left(\prod_{j=0}^{N-1} b_{N-j-1} \right) x_0 \right], \tag{46}$$

where

$$\Lambda = \alpha^2 \sum_{i=0}^{N-1} \left\{ (r+\alpha^2 k_{i+1})^{-1} \left(\prod_{k=0}^{N-i-2} b_{N-k-1} \right)^2 \right\}. \tag{47}$$

The corresponding optimal state is given by

$$z_i = \left(\prod_{j=0}^{i-1} b_{i-j-1} \right) x_0 + \alpha \left\{ \sum_{j=0}^{i-2} \left(\prod_{k=0}^{i-j-2} b_{i-k-1} \right) v_j^E + v_{i-1}^E \right\}, \quad i \in \sigma_0^N. \tag{48}$$

Finally, the optimal control and cost production are respectively given by (cf. Theorem 3)

$$u_i^P = v_i^E - \frac{\alpha}{r} k_{i+1} b_i z_i, \quad i \in \sigma_0^{N-1} \tag{49}$$

and

$$J(u_P) = k_0 x_0^2 + \sum_{i=0}^{N-1} \left(r + \alpha^2 k_{i+1} \right) \left(v_i^E \right)^2. \tag{50}$$

For numerical simulation, we set $\alpha = 0.1$, $r = 0.05$, $x_0 = 0$ and $d = 10$. Figures 1 and 2 describe respectively the optimal state $z = (z_0, z_1, \dots, z_{N-1})$ and the optimal control $u_P = (u_0, \dots, u_{N-1})$ for $N = 30$.

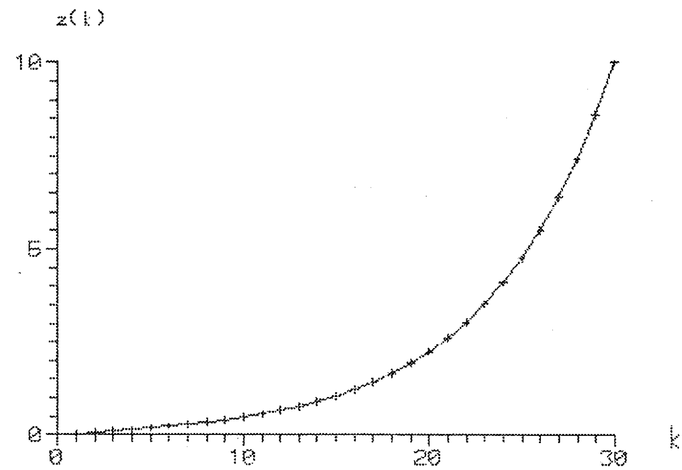

Fig. 1. Optimal trajectory for $N = 30$.

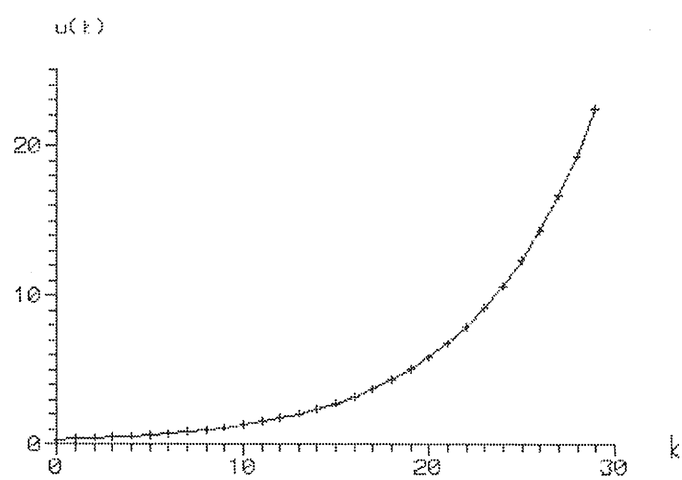

Fig. 2. Optimal control for $N = 30$.

6. Final Remarks

In this paper, we have examined the linear-quadratic optimal-control problem for discrete distributed-parameter systems with equality final constraints. A close relationship between the original problem and a minimum energy one was combined with the HUM approach to provide an explicit expression for the optimal control. The inequality-constrained case and the minimum-time problem are currently under investigation.

Appendix

Proof of Lemma 1. It is easy to check that (20) is true for $r = 1$. Suppose now that it holds for r. Then we have

$$\prod_{p=0}^{r} \left(\Phi - G_{N-p-1}\right) = \left[\prod_{p=0}^{r-1} \left(\Phi - G_{N-p-1}\right)\right] \left(\Phi - G_{N-r-1}\right),$$

which gives

$$\prod_{p=0}^{r} \left(\Phi - G_{N-p-1}\right)$$
$$= \left\{\Phi^r - \sum_{k=N-r}^{N-1} \Phi^{N-k-1} G_k \left[\prod_{p=0}^{k+r-N-1} \left(\Phi - G_{k-p-1}\right)\right]\right\} \left(\Phi - G_{N-r-1}\right).$$

Hence

$$\prod_{p=0}^{r} \left(\Phi - G_{N-p-1}\right) = \Phi^{r+1} - \Phi^r G_{N-r-1}$$
$$- \sum_{k=N-r}^{N-1} \Phi^{N-k-1} G_k \left[\prod_{p=0}^{k+r-N} \left(\Phi - G_{k-p-1}\right)\right],$$

which implies

$$\prod_{p=0}^{r} \left(\Phi - G_{N-p-1}\right) = \Phi^{r+1} - \sum_{k=N-r-1}^{N-1} \Phi^{N-k-1} G_k \left[\prod_{p=0}^{k+r-N} \left(\Phi - G_{k-p-1}\right)\right].$$

and (20) is also satisfied for $r + 1$. ■

Proof of Lemma 2. For $y \in \text{range}\, H$, there exists $v \in l^2(\sigma_0^{N-1}, U)$ such that $y = H(v)$. Then, for every $z \in X$, we have

$$\langle H(v), z\rangle = \sum_{i=0}^{N-1} \langle v_i, \left(H^*(z)\right)_i\rangle$$
$$= \sum_{i=0}^{N-1} \langle R_i^{\frac{1}{2}}(v_i), R_i^{-\frac{1}{2}}(H^*(z))_i\rangle$$
$$= \sum_{i=0}^{N-1} \langle R_i^{\frac{1}{2}}(v_i), (H_R^*(z))_i\rangle.$$

Hence

$$\left|\langle H(v), z\rangle\right| \leq \left\|\bar{R}(v)\right\| \left\|H_R^*(z)\right\|,$$

which gives

$$\left|\,\langle H(v), z\rangle\,\right| \le \left\|\bar{R}(v)\right\| \, \|z\|_{X_E}, \quad \forall z \in X,$$

and by density we obtain

$$\left|\langle H(v), z\rangle_{X_E, X'_E}\right| \le \left\|\bar{R}(v)\right\| \, \|z\|_{X_E}, \quad \forall z \in X_E.$$

Therefore $z \longrightarrow \langle H(v), z\rangle_{X_E, X'_E}$ is a continuous linear form on X_E, which, by the Riesz theorem, implies $y = H(v) \in X'_E$. ■

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