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TIME-VARIANT DARLINGTON SYNTHESIS AND INDUCED REALIZATIONS

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For a block lower triangular contraction T, necessary and sufficient conditions are given in order that there exist block lower triangular contractions $T_{1,1}$, $T_{2,1}$ and $T_{2,2}$ such that

$$U_T = \begin{bmatrix} T_{1,1} & T \\ T_{2,1} & T_{2,2} \end{bmatrix}$$

is unitary. For the case when $T_{1,1}^*$ and $T_{2,2}$ have dense ranges, all such embeddings are described. Each unitary embedding of U_T induces a contractive realization of T, and various properties of this realization are characterized in terms of the unitary embedding.

Keywords: contractive linear systems, Darlington synthesis, time-variant linear systems

1. Introduction

Let T be a block lower triangular contraction, i.e., a contractive operator $T = (t_{i,j})_{i,j=-\infty}^{\infty}$ acting from a doubly infinite Hilbert space direct sum $\bigoplus_{j=-\infty}^{\infty} K_j$ into a doubly infinite Hilbert space direct sum $\bigoplus_{j=-\infty}^{\infty} L_j$. The operator $t_{i,j}$, which maps K_j into L_i, is the (i,j)-th entry in the operator matrix representation of T relative to the natural Hilbert space direct sum decompositions. In this paper we study the problem of finding block lower triangular contractions $T_{1,1}$, $T_{2,1}$ and $T_{2,2}$ such that the operator matrix

$$U_T = \begin{bmatrix} T_{1,1} & T \\ T_{2,1} & T_{2,2} \end{bmatrix} \tag{1}$$

is unitary. If such operators $T_{1,1}$, $T_{2,1}$ and $T_{2,2}$ exist, then the embedding (1) is referred to as a $\tilde{D}$-*embedding* of the block lower triangular contraction T. If, in addition, both the image of $T_{1,1}^*$ and the image of $T_{2,2}$ are dense, then the embedding (1) is referred to as a *Darlington embedding* or a $\mathcal{D}$-*embedding* of the block lower triangular contraction T.

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The problem of embedding a block lower triangular contraction into a block lower triangular unitary operator has its roots in the Darlington synthesis problem from electrical network theory (Belevitch, 1968, Sec. 9.13). In this original setting, the block lower triangular contractions are replaced by operator-valued Schur class functions. To state the corresponding problem explicitly, let a Schur class function $\theta(\cdot)$ acting from a Hilbert space V into a Hilbert space W be given. The $\mathcal{D}$-embedding problem consists in finding appropriate Hilbert spaces V° and W° as well as Schur class functions $\theta_{1,1}(\cdot)$, $\theta_{2,1}(\cdot)$, $\theta_{2,2}(\cdot)$, such that

$$\Theta(e^{it}) = \begin{bmatrix} \theta_{1,1}(e^{it}) & \theta(e^{it}) \\ \theta_{2,1}(e^{it}) & \theta_{2,2}(e^{it}) \end{bmatrix} : V^\circ \oplus V \to W \oplus W^\circ \tag{2}$$

is bi-inner, i.e., $\Theta(\cdot)$ is a Schur class function, which takes unitary values almost everywhere on the unit circle, and

$$\overline{M_{\theta_{2,2}}[L_2(V)]} = L_2(W^\circ), \quad \overline{M^*_{\theta_{1,1}}[L_2(W)]} = L_2(V^\circ). \tag{3}$$

Here $M_{\theta_{2,2}} : L_2(V) \to L_2(W^\circ)$ and $M_{\theta_{1,1}} : L_2(V^\circ) \to L_2(W)$ are the multiplication operators by $\theta_{2,2}$ and $\theta_{1,1}$, respectively.

Arov (1971) and Dewilde (1971) independently obtained necessary and sufficient conditions such that a matrix-valued Schur class function θ admits a $\mathcal{D}$-embedding; these conditions are stated in terms of the pseudo-continuability of θ. In both the papers the results from (Douglas *et al.*, 1970) were used as a starting point. The paper (Arov, 1971) also treats the operator-valued case. Moreover, in (Arov, 1971) various different properties such as the minimality and optimality of a contractive realization of θ induced by the embedding (2) are characterized in terms of the $\mathcal{D}$-embedding. In (Dewilde, 1971), the analysis is done with the upper half plane, and the fundamental property which ensures a matrix-valued Schur class function θ to admit a $\mathcal{D}$-embedding is the requirement that θ allows for a so-called roomy contractive realization. Independently of (Arov, 1971), Douglas and Helton (1973) have also constructed a unitary embedding for an operator-valued Schur class function which allows for pseudo-continuation. Here only functional-theoretic methods are used. In (Douglas and Helton, 1973) it is also shown that for the matrix case this condition is sufficient as well.

The problem of embedding a block lower triangular contraction into a block lower triangular unitary operator was solved in (Dewilde, 1999; Dewilde and Van der Veen, 1998) for the class of block lower triangular contractions which appear as the input-output map of an exponentially stable contractive system (see Section 8 for the definition). The input-output map of an exponentially stable contractive system is *exponentially decaying off the main diagonal*, i.e.,

$$\|t_{i,j}\| \leq M\alpha^{i-j} \tag{4}$$

for some numbers $M > 0$ and $0 < \alpha < 1$. As a by-product of the main results of the present paper it is shown that the converse of the latter statement is not true (see Section 9). In fact, we will give necessary and sufficient conditions in order that a block lower triangular contraction which is exponentially decaying off the main diagonal is the input-output map of an exponentially stable system. These conditions are stated in terms of $\tilde{\mathcal{D}}$-embeddings.

The present paper does not require the condition (4). We shall show (see Proposition 1) that a block lower triangular contraction T admits a $\tilde{\mathcal{D}}$-embedding if and only if T admits a contractive realization which is both pointwise stable and pointwise star-stable. (For the terminology concerning contractive systems, see Section 1 below.) A necessary condition for the operator U_T in (1) to be unitary is the existence of block lower triangular contractions $T_{1,1}$ and $T_{2,2}$ such that $I - TT^* = T_{1,1}T_{1,1}^*$ and $I - T^*T = T_{2,2}^*T_{2,2}$. It is well-known (Constantinescu, 1995, p.128) that in this case there exist an outer block lower triangular contraction F and a star-outer block lower triangular contraction G such that

$$I - T^*T = F^*F, \quad I - TT^* = GG^*. \tag{5}$$

Recall that an operator $F : \bigoplus_{j=-\infty}^{\infty} K_j \to \bigoplus_{j=-\infty}^{\infty} N_j$ is called *outer* if it is block lower triangular and the image of $\bigoplus_{j=n}^{\infty} K_j$ under F is dense in $\bigoplus_{j=n}^{\infty} N_j$, for each integer n. An operator $G : \bigoplus_{j=-\infty}^{\infty} M_j \to \bigoplus_{j=-\infty}^{\infty} L_j$ is called *star-outer* if it is block lower triangular and the image of $\bigoplus_{j=-\infty}^{m} L_j$ under G^* is dense in $\bigoplus_{j=-\infty}^{m} N_j$, for each integer m. Once F and G in (5) have been chosen, there exists (see Lemma 1) a unique contraction H such that the operator block matrix

$$\begin{bmatrix} G & T \\ H & F \end{bmatrix}$$

is unitary. However, the operator H does not need to be block lower triangular. If we can find block lower triangular unitary operators U_1, U_2 such that U_2HU_1 is block lower triangular, then the operator

$$\tilde{T} = \begin{bmatrix} GU_1 & T \\ U_2HU_1 & U_2F \end{bmatrix} \tag{6}$$

is unitary, each of the blocks GU_1, U_2HU_1, U_2F is block lower triangular, and both $(GU_1)^*$ and U_2F have dense range. Thus (6) is a Darlington embedding or $\mathcal{D}$-embedding of T. The following, first main theorem of the present paper shows that each $\mathcal{D}$-embedding is obtained in this way.

Theorem 1. *A block lower triangular contraction T admits a $\mathcal{D}$-embedding if and only if the following two conditions are satisfied:*

(i) there exists an outer operator F such that $I - T^*T = F^*F$ and there exists a star-outer operator G such that $I - TT^* = GG^*$,

(ii) for the unique contraction H satisfying $H^*F = -G^*T$ there exist block lower triangular unitary operators U_1 and U_2 such that U_2HU_1 is a block lower triangular.

If conditions (i) and (ii) are satisfied, then each $\mathcal{D}$-embedding is obtained by (6), where U_1 and U_2 are block lower triangular unitary operators such that U_2HU_1 is block lower triangular.

The second part of the paper concerns realizations induced by $\mathcal{D}$-embeddings. The operator U_T in (1) is, after an appropriate reordering of the coordinate spaces, a block lower triangular unitary operator . By Theorem 3.1 from (Kaashoek and Pik, 1998) it appears as the input-output map of a controllable, observable and unitary system

$$\tilde{\Sigma} = \left(A(n), \begin{bmatrix} B_1(n) & B(n) \end{bmatrix}, \begin{bmatrix} C(n) \\ C_1(n) \end{bmatrix}, \begin{bmatrix} D_{1,1}(n) & D(n) \\ D_{2,1}(n) & D_{2,2}(n) \end{bmatrix}; \right.$$
$$\left. H_n, K_n^\circ \oplus K_n, L_n \oplus L_n^\circ \right).$$

(See Section 4 for the notions of observability and controllability.) It follows that the system $\Sigma = (A(n), B(n),\ C(n),\ D(n);\ H_n,\ K_n,\ L_n)$ is a contractive realization of T. We refer to Σ as the *realization of T induced by the $\mathcal{D}$-embedding U_T*. It will be shown that the realizations of T induced by the $\mathcal{D}$-embedding U_T are unitarily equivalent.

As in the time-invariant case, see (Arov, 1979), many properties of the system Σ are reflected by properties of the blocks $T_{1,1}$, $T_{2,1}$, and $T_{2,2}$ of the embedding U_T. To describe these we introduce the notion of a minimal $\mathcal{D}$-embedding. We will call a pair of block lower triangular unitary operators (U_2, U_1) such that $U_2 H U_1$ is block lower triangular a *denominator* of H. A pair of block lower triangular unitary operators $(\tilde{U}_2, \tilde{U}_1)$ is a *divisor of* (U_2, U_1) *with respect to* H if there exist block lower triangular unitary operators B_1 and B_2 such that $U_2 = B_2 \tilde{U}_2$ and $U_1 = \tilde{U}_1 B_1$, and $\tilde{U}_2 H \tilde{U}_1$ is block lower triangular. A denominator (U_2, U_1) of H is called *minimal*, or a *minimal denominator*, if for each divisor $(\tilde{U}_2, \tilde{U}_1)$ of (U_2, U_1) with respect to H we have $U_2 = B_2 \tilde{U}_2$ and $U_1 = \tilde{U}_1 B_1$, where B_1 and B_2 are diagonal unitary operators. A $\mathcal{D}$-embedding (6) will be called *minimal* if (U_2, U_1) is a minimal denominator of H. We shall show that such a definition makes sense, and is independent of the particular choice of the outer operator F, the star-outer operator G and the contraction H in the embedding (6). The controllability and observability of Σ can now be characterized in terms of the minimality of the $\mathcal{D}$-embedding U_T.

Theorem 2. *Let T be a block lower triangular operator which admits a $\mathcal{D}$-embedding U_T as in (1), and let Σ be a realization induced by the $\mathcal{D}$-embedding U_T. Then U_T is a minimal $\mathcal{D}$-embedding if and only if Σ is controllable and observable.*

The above theorem appears in this paper as Theorem 5. The property that Σ is optimal (star-optimal) can be also seen from the $\mathcal{D}$-embedding. (For the definition of an optimal system, see Section 8.)

Theorem 3. *Let T be a block lower triangular operator which admits a minimal $\mathcal{D}$-embedding U_T as in (1), and let Σ be a realization of T induced by the $\mathcal{D}$-embedding U_T. Then Σ is controllable and observable. In this case,*

(i) $T_{2,2}$ *is outer if and only if Σ is an optimal system,*

(ii) $T_{1,1}$ *is star-outer if and only if Σ is a star-optimal system.*

The main results are taken from the author's thesis (Pik, 1999). The analogues of Theorems 1–3 for Schur class functions can be found in (Arov, 1979; 1985). The research which lead to this paper was inspired by these two papers.

The present paper consists of eight sections, not counting this introduction. In Section 2 we explain some notions from systems theory. We will introduce contractive systems and their input-output map. In Section 3 the basic problem how to embed a block lower triangular contraction in a block lower triangular unitary operator is discussed. Necessary and sufficient conditions to admit such an embedding are given. In Section 4 Darlington embeddings are introduced, and in this section we prove Theorem 1. Section 5 deals deal with realizations induced by $\tilde{\mathcal{D}}$-, and $\mathcal{D}$-embeddings. In Section 6 the notion of a minimal Darlington embedding is introduced, and elementary properties of a such an embedding are given. In Section 7 we prove Theorem 2, and Theorem 3 is proved in Section 8. Section 9 specifies the theory on Darlington embeddings for the case when the operator $T = (t_{i,j})_{i,j=-\infty}^{\infty}$ is exponentially decaying off the main diagonal.

We conclude this introduction with some notation. The symbol $\ell^2(\mathcal{K})$ denotes the Hilbert space consisting of all square norm summable sequences $(k_j)_{j\in\mathbb{Z}}$ with $k_j \in K_j$. In other words,

$$\ell^2(\mathcal{K}) = \bigoplus_{j=-\infty}^{\infty} K_j. \tag{7}$$

Let $\mathcal{G} = (G_n)_{j\in\mathbb{Z}}$ and $\mathcal{H} = (H_n)_{j\in\mathbb{Z}}$ be two doubly infinite sequences of Hilbert spaces. We define the operator $W_{\mathcal{G},\mathcal{H}}$ acting from $\bigoplus_{n\in\mathbb{Z}}(G_n \oplus H_n)$ into $(\bigoplus_{n\in\mathbb{Z}} G_n) \oplus (\bigoplus_{n\in\mathbb{Z}} H_n)$ by

$$W_{\mathcal{G},\mathcal{H}}\big((g_j, h_j)_{j\in\mathbb{Z}}\big) = \big((g_j)_{j\in\mathbb{Z}}, (h_j)_{j\in\mathbb{Z}}\big), \tag{8}$$

and the operator $Z_{\mathcal{G},\mathcal{H}}$ acting from $\bigoplus_{n\in\mathbb{Z}}(G_n \oplus H_n)$ into $(\bigoplus_{n\in\mathbb{Z}} H_n) \oplus (\bigoplus_{n\in\mathbb{Z}} G_n)$ by

$$Z_{\mathcal{G},\mathcal{H}}\big((g_j, h_j)_{j\in\mathbb{Z}}\big) = \big((h_j)_{j\in\mathbb{Z}}, (g_j)_{j\in\mathbb{Z}}\big). \tag{9}$$

Notice that both $W_{\mathcal{G},\mathcal{H}}$ and $Z_{\mathcal{G},\mathcal{H}}$ are unitary operators. These two operators will be used throughout the paper to transform an array of four block lower triangular operators into one block lower triangular operator (see, e.g., (15)).

2. Preliminaries about Contractive Systems

In this section we will review some basic facts about time-variant contractive systems. For a more extensive treatment we refer to the papers (Arov *et al.*, 1998; Gohberg *et al.*, 1992), and the books (Constantinescu, 1995; Dewilde and Van der Veen, 1998; Foias *et al.*, 1998; Halanay and Ionescu, 1994).

Consider the time-variant system with discrete time n:

$$\Sigma \begin{cases} x_{n+1} = A(n)x_n + B(n)u_n, \\ \quad\; y_n = C(n)x_n + D(n)u_n \end{cases} \tag{10}$$

for $n \in \mathbb{Z}$. Here $A(n) : H_n \to H_{n+1}$, $B(n) : K_n \to H_{n+1}$, $C(n) : H_n \to L_n$ and $D(n) : K_n \to L_n$ are bounded linear operators acting between Hilbert spaces. It will be convenient to use the notation $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ instead of (10).

With a system Σ we associate the operator matrix $T_\Sigma = (t_{i,j})_{i,j=-\infty}^{\infty}$, where

$$t_{i,j} = \begin{cases} 0, & i < j, \\ D(n), & i = j, \\ C(i)\tau_{\mathcal{A}}(i, j+1)B(j), & i > j. \end{cases} \tag{11}$$

Here the operator $\tau_{\mathcal{A}}(k,l)$ is defined by

$$\tau_{\mathcal{A}}(k,l) = \begin{cases} A(k-1)A(k-2)\cdots A(l+1)A(l), & k > l, \\ I_{H_l}, & k = l, \\ 0, & k < l. \end{cases} \tag{12}$$

Starting the system with initial state $x_n = 0$, the vector $y = (\dots, 0, 0, y_n, \ y_{n+1}, \dots)^{\mathrm{tr}}$ containing the outputs of the system (11) can be obtained by multiplication of T_Σ with the vector of inputs $(\dots, 0, 0, u_n, u_{n+1}, \dots)^{\mathrm{tr}}$ (in formula, $y = T_\Sigma u$).

The system Σ is called *contractive (isometric, co-isometric, or unitary)* if the system matrix

$$M_\Sigma(n) = \begin{bmatrix} A(n) & B(n) \\ C(n) & D(n) \end{bmatrix} : H_n \oplus K_n \to H_{n+1} \oplus L_n$$

is a contraction (isometry, co-isometry, or unitary operator) for each integer n.

If Σ is contractive, then the input-output operator T_Σ induces a contractive linear operator acting from $\ell^2(\mathcal{K})$ into $\ell^2(\mathcal{L})$, which is again denoted by T_Σ (Arov *et al.*, 1998, Thm. 4.1). The operator T_Σ is referred to as the *input-output map* of Σ. On the other hand, each block lower triangular contraction T, acting from $\ell^2(\mathcal{K})$ into $\ell^2(\mathcal{L})$, appears as the input-output map of a contractive system Σ (see, for instance (Arov *et al.*, 1998, Thm. 6.1). Such a system is called a *realization* of T.

To give a characterization of the property that a block lower triangular contraction T admits a $\tilde{\mathcal{D}}$-embedding, we will introduce the notion of pointwise stability. A system $\Sigma = (A(n), \ B(n), \ C(n), \ D(n); \ H_n, \ K_n, L_n)$ is called *pointwise stable* if its sequence of main operators $(A(n) : H_n \to H_{n+1})_{n \in \mathbb{Z}}$ is pointwise stable, i.e.,

$$\lim_{p \to \infty} \left\| \tau_{\mathcal{A}}(n+p, n)x \right\| = 0$$

for each integer n and each vector $x \in H_n$. The system Σ is called a *pointwise star-stable* if its sequence of main operators is pointwise star-stable, i.e.,

$$\lim_{p \to \infty} \left\| \tau_{\mathcal{A}}(n, n-p)^* x \right\| = 0$$

for each integer n and each vector $x \in H_n$.

3. Unitary Embedding of a Block Lower Triangular Contraction

In this section we will consider the embedding of a block lower triangular contraction $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ into a unitary operator matrix

$$\begin{bmatrix} T_{1,1} & T \\ T_{2,1} & T_{2,2} \end{bmatrix} : \ell^2(\mathcal{K}^\circ) \oplus \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L}) \oplus \ell^2(\mathcal{L}^\circ), \tag{13}$$

where $\mathcal{K}^\circ = (K_i^\circ)_{i=-\infty}^{\infty}$ and $\mathcal{L}^\circ = (L_i^\circ)_{i=-\infty}^{\infty}$ are sequences of Hilbert spaces, and $T_{1,1},\ T_{2,1},\ T_{2,2}$ are required to be block lower triangular contractions. A unitary embedding of the form (13) will be called a *$\tilde{\mathcal{D}}$-embedding* of the block lower triangular contraction T. First we will characterize the property that T admits a $\tilde{D}$-embedding in terms of systems.

Proposition 1. *A block lower triangular contraction T admits a $\tilde{D}$-embedding if and only if T admits a pointwise stable and pointwise star-stable contractive realization.*

Proof. Part (a). Let $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ be a pointwise stable and pointwise star-stable contractive realization of T. Denote by $M_\Sigma(n)$ the system matrix at time n. Moreover, for each integer n define the defect operators

$$\begin{aligned} D_{M_\Sigma(n)} &= \big(I - M_\Sigma(n)^* M_\Sigma(n)\big)^{1/2} : H_n \oplus K_n \to H_n \oplus K_n, \\ D_{M_\Sigma(n)^*} &= (I - M_\Sigma(n) M_\Sigma(n)^*)^{1/2} : H_{n+1} \oplus L_n \to H_{n+1} \oplus L_n, \end{aligned} \tag{14}$$

and the defect spaces $\mathcal{D}_{M_\Sigma(n)} = \overline{\operatorname{Im} D_{M_\Sigma(n)}}$ and $\mathcal{D}_{M_\Sigma(n)^*} = \overline{\operatorname{Im} D_{M_\Sigma(n)^*}}$. Set

$$\begin{aligned} \tilde{A}(n) &= A(n), \\ \tilde{B}(n) &= \begin{bmatrix} B(n) & \tilde{\tau}_{H_{n+1}}^* D_{M_\Sigma(n)^*} \mid \mathcal{D}_{M_\Sigma(n)^*} \end{bmatrix} : K_n \oplus \mathcal{D}_{M_\Sigma(n)^*} \to H_{n+1}, \\ \tilde{C}(n) &= \begin{bmatrix} C(n) \\ D_{M_\Sigma(n)} \tau_{H_n} \end{bmatrix} : H_n \to L_n \oplus \mathcal{D}_{M_\Sigma(n)}, \\ \tilde{D}(n) &= \begin{bmatrix} D(n) & \tilde{\tau}_{L_n}^* D_{M_\Sigma(n)^*} \mid \mathcal{D}_{M_\Sigma(n)^*} \\ D_{M_\Sigma(n)} \tau_{K_n} & -M_\Sigma(n)^* \mid \mathcal{D}_{M_\Sigma(n)^*} \end{bmatrix} : K_n \oplus \mathcal{D}_{M_\Sigma(n)^*} \to L_n \oplus \mathcal{D}_{M_\Sigma(n)}, \end{aligned}$$

where

$$\begin{aligned} &\tau_{H_n} : H_n \to H_n \oplus K_n, && \tilde{\tau}_{H_{n+1}} : H_{n+1} \to H_{n+1} \oplus L_n, \\ &\tau_{K_n} : K_n \to H_n \oplus K_n, && \tilde{\tau}_{L_n} : L_n \to H_{n+1} \oplus L_n \end{aligned}$$

are the canonical embeddings. Then

$$\tilde{\Sigma} = \big(\tilde{A}(n), \tilde{B}(n), \tilde{C}(n), \tilde{D}(n); H_n, K_n \oplus \mathcal{D}_{M_\Sigma(n)^*}, L_n \oplus \mathcal{D}_{M_\Sigma(n)}\big)$$

is a unitary system. Denote by $T_{\tilde{\Sigma}}$ its input-output map. Since Σ is pointwise stable and pointwise star-stable, the system $\tilde{\Sigma}$, having the same sequence of main operators $A(n)$, is pointwise stable and pointwise star-stable too. From Theorem 3.1 from (Kaashoek and Pik, 1998) it follows that $T_{\tilde{\Sigma}}$ is unitary. Denote by $\mathcal{D}$ the doubly infinite sequence of Hilbert spaces $(\mathcal{D}_{M_\Sigma(n)})_{n\in\mathbb{Z}}$, and by $\mathcal{D}^*$ the doubly infinite sequence of Hilbert spaces $(\mathcal{D}_{M_\Sigma(n)^*})_{n\in\mathbb{Z}}$, and let the operators $W_{\mathcal{L},\mathcal{D}}$ and $Z_{\mathcal{K},\mathcal{D}^*}$ be defined by (8) and (9), respectively. Let $T_{1,1}$, $T_{2,1}$ and $T_{2,2}$ be defined by

$$W_{\mathcal{L},\mathcal{D}}T_{\tilde{\Sigma}}Z^*_{\mathcal{K},\mathcal{D}^*} = \begin{bmatrix} T_{1,1} & T \\ T_{2,1} & T_{2,2} \end{bmatrix} : \ell^2(\mathcal{D}^*) \oplus \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L}) \oplus \ell^2(\mathcal{D}). \tag{15}$$

The operators $T_{1,1}$, $T_{2,1}$ and $T_{2,2}$ are block lower triangular contractions, because $T_{\tilde{\Sigma}}$ is a block lower triangular operator. Since $W_{\mathcal{L},\mathcal{D}}$, $Z_{\mathcal{K},\mathcal{D}^*}$ and $T_{\tilde{\Sigma}}$ are unitary, $W_{\mathcal{L},\mathcal{D}}T_{\tilde{\Sigma}}Z^*_{\mathcal{K},\mathcal{D}^*}$ is unitary, and thus we have shown that T admits a $\tilde{D}$-embedding.

Part (b). Suppose that T admits a $\tilde{D}$-embedding. So there are sequences of Hilbert spaces $\mathcal{K}^\circ = (K^\circ_n)_{n\in\mathbb{Z}}$ and $\mathcal{L}^\circ = (L^\circ_n)_{n\in\mathbb{Z}}$, and block lower triangular contractions $T_{1,1}$, $T_{2,1}$ and $T_{2,2}$ such that

$$U_T = \begin{bmatrix} T_{1,1} & T \\ T_{2,1} & T_{2,2} \end{bmatrix} : \ell^2(\mathcal{K}^\circ) \oplus \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L}) \oplus \ell^2(\mathcal{L}^\circ)$$

is a unitary operator. The operator

$$V_T = W^*_{\mathcal{L},\mathcal{L}^\circ} U_T W_{\mathcal{K}^\circ,\mathcal{K}} : \bigoplus_{n\in\mathbb{Z}} (K^\circ_n \oplus K_n) \to \bigoplus_{n\in\mathbb{Z}} (L_n \oplus L^\circ_n) \tag{16}$$

is a unitary block lower triangular operator. By Theorem 4.1 from (Kaashoek and Pik, 1998), the operator V_T admits a unitary realization

$$\tilde{\Sigma} = \Bigg(A(n), \begin{bmatrix} \tilde{B}(n) & B(n) \end{bmatrix}, \begin{bmatrix} C(n) \\ \tilde{C}(n) \end{bmatrix}, \begin{bmatrix} \tilde{D}_{1,1}(n) & D(n) \\ \tilde{D}_{2,1}(n) & \tilde{D}_{2,2}(n) \end{bmatrix}; \\ H_n, K^\circ_n \oplus K_n, L_n \oplus L^\circ_n \Bigg),$$

which is pointwise stable and pointwise star-stable. The system $\Sigma = (A(n),\ B(n),\ C(n),\ D(n);\ H_n,\ K_n, L_n)$ is a contractive pointwise stable and pointwise star-stable realization of T. ■

Suppose that T admits a $\tilde{\mathcal{D}}$-embedding. Then by Theorems 2.1 and 3.1 from (Arov *et al.*, 2000) and by Proposition 1 there exist block lower triangular operators

$$F : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{N}), \quad G : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{L})$$

such that F is outer, G is star-outer, and

$$I - T^*T = F^*F, \quad I - TT^* = GG^*.$$

(For a definition of outer and star-outer, see below formula (5) in the introduction). As the next lemma shows, the existence of such operators F and G allows us to find a unique operator $H : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{N})$ such that

$$\begin{bmatrix} G & T \\ H & F \end{bmatrix} : \ell^2(\mathcal{M}) \oplus \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L}) \oplus \ell^2(\mathcal{N}) \tag{17}$$

is unitary. However, the operator H in (17) is not necessarily block lower triangular, and hence (17) may not be a $\tilde{\mathcal{D}}$-embedding.

Lemma 1. *Let $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ be a block lower triangular contraction, $F : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{N})$ be an outer operator satisfying $I - T^*T = F^*F$, and $G : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{L})$ a star-outer operator satisfying $I - TT^* = GG^*$. Then there exists a unique contraction $H : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{N})$ such that $HG^* = -FT^*$. Moreover, the operator H is also uniquely determined by the operator equation $F^*H = -T^*G$, and the operator*

$$V = \begin{bmatrix} G & T \\ H & F \end{bmatrix} : \ell^2(\mathcal{M}) \oplus \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L}) \oplus \ell^2(\mathcal{N}) \tag{18}$$

is unitary.

Proof. Part (a). Take $v \in \ell^2(\mathcal{L})$. Then

$$\begin{aligned} \|FT^*v\|^2_{\ell^2(\mathcal{N})} &= \langle F^*FT^*v, T^*v\rangle_{\ell^2(\mathcal{K})} = \langle (I - T^*T)T^*v, T^*v\rangle_{\ell^2(\mathcal{K})} \\ &= \left\|(I - T^*T)^{1/2}T^*v\right\|^2_{\ell^2(\mathcal{K})} = \left\|T^*(I - TT^*)^{1/2}v\right\|^2_{\ell^2(\mathcal{K})} \\ &\leq \left\|(I - TT^*)^{1/2}v\right\|^2_{\ell^2(\mathcal{L})} = \langle (I - TT^*)v, v\rangle_{\ell^2(\mathcal{L})} \\ &= \left\|G^*v\right\|^2_{\ell^2(\mathcal{M})}. \end{aligned}$$

Hence the operator $H : \mathrm{Im}\, G^* \to \ell^2(\mathcal{N})$ defined by

$$HG^*v = -FT^*v, \quad v \in \ell^2(\mathcal{L}) \tag{19}$$

is a well-defined contraction. Since G is star-outer, $\mathrm{Im}\, G^*$ is dense in $\ell^2(\mathcal{M})$. The operator H extends to a contraction $H : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{N})$ by continuity. If another operator $\tilde{H} : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{N})$ satisfies the equation $\tilde{H}G^* = -FT^*$, then $(H - \tilde{H})|\mathrm{Im}\, G^* = 0$. We conclude that $H = \tilde{H}$, because G is star-outer.

Part (b). Let the operator $H : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{N})$ satisfy the equation $HG^* = -FT^*$. So

$$GH^*F = -TF^*F = -T(I - T^*T) = -(I - TT^*)T = -GG^*T.$$

Since G is star-outer, $\mathrm{Ker}\, G = (0)$, and hence it follows that $H^*F = -G^*T$.

On the other hand, if H satisfies $H^*F = -G^*T$, then

$$GH^*F = -GG^*T = -(I - TT^*)T = -T(I - T^*T) = -TF^*F.$$

Since F is outer, $\operatorname{Im} F$ is dense in $\ell^2(\mathcal{N})$, and hence it follows that $GH^* = -TF^*$. We have shown that the equations $GH^* = -TF^*$ and $H^*F = -G^*T$ are equivalent.

Part (c). Now we will show that the operator V in (18) is unitary. Let $H : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{N})$ be the unique contraction satisfying $H^*F = -G^*T$. By the arguments above H satisfies $GH^* = -TF^*$. Since

$$(HH^* + FF^*)F = -HG^*T + F - FT^*T = F$$

and since F is outer, it follows that $HH^* + FF^* = I_{\ell^2(\mathcal{N})}$. Thus

$$VV^* = \begin{bmatrix} GG^* + TT^* & GH^* + TF^* \\ HG^* + FT^* & HH^* + FF^* \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Since

$$G(G^*G + H^*H) = G - TT^*G - TF^*H = G,$$

and G is star-outer, it follows that $G^*G + H^*H = I_{\ell^2(\mathcal{M})}$. Hence it follows that

$$V^*V = \begin{bmatrix} G^*G + H^*H & G^*T + H^*F \\ T^*G + F^*H & T^*T + F^*F \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

■

Using the above lemma, we can say more about the properties of $T_{1,1}$, $T_{2,1}$, and $T_{2,2}$ in a $\tilde{\mathcal{D}}$-embedding.

Proposition 2. *Let $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ be a block lower triangular contraction, which admits a $\tilde{\mathcal{D}}$-embedding*

$$U_T = \begin{bmatrix} T_{1,1} & T \\ T_{2,1} & T_{2,2} \end{bmatrix} : \ell^2(\mathcal{K}^\circ) \oplus \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L}) \oplus \ell^2(\mathcal{L}^\circ). \tag{20}$$

Then there exists: an outer operator $F : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{N})$ such that $I - T^*T = F^*F$, a star-outer operator $G : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{L})$ such that $I - TT^* = GG^*$, and a unique contraction $H : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{N})$ satisfying $H^*F = -G^*T$. Moreover, there exist a block lower triangular co-isometry B_1 and a block lower triangular isometry B_2 such that $T_{1,1} = GB_1$, $T_{2,2} = B_2F$ and $B_2^*T_{2,1}B_1^* = H$.

Proof. From the fact that U_T is unitary, it follows that

$$I_{\ell^2(\mathcal{L})} - TT^* = T_{1,1}T_{1,1}^*, \quad I_{\ell^2(\mathcal{K})} - T^*T = T_{2,2}^*T_{2,2}. \tag{21}$$

By Theorem 2.1 from (Arov *et al.*, 2000) there exists an outer operator $F : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{N})$ such that $I - T^*T \geq F^*F$, and $T_{2,2} = B_2F$ where $B_2 : \ell^2(\mathcal{N}) \to \ell^2(\mathcal{L}^\circ)$ is a block lower triangular contraction. Since

$$F^*F \geq F^*B_2^*B_2F = T_{2,2}^*T_{2,2} = I - T^*T \geq F^*F,$$

it follows that B_2 is an isometry on $\operatorname{Im} F$. The operator F is outer, so B_2 is an isometry on $\overline{\operatorname{Im} F} = \ell^2(\mathcal{N})$. By Theorem 2.3 from (Arov *et al.*, 2000) there exists a star-outer operator $G : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{L})$ such that $I - TT^* \geq GG^*$, and $T_{1,1} = GB_1$ for a block lower triangular contraction B_1. Since

$$GG^* \geq GB_1B_1^*G^* = T_{1,1}T_{1,1}^* = I - TT^* \geq GG^*,$$

it follows that B_1^* acts as an isometry on $\operatorname{Im} G^*$. As G is star-outer, B_1^* is an isometry on $\ell^2(\mathcal{M})$. Since U_T is unitary, we conclude that

$$0 = (T^*T_{1,1} + T_{2,2}^*T_{2,1})B_1^* = T^*G + F^*B_2^*T_{2,1}B_1^*.$$

By Lemma 1 there exists a unique contraction $H : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{N})$ such that $F^*H = -T^*G$. Hence $H = B_2^*T_{2,1}B_1^*$. ■

4. Darlington Embeddings

Next we will consider a Darlington embedding or $\mathcal{D}$-embedding of a block lower triangular contraction, i.e., a unitary embedding (13) with the additional property that

$$\overline{\operatorname{Im} T_{1,1}^*} = \ell^2(\mathcal{K}^\circ), \quad \overline{\operatorname{Im} T_{2,2}} = \ell^2(\mathcal{L}^\circ). \tag{22}$$

First we will deduce necessary and sufficient conditions for the existence of such an embedding.

Theorem 4. *A block lower triangular contraction $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ admits a $\mathcal{D}$-embedding if and only if the following two conditions are satisfied:*

(i) there exists an outer operator $F : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{N})$ such that $I - T^*T = F^*F$ and there exists a star-outer operator $G : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{L})$ such that $I - TT^* = GG^*$,

(ii) for the unique contraction $H : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{N})$ satisfying $H^*F = -G^*T$ there exist block lower triangular unitary operators $U_1 : \ell^2(\mathcal{P}_1) \to \ell^2(\mathcal{M})$ and $U_2 : \ell^2(\mathcal{N}) \to \ell^2(\mathcal{P}_2)$, where $\mathcal{P}_i$, $i = 1, 2$ is a doubly infinite sequence of Hilbert spaces, such that U_2HU_1 is a block lower triangular.

If conditions (i) and (ii) are satisfied, then each $\mathcal{D}$-embedding is obtained by

$$U_T = \begin{bmatrix} GU_1 & T \\ U_2HU_1 & U_2F \end{bmatrix}, \tag{23}$$

where U_2 and U_1 are block lower triangular unitary operators such that U_2HU_1 is a block lower triangular operator.

Proof. Suppose that conditions (i) and (ii) hold. From Lemma 1 it follows that the operator

$$\begin{bmatrix} I & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} G & T \\ H & F \end{bmatrix} \begin{bmatrix} U_1 & 0 \\ 0 & I \end{bmatrix}$$

is unitary. By assumption, each of the operator blocks in the above product is block lower triangular. Hence conditions (i) and (ii) imply that T admits a $\mathcal{D}$-embedding. Now we will show the reverse implication. Suppose that T admits a $\mathcal{D}$-embedding

$$U_T = \begin{bmatrix} T_{1,1} & T \\ T_{2,1} & T_{2,2} \end{bmatrix} : \ell^2(\mathcal{K}^\circ) \oplus \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L}) \oplus \ell^2(\mathcal{L}^\circ). \tag{24}$$

By Proposition 2 there exist: an outer operator $F : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{N})$ such that $I - T^*T = F^*F$, a star-outer operator $G : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{L})$ such that $I - TT^* = GG^*$, a unique contraction $H : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{N})$ satisfying $H^*F = -G^*T$, a block lower triangular co-isometry B_1 and a block lower triangular isometry B_2 such that $T_{1,1} = GB_1$, $T_{2,2} = B_2F$, and $B_2^*T_{2,1}B_1^* = H$. By assumption, the image of $T_{2,2}$ is dense in $\ell^2(\mathcal{L}^\circ)$, so

$$\overline{\text{Im } B_2} = \overline{\text{Im } B_2F} = \overline{\text{Im } T_{2,2}} = \ell^2(\mathcal{L}^\circ).$$

It follows that B_2 is unitary. Also by assumption, the image of $T_{1,1}^*$ is dense in $\ell^2(\mathcal{K}^\circ)$, so

$$\overline{\text{Im } B_1^*} = \overline{\text{Im } B_1^*G^*} = \overline{\text{Im } T_{1,1}^*} = \ell^2(\mathcal{K}^\circ).$$

It follows that B_1 is unitary. Also from Proposition 2 we may conclude that

$$T_{2,1} = B_2B_2^*T_{2,1}B_1^*B_1 = B_2HB_1,$$

so B_2HB_1 is block lower triangular. The theorem is proved. ■

5. Realizations Induced by $\tilde{\mathcal{D}}$- and $\mathcal{D}$-Embeddings

In this section we will show how a $\tilde{\mathcal{D}}$-embedding of a block lower triangular contraction T brings forth an essentially unique contractive realization of T. To guarantee the uniqueness we need the notions of controllability and observability. In the remaining sections we will see that many properties of this realization can be found back in properties of the $\tilde{\mathcal{D}}$-embedding.

Let $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ be a time-variant system. The subspace Ker $(C|A; n)$ of H_n, defined by

$$\text{Ker } (C|A; n) = \bigcap_{j \geq n} \text{Ker } C(j)\tau_{\mathcal{A}}(j, n), \tag{25}$$

is called the *unobservable subspace at time* n. The closure of the linear manifold Im $(A|B; n)$, defined by

$$\text{Im } (A|B; n) = \operatorname*{span}_{j \leq n-1} \text{Im } \tau_{\mathcal{A}}(n, j+1)B(j),$$

is called the *controllable subspace at time* n. A system $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ is called *observable at time* n if Ker $(C|A; n) = \{0\}$, and *controllable at time* n if the linear manifold Im $(A|B; n)$ is dense in H_n. The system Σ will be called *(completely) observable* if Σ is observable at each time n, and *(completely) controllable* if Σ is controllable at each time n.

Let a block lower triangular operator T admit a $\tilde{\mathcal{D}}$-embedding

$$U_T = \begin{bmatrix} T_{1,1} & T \\ T_{2,1} & T_{2,2} \end{bmatrix} : \ell^2(\mathcal{K}^\circ) \oplus \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L}) \oplus \ell^2(\mathcal{L}^\circ). \tag{26}$$

By Theorem 4.1 from (Kaashoek and Pik, 1998), the block lower triangular unitary operator

$$V_T = W^*_{\mathcal{L},\mathcal{L}^\circ} U_T W_{\mathcal{K}^\circ,\mathcal{K}} : \bigoplus_{n\in\mathbb{Z}} (K_n^\circ \oplus K_n) \to \bigoplus_{n\in\mathbb{Z}} (L_n \oplus L_n^\circ), \tag{27}$$

where $W_{\mathcal{K}^\circ,\mathcal{K}}$ and $W_{\mathcal{L},\mathcal{L}^\circ}$ are defined in (8), admits a controllable, observable and unitary realization

$$\hat{\Sigma} = \Bigg(A(n), \begin{bmatrix} B_1(n) & B(n) \end{bmatrix}, \begin{bmatrix} C(n) \\ C_1(n) \end{bmatrix}, \begin{bmatrix} D_{1,1}(n) & D(n) \\ D_{2,1}(n) & D_{2,2}(n) \end{bmatrix}; \\ H_n, K_n^\circ \oplus K_n, L_n \oplus L_n^\circ \Bigg). \tag{28}$$

The system $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ will be called a *realization of T induced by the $\tilde{\mathcal{D}}$-embedding of T.*

Let us denote by $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ and $\tilde{\Sigma} = (\tilde{A}(n), \tilde{B}(n), \tilde{C}(n), D(n);\ \tilde{H}_n,\ K_n,\ L_n)$ two realizations of T, induced by the $\tilde{\mathcal{D}}$-embedding U_T. In the next proposition we will show that the systems Σ and $\tilde{\Sigma}$ are *unitarily equivalent*, i.e., there exist unitary operators $U_n : H_n \to \tilde{H}_n$ such that

$$U_{n+1}A(n) = \tilde{A}(n)U_n, \quad U_{n+1}B(n) = \tilde{B}(n), \quad C(n) = \tilde{C}(n)U_n.$$

Proposition 3. *Let T be a block lower triangular contraction which admits a $\tilde{\mathcal{D}}$-embedding U_T. Then all the realizations of T induced by the $\tilde{\mathcal{D}}$-embedding U_T are unitarily equivalent. Moreover, each realization of T induced by the $\tilde{\mathcal{D}}$-embedding U_T is pointwise stable and pointwise star-stable.*

Proof. Consider the $\tilde{\mathcal{D}}$-embedding $U_T : \ell^2(\mathcal{K}^\circ) \oplus \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L}) \oplus \ell^2(\mathcal{L}^\circ)$ of T given by (26). For $i = 1, 2$ let

$$\hat{\Sigma}^{(i)} = \Bigg(A^{(i)}(n), \begin{bmatrix} B_1^{(i)}(n) & B^{(i)}(n) \end{bmatrix}, \begin{bmatrix} C^{(i)}(n) \\ C_1^{(i)}(n) \end{bmatrix}, \begin{bmatrix} D_{1,1}(n) & D(n) \\ D_{2,1}(n) & D_{2,2}(n) \end{bmatrix}; \\ H_n^{(i)}, K_n^\circ \oplus K_n, L_n \oplus L_n^\circ \Bigg)$$

be a controllable, observable and unitary realization of the block lower triangular unitary operator V_T given by formula (27). Then $\hat{\Sigma}^{(i)}$ are unitarily equivalent by

Proposition 2.3.3 in (Constantinescu, 1995). Let the unitary equivalence be given by $U_n : H_n^{(1)} \to H_n^{(2)},\ n \in \mathbb{Z}$. So

$$U_{n+1} A^{(1)}(n) U_n^* = A^{(2)}(n),$$

$$U_{n+1} \begin{bmatrix} B_1^{(1)}(n) & B^{(1)}(n) \end{bmatrix} = \begin{bmatrix} B_1^{(2)}(n) & B^{(2)}(n) \end{bmatrix},$$

$$\begin{bmatrix} C^{(1)}(n) \\ C_1^{(1)}(n) \end{bmatrix} U_n^* = \begin{bmatrix} C^{(2)}(n) \\ C_1^{(2)}(n) \end{bmatrix}.$$

In particular, the systems $\Sigma^{(i)} = (A^{(i)}(n), B^{(i)}(n), C^{(i)}(n), D(n); H_n^{(i)},\ K_n,\ L_n)$, $i = 1, 2$ are unitarily equivalent.

Now we will show that each realization of T induced by the $\tilde{\mathcal{D}}$-embedding U_T is pointwise stable and pointwise star-stable. Let $\Sigma = (A(n), B(n),\ C(n),\ D(n);\ H_n,\ K_n,\ L_n)$ be a realization of T induced by the $\tilde{\mathcal{D}}$-embedding (26). By Theorem 4.1 from (Kaashoek and Pik, 1998) it follows that the block lower triangular unitary operator V_T, given in (27), admits a controllable, observable and unitary realization

$$\hat{\Sigma} = \left(\alpha(n), \begin{bmatrix} \beta_1(n) & \beta(n) \end{bmatrix}, \begin{bmatrix} \gamma(n) \\ \gamma_1(n) \end{bmatrix}, \begin{bmatrix} D_{1,1}(n) & D(n) \\ D_{2,1}(n) & D_{2,2}(n) \end{bmatrix};\right.$$
$$\left. X_n, K_n^\circ \oplus K_n, L_n \oplus L_n^\circ \right),$$

which is pointwise stable and pointwise star-stable. Both Σ and the system $\Upsilon = (\alpha(n), \beta(n),\ \gamma(n),\ D(n);\ X_n, K_n, L_n)$ are realizations of T induced by the $\tilde{\mathcal{D}}$-embedding (13). According to the first part of the proof, Σ and Υ are unitarily equivalent. By unitary equivalence, the system Σ is pointwise stable and pointwise star-stable. ∎

6. Minimal $\mathcal{D}$-Embeddings

Let $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ be a block lower triangular contraction which admits a $\mathcal{D}$-embedding U_T. In this case the operator T has many $\mathcal{D}$-embeddings, as follows from Theorem 4. In this section we shall identify among all $\mathcal{D}$-embeddings of T certain minimal ones. For this purpose we will use the following terminology.

Let $H : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{N})$ be a bounded operator. A pair (U_2, U_1) of block lower triangular unitary operators, where U_2 acts from $\ell^2(\mathcal{N})$ into $\ell^2(\mathcal{R})$ and U_1 acts from $\ell^2(\mathcal{Q})$ into $\ell^2(\mathcal{M})$, is called a *denominator of* H if $U_2 H U_1$ is block lower triangular. If H has a denominator (U_2, U_1), then it has many denominators. For any pair of block lower triangular unitary operators (W_2, W_1), acting on appropriate spaces, the operator $W_2 U_2 H U_1 W_1$ is again block lower triangular, and the pair $(W_2 U_2, U_1 W_1)$ is a denominator of H.

Let $F : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{N})$ be an outer operator satisfying $I - T^*T = F^*F$, $G : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{L})$ be a star-outer operator satisfying $I - TT^* = GG^*$, and let $H : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{N})$ be the unique contraction such that $H^*F = -G^*T$. Then, by Theorem 4, the $\mathcal{D}$-embedding U_T of T equals

$$U_T = \begin{bmatrix} GU_1 & T \\ U_2HU_1 & U_2F \end{bmatrix}, \tag{29}$$

where U_2 and U_1 are block lower triangular unitary operators such that U_2HU_1 is a block lower triangular operator. Thus (U_2, U_1) is a denominator of H.

A denominator $(\tilde{U}_2, \tilde{U}_1)$ of H is called a *divisor* of the denominator (U_2, U_1) of H if there exist block lower triangular unitary operators B_1 and B_2 such that

(i) $U_2 = B_2\tilde{U}_2$

(ii) $U_1 = \tilde{U}_1 B_1$,

(iii) $\tilde{U}_2 H \tilde{U}_1$ is block lower triangular.

A denominator (U_2, U_1) of H is called *minimal from the left* if for each divisor $(\tilde{U}_2, \tilde{U}_1)$ of (U_2, U_1) we have $U_2 = B_2\tilde{U}_2$ with B_2 a diagonal unitary operator. A denominator (U_2, U_1) of H is called *minimal from the right* if for each divisor $(\tilde{U}_2, \tilde{U}_1)$ of (U_2, U_1) with respect to H we have $U_1 = \tilde{U}_1 B_1$ with B_1 a diagonal unitary operator. A denominator (U_2, U_1) of H is called *minimal* if it is minimal from the left and minimal from the right. We call U_T a *(left, right) minimal $\mathcal{D}$-embedding* of T if the denominator (U_2, U_1) of H is (left, right) minimal. The next proposition shows that the definition does not depend on the particular choice of F and G in (29).

Proposition 4. *Let $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ be a block lower triangular contraction which admits a $\mathcal{D}$-embedding*

$$U_T = \begin{bmatrix} T_{1,1} & T \\ T_{2,1} & T_{2,2} \end{bmatrix}. \tag{30}$$

Suppose that

$$U_T = \begin{bmatrix} G^{(i)}U_1^{(i)} & T \\ U_2^{(i)}H^{(i)}U_1^{(i)} & U_2^{(i)}F^{(i)} \end{bmatrix}, \quad i = 1, 2, \tag{31}$$

where $F^{(i)} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{M}^{(i)})$ is an outer operator satisfying

$$I - T^*T = \big(F^{(i)}\big)^*\big(F^{(i)}\big), \quad i = 1, 2$$

and $G^{(i)} : \ell^2(\mathcal{N}^{(i)}) \to \ell^2(\mathcal{L})$ is a star-outer operator satisfying

$$I - TT^* = G^{(i)}\big(G^{(i)}\big)^*, \quad i = 1, 2.$$

The operator $H^{(i)} : \ell^2(\mathcal{M}^{(i)}) \to \ell^2(\mathcal{N}^{(i)})$, $i = 1, 2$ *is the unique contraction satisfying* $H^{(i)}F^{(i)} = -(G^{(i)})^*T$. *Then the denominator* $(U_2^{(1)}, U_1^{(1)})$ *is minimal from the left (resp. from the right) if and only if* $(U_2^{(2)}, U_1^{(2)})$ *is minimal from the left (resp. from the right).*

Proof. Assume that $(U_2^{(2)}, U_1^{(2)})$ is a denominator of $H^{(2)}$ which is minimal from the left. We will show that $(U_2^{(1)}, U_1^{(1)})$ is a denominator of $H^{(1)}$ which is minimal from the left. Let $(\tilde{U}_2^{(1)}, \tilde{U}_1^{(1)})$ be a divisor of $(U_2^{(1)}, U_1^{(1)})$. So there exist block lower triangular unitary operators B_2 and B_1 such that $U_2^{(1)} = B_2\tilde{U}_2^{(1)}$ and $U_1^{(1)} = \tilde{U}_1^{(1)}B_1$, and $\tilde{U}_2^{(1)}H^{(1)}\tilde{U}_1^{(1)}$ is block lower triangular. We will show that B_2 is a diagonal unitary operator. Since $F^{(1)}$ and $F^{(2)}$ are outer operators satisfying

$$F^{(1)*}F^{(1)} = F^{(2)*}F^{(2)},$$

we may define the isometry $Q : \mathrm{Im}\ F^{(1)} \to \ell^2(\mathcal{M}^{(2)})$ by $QF^{(1)}x = F^{(2)}x$, which extends by continuity to an isometry from $\overline{\mathrm{Im}\ F^{(1)}} = \ell^2(\mathcal{M}^{(1)})$ into $\ell^2(\mathcal{M}^{(2)})$. We will show that Q is diagonal and unitary. It is surjective because $\overline{\mathrm{Im}\ F^{(2)}} = \ell^2(\mathcal{M}^{(2)})$, and block lower triangular because both $F^{(1)}$ and $F(2)$ are block lower triangular:

$$Q\big[\bigoplus_{j\geq n} M_j^{(1)}\big] = Q\big[\overline{F^{(1)}\bigoplus_{j\geq n} K_j}\big] = \overline{Q\big[F^{(1)}\bigoplus_{j\geq n} K_j\big]} = \overline{F^{(2)}\bigoplus_{j\geq n} K_j} = \bigoplus_{j\geq n} M_j^{(1)}.$$

In the same way we can construct a block lower triangular unitary operator $\tilde{Q}$ such that $\tilde{Q}F^{(2)} = F^{(1)}$. It follows that $\tilde{Q} = Q^*$, so Q is diagonal. We have shown that there exists a diagonal unitary operator Q such that $F^{(1)} = QF^{(2)}$.

Since $G^{(1)}$ and $G^{(2)}$ are star-outer operators satisfying

$$G^{(1)}G^{(1)*} = G^{(2)}G^{(2)*},$$

there exists a diagonal unitary operator R such that $G^{(1)} = G^{(2)}R$. From Lemma 1 we see that the operator $H^{(i)}$, $i = 1, 2$ is uniquely defined by the equation $F^{(i)*}H^{(i)} = -T^*G^{(i)}$. Hence

$$F^{(1)*}QH^{(2)}R = F^{(2)*}H^{(2)}R = -T^*G^{(2)}R = -T^*G^{(1)}.$$

Thus we obtain $QH^{(2)}R = H^{(1)}$. The pair of operators $(\tilde{U}_2^{(1)}Q, RU_1^{(1)})$ is a denominator of $H^{(2)}$, because $\tilde{U}_2^{(1)}Q$ is block lower triangular, $RU_1^{(1)}$ is block lower triangular, and so is

$$\tilde{U}_2^{(1)}QH^{(2)}RU_1^{(1)} = \tilde{U}_2^{(1)}H^{(1)}U_1^{(1)} = \tilde{U}_2^{(1)}H^{(1)}\tilde{U}_1^{(1)}B_1$$

as a product of block lower triangular operators $\tilde{U}_2^{(1)}H^{(1)}\tilde{U}_1^{(1)}$ and B_1. From

$$U_2^{(2)}F^{(2)} = T_{2,2} = U_2^{(1)}F^{(1)} = U_2^{(1)}QF^{(2)},$$

and the property that $F^{(2)}$ is outer, it follows that $U_2^{(2)} = U_2^{(1)}Q$. Since

$$G^{(2)}U_1^{(2)} = T_{1,1} = G^{(1)}U_1^{(1)} = G^{(2)}RU_1^{(1)}$$

and $G^{(2)}$ is star-outer, we see that $U_1^{(2)} = RU_1^{(1)}$. We conclude that the denominator $(\tilde{U}_2^{(1)}Q, RU_1^{(1)})$ of $H^{(2)}$ is a divisor of $(U_2^{(2)}, U_1^{(2)})$, since $B_2\tilde{U}_2^{(1)}Q = U_2^{(2)}$ and $RU_1^{(1)} = U_1^{(2)}$. But $(U_2^{(2)}, U_1^{(2)})$ is minimal from the left, so B_2 is a block diagonal unitary operator. The statement about the minimality from the right is proved in the same way. ■

7. Minimal $\mathcal{D}$-Embeddings and Their Induced Realizations

In this section we will characterize the minimality of a $\mathcal{D}$-embedding in terms of systems. Let $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ be a block lower triangular contraction admitting a $\mathcal{D}$-embedding

$$U_T = \begin{bmatrix} T_{1,1} & T \\ T_{2,1} & T_{2,2} \end{bmatrix} : \ell^2(\mathcal{K}^\circ) \oplus \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L}) \oplus \ell^2(\mathcal{L}^\circ),$$

and let

$$\tilde{\Sigma} = \Bigg(A(n), \begin{bmatrix} B_1(n) & B(n) \end{bmatrix}, \begin{bmatrix} C(n) \\ C_1(n) \end{bmatrix}, \begin{bmatrix} D_{1,1}(n) & D(n) \\ D_{2,1}(n) & D_{2,2}(n) \end{bmatrix};$$
$$H_n, K_n^\circ \oplus K_n, L_n \oplus L_n^\circ\Bigg)$$

be a controllable, observable and unitary realization of $W^*_{\mathcal{L},\mathcal{L}^\circ} U_T W_{\mathcal{K}^\circ,\mathcal{K}}$ (which exists by Theorem 4.1 from (Kaashoek and Pik, 1998)). Here $W_{\mathcal{L},\mathcal{L}^\circ}$ and $W_{\mathcal{K}^\circ,\mathcal{K}}$ are defined in (8). The system $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ is called a *realization of T induced by the $\mathcal{D}$-embedding U_T* (cf. Section 5 where this notion is introduced for $\tilde{\mathcal{D}}$-embeddings). The main theorem of this section relates the minimality from the left (resp. from the right) of the $\mathcal{D}$-embedding to the observability (resp. controllability) of a realization induced by a $\mathcal{D}$-embedding.

Theorem 5. *Let T be a block lower triangular operator which admits a $\mathcal{D}$-embedding U_T, and let Σ be a realization induced by the $\mathcal{D}$-embedding U_T. Then*

(i) U_T is a left minimal $\mathcal{D}$-embedding if and only if Σ is observable,

(ii) U_T is a right minimal $\mathcal{D}$-embedding if and only if Σ is controllable,

(iii) U_T is a minimal $\mathcal{D}$-embedding if and only if Σ is observable and controllable.

To prove this theorem, we need the notion of a cascade connection or product of two systems. Consider two contractive time-variant systems

$$\Sigma_\nu = \big(A_\nu(n), B_\nu(n), C_\nu(n), D_\nu(n); H_{\nu,n}, K_{\nu,n}, L_{\nu,n}\big), \quad \nu = 1, 2.$$

We assume that for each n the output space $L_{2,n}$ of Σ_2 at time n is equal to the input space $K_{1,n}$ of Σ_2 at time n. Define a new system Σ by

$$\begin{bmatrix} x_{n+1}^{(1)} \\ x_{n+1}^{(2)} \\ y_n^{(1)} \end{bmatrix} = \begin{bmatrix} A_1(n) & B_1(n)C_2(n) & B_1(n)D_2(n) \\ 0 & A_2(n) & B_2(n) \\ C_1(n) & D_1(n)C_2(n) & D_1(n)D_2(n) \end{bmatrix} \begin{bmatrix} x_n^{(1)} \\ x_n^{(2)} \\ u_n^{(2)} \end{bmatrix}. \tag{32}$$

The system Σ in (32) is called the *cascade connection* or *product* of Σ_1 and Σ_2. It will be denoted by $\Sigma_1\Sigma_2$. In fact, we have

$$\Sigma_1\Sigma_2 = \left(\begin{bmatrix} A_1(n) & B_1(n)C_2(n) \\ 0 & A_2(n) \end{bmatrix}, \begin{bmatrix} B_1(n)D_2(n) \\ B_2(n) \end{bmatrix}, \begin{bmatrix} C_1(n) & D_1(n)C_2(n) \end{bmatrix}, \right.$$

$$\left. D_1(n)\, D_2(n);\; H_{1,n} \oplus H_{2,n}, K_{2,n}, L_{1,n} \right).$$

The equality $T_{\Sigma_1\Sigma_2} = T_{\Sigma_1}T_{\Sigma_2}$ follows from the construction of the cascade connection. A cascade connection of unitary time-variant systems enjoys the following property:

Lemma 2. *Let Σ_i $(i = 1, 2)$ be a unitary system with a unitary input-output map T_{Σ_i}, and let the product $\Sigma = \Sigma_1\Sigma_2$ be well-defined. Then Σ is controllable and observable if and only if Σ_i $(i = 1, 2)$ is controllable and observable.*

In the proof of Theorem 5 and this lemma we use the notion of a simple system. A system $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ is called *simple* if

$$\mathrm{Ker}\ (C|A; n) \cap \mathrm{Im}\ (A|B; n)^{\perp} = \{0\}, \quad n \in \mathbb{Z}.$$

Proof. Let $\Sigma_i = (A_i(n), B_i(n), C_i(n), D_i(n); H_{i,n}, K_{i,n}, L_{i,n})$, $i = 1, 2$, and let the product Σ be given by $\Sigma = (A(n), B(n), C(n), D(n);\ H_n,\ K_{2,n},\ L_{1,n})$, where $H_n = H_{1,n} \oplus H_{2,n}$, and

$$A(n) = \begin{bmatrix} A_1(n) & B_1(n)C_2(n) \\ 0 & A_2(n) \end{bmatrix}, \quad B(n) = \begin{bmatrix} B_1(n)D_2(n) \\ B_2(n) \end{bmatrix},$$

$$C(n) = \begin{bmatrix} C_1(n) & D_1(n)C_2(n) \end{bmatrix}, \quad D(n) = D_1(n)D_2(n).$$

Part (a). Assume first that Σ_i is a controllable, observable and unitary time-variant system for $i = 1, 2$. Since Σ_i is unitary for $i = 1, 2$, the product Σ is a unitary time-variant system by Theorem 5.1 of (Kaashoek and Pik, 1998). The input-output map T_Σ of the product Σ equals $T_{\Sigma_1}T_{\Sigma_2}$ by the construction of the product $\Sigma_1\Sigma_2$. First we will show that the system Σ is simple. Fix $n \in \mathbb{Z}$. Take $x \in \mathrm{Ker}\ (C|A; n) \cap$

Im $(A|B;n)^{\perp}$. Let us write $x = (x_1, x_2)^{\mathrm{tr}} \in H_{1,n} \oplus H_{2,n}$. Since $x \in \mathrm{Ker}\ (C|A;n)$, the vector

$$\left[\begin{array}{cc} C_1(n+j) & D_1(n+j)C_2(n+j) \end{array}\right] \left[\begin{array}{cc} \tau_{\mathcal{A}_1}(n+j,n) & * \\ 0 & \tau_{\mathcal{A}_2}(n+j,n) \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]$$

equals 0 for each $j \geq 0$, and hence it follows that $x_1 \in \mathrm{Ker}\ (C_1|A_1;n)$. The system Σ_1 is observable by assumption, so $x_1 = 0$.

Since $x \perp \mathrm{Im}\ (A|B;n)$, it follows that

$$0 = \left\langle \left[\begin{array}{c} 0 \\ x_2 \end{array}\right], \tau_{\mathcal{A}}(n, n-j+1)B(n-j)u_j \right\rangle = \left\langle x_2, \tau_{\mathcal{A}_2}(n, n-j+1)B_2(n-j)u_j \right\rangle$$

for each $j \geq 1$ and each vector $u_j \in K_{2,n-j}$. The system Σ_2 is controllable by assumption, so $x_2 = 0$. It follows that the system Σ is simple.

The input-output map of Σ is unitary. The unitary realization of a block lower triangular unitary operator constructed from Theorem 4.1 from (Kaashoek and Pik, 1998) is observable and controllable. All simple unitary realizations of a block lower triangular contraction are unitarily equivalent (Constantinescu, 1995, p.44, Prop. 3.3). It follows that the realization Σ is observable and controllable by unitary equivalence.

Part (b). Let us now assume that Σ is controllable and observable. Fix $n \in \mathbb{Z}$ and take $x_1 \in \mathrm{Ker}\ (C_1|A_1;n) \cap \mathrm{Im}\ (A_1|B_1;n)^{\perp}$. Then $(x_1, 0)^{\mathrm{tr}} \in \mathrm{Ker}\ (C|A;n)$. Since Σ is observable, it follows that $x_1 = 0$. The system Σ_1 is simple.

Fix $m \in \mathbb{Z}$. Take $x_2 \in \mathrm{Ker}\ (C_2|A_2;m) \cap \mathrm{Im}\ (A_2|B_2;m)^{\perp}$. Then $(0, x_2)^{\mathrm{tr}} \perp \mathrm{Im}\ (A|B;n)$. Since Σ is observable, it follows that $x_2 = 0$. The system Σ_2 is simple. The input-output map of the system $\Sigma_i,\ i = 1, 2$ is unitary. By the same argument as in Part (a) of the proof it follows that the systems $\Sigma_i,\ i = 1, 2$ are controllable and observable. ■

The following simple lemma is used in the proof of the main theorem:

Lemma 3. *A controllable and observable realization of a block diagonal contraction has a trivial state space sequence.*

Proof. Let $T : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L})$ be a diagonal contraction, and let $\Sigma = (A(n), B(n), C(n),\ D(n);\ H_n, K_n, L_n)$ be a controllable and observable realization of T. Fix $n \in \mathbb{Z}$. Take $x \in \mathrm{Im}\ (A|B;n)$. Now $C(n+m)\tau_{\mathcal{A}}(n+m, n)x = 0$ for each $m \geq 0$, because T is diagonal. Since Σ is observable, this implies that $x = 0$. Hence $H_n = \overline{\mathrm{Im}\ (A|B;n)} = \{0\}$. ■

Proof of Theorem 5. Let

$$\tilde{\Sigma} = \left(A(n), \left[\begin{array}{cc} B_1(n) & B(n) \end{array}\right], \left[\begin{array}{c} C(n) \\ C_1(n) \end{array}\right], \left[\begin{array}{cc} D_{1,1}(n) & D(n) \\ D_{2,1}(n) & D_{2,2}(n) \end{array}\right]; \right.$$

$$\left. H_n, K_n^{\circ} \oplus K_n, L_n \oplus L_n^{\circ} \right) \tag{33}$$

be a controllable, observable and unitary realization of the block lower triangular unitary operator $W^*_{\mathcal{L},\mathcal{L}^\circ} U_T W_{\mathcal{K}^\circ,\mathcal{K}}$, where U_T is the $\mathcal{D}$-embedding

$$U_T = \begin{bmatrix} T_{1,1} & T \\ T_{2,1} & T_{2,2} \end{bmatrix} : \ell^2(\mathcal{K}^\circ) \oplus \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L}) \oplus \ell^2(\mathcal{L}^\circ) \tag{34}$$

of T. So $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ is a realization of T induced by the $\mathcal{D}$-embedding U_T. From Theorem 4 it follows that the $\mathcal{D}$-embedding (34) of T is of the form

$$U_T = \begin{bmatrix} GU_1 & T \\ U_2 H U_1 & U_2 F \end{bmatrix} : \ell^2(\mathcal{K}^\circ) \oplus \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L}) \oplus \ell^2(\mathcal{L}^\circ), \tag{35}$$

where $F : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{N})$ is an outer operator satisfying $I - T^*T = F^*F$, $G : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{L})$ is a star-outer operator satisfying $I - TT^* = GG^*$, $H : \ell^2(\mathcal{M}) \to \ell^2(\mathcal{N})$ is the unique contraction satisfying $H^*F = -G^*T$, and (U_2, U_1) is a denominator of H, where $U_2 : \ell^2(\mathcal{N}) \to \ell^2(\mathcal{L}^\circ)$ and $U_1 : \ell^2(\mathcal{K}^\circ) \to \ell^2(\mathcal{M})$.

Part (a). We will show that if Σ is an observable system, then U_T is a left minimal $\mathcal{D}$-embedding. Let $(\hat{U}_2, \hat{U}_1)$ be a divisor of (U_2, U_1), where $\hat{U}_2 : \ell^2(\mathcal{N}) \to \ell^2(\mathcal{P})$ and $\hat{U}_1 : \ell^2(\mathcal{Q}) \to \ell^2(\mathcal{N})$. So there exist block lower triangular unitary operators $B_2 : \ell^2(\mathcal{P}) \to \ell^2(\mathcal{L}^\circ)$ and $B_1 : \ell^2(\mathcal{K}^\circ) \to \ell^2(\mathcal{Q})$ such that $U_2 = B_2 \hat{U}_2$, $U_1 = \hat{U}_1 B_1$, and $\hat{U}_2 H \hat{U}_1$ is block lower triangular. We have to show that if Σ is observable, then B_2 is diagonal. Set $\hat{U}_2 = (\hat{u}_{i,j})_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{N}) \to \ell^2(\mathcal{P})$ and $B_2 = (b_{i,j})_{i,j=-\infty}^{\infty} : \ell^2(\mathcal{P}) \to \ell^2(\mathcal{L}^\circ)$. Let

$$\Sigma_{B_2} = \Bigg(A_{B_2}(n), \begin{bmatrix} 0 & B_{B_2}(n) \end{bmatrix}, \begin{bmatrix} 0 \\ C_{B_2}(n) \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & b_{n,n} \end{bmatrix}; \\ X_{B_2,n}, L_n \oplus P_n, L_n \oplus L_n^\circ \Bigg)$$

be a controllable, observable and unitary realization of the block lower triangular unitary operator

$$\left(\begin{bmatrix} \delta_{i,j} I & 0 \\ 0 & b_{i,j} \end{bmatrix} \right)_{i,j=-\infty}^{\infty} : \bigoplus_{j\in\mathbb{Z}} (L_j \oplus P_j) \to \bigoplus_{j\in\mathbb{Z}} (L_j \oplus L_j^\circ). \tag{36}$$

Here

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

is the Kronecker delta. Let

$$\hat{\Sigma} = \Bigg(\hat{A}(n), \begin{bmatrix} \hat{B}_1(n) & \hat{B}(n) \end{bmatrix}, \begin{bmatrix} \hat{C}(n) \\ \hat{C}_1(n) \end{bmatrix}, \begin{bmatrix} \hat{D}_{1,1}(n) & \hat{D}_{1,2}(n) \\ \hat{D}_{2,1}(n) & \hat{D}_{2,2}(n) \end{bmatrix}; \\ \hat{X}_n, K_n^\circ \oplus K_n, L_n \oplus P_n \Bigg)$$

be a controllable, observable and unitary realization of the block lower triangular unitary operator $W^*_{\mathcal{L},\mathcal{P}}\hat{T}W_{\mathcal{K}^\circ,\mathcal{K}}$, where

$$\hat{T} = \begin{bmatrix} I & 0 \\ 0 & \hat{U}_2 \end{bmatrix} \begin{bmatrix} G & T \\ H & F \end{bmatrix} \begin{bmatrix} U_1 & 0 \\ 0 & I \end{bmatrix},$$

and $W^*_{\mathcal{L},\mathcal{P}}$, $W_{\mathcal{K}^\circ,\mathcal{K}}$ are the unitary operators defined by (8). The product of Σ_{B_2} and $\hat{\Sigma}$ is a well-defined unitary system, because the output space of $\hat{\Sigma}$ equals the input space of Σ_{B_2} at each time instant. The product $\Sigma_{B_2}\hat{\Sigma} = \tilde{\Upsilon}$ is given by

$$\begin{aligned} \tilde{\Upsilon} \;=\; \Bigg(& \begin{bmatrix} A_{B_2}(n) & B_{B_2}(n)\hat{C}_1(n) \\ 0 & \hat{A}(n) \end{bmatrix}, \begin{bmatrix} B_{B_2}(n)\hat{D}_{2,1}(n) & B_{B_2}(n)\hat{D}_{2,2}(n) \\ \hat{B}_1(n) & \hat{B}(n) \end{bmatrix}, \\ & \begin{bmatrix} 0 & \hat{C}(n) \\ C_{B_2}(n) & b_{n,n}\hat{C}_1(n) \end{bmatrix}, \begin{bmatrix} \hat{D}_{1,1}(n) & \hat{D}(n) \\ \hat{D}_{2,1}(n) & \hat{D}_{2,2}(n) \end{bmatrix}; \\ & X_{B_2,n} \oplus \hat{X}_n, K^\circ_n \oplus K_n, L_n \oplus L^\circ_n \Bigg). \end{aligned} \tag{37}$$

Since both Σ_{B_2} and $\hat{\Sigma}$ are controllable, observable and unitary, by Lemma 2 the system $\tilde{\Upsilon}$ is controllable, observable and unitary. On the other hand, the system $\tilde{\Upsilon}$ is a realization of the block lower triangular unitary operator $W^*_{\mathcal{L},\mathcal{L}^\circ}U_T W_{\mathcal{K}^\circ,\mathcal{K}}$, since

$$\begin{aligned} U_T \;&=\; \begin{bmatrix} I & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} G & T \\ H & F \end{bmatrix} \begin{bmatrix} U_1 & 0 \\ 0 & I \end{bmatrix} \\ &=\; \begin{bmatrix} I & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \hat{U}_2 \end{bmatrix} \begin{bmatrix} G & T \\ H & F \end{bmatrix} \begin{bmatrix} U_1 & 0 \\ 0 & I \end{bmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} \Upsilon \;=\; \Bigg(& \begin{bmatrix} A_{B_2}(n) & B_{B_2}(n)\hat{C}_1(n) \\ 0 & \hat{A}(n) \end{bmatrix}, \begin{bmatrix} B_{B_2}(n)\hat{D}_{2,2}(n) \\ \hat{B}(n) \end{bmatrix}, \begin{bmatrix} 0 & \hat{C}(n) \end{bmatrix}, \hat{D}(n); \\ & X_{B_2,n} \oplus \hat{X}_n, K^\circ_n \oplus K_n, L_n \oplus L^\circ_n \Bigg) \end{aligned}$$

is a realization of T induced by the $\mathcal{D}$-embedding U_T. By Proposition 3 the systems Υ and Σ are unitarily equivalent. The system Σ is observable by assumption, so the same holds for Υ by unitary equivalence. This implies that the subspaces $X_{B_2,n}$ are trivial for each integer n. The state space sequence of the system Σ_{B_2} is trivial, so its input-output map (36) is diagonal. It follows that B_2 is a diagonal unitary operator.

Part (b). Assume that Σ is not observable. We will construct a divisor $(\hat{U}_2, \hat{U}_1)$ of (U_2, U_1) such that $U_2 = B_2\hat{U}_2$, where B_2 is a block lower triangular unitary operator which is not diagonal. For each integer n consider the orthogonal decomposition of the sequence of state spaces

$$H_n = \text{Ker } (C|A;n) \oplus \text{Ker } (C|A;n)^{\perp}$$

of the system $\tilde{\Sigma}$, given in (33). Relative to this decomposition of the sequence of state spaces, the system matrix at time n of the system $\tilde{\Sigma}$ is given by

$$M_{\tilde{\Sigma}}(n) = \left[\begin{array}{cc|cc} A_{1,1}(n) & A_{1,2}(n) & B_{1,1}(n) & B_{1,2}(n) \\ 0 & A_{2,2}(n) & B_{2,1}(n) & B_{2,2}(n) \\ \hline 0 & C_{1,2}(n) & D_{1,1}(n) & D(n) \\ C_{2,1}(n) & C_{2,2}(n) & D_{2,1}(n) & D_{2,2}(n) \end{array}\right]:$$

$$\text{Ker } (C|A;n) \oplus \text{Ker } (C|A;n)^{\perp} \oplus K_n^{\circ} \oplus K_n$$
$$\to \text{Ker } (C|A;n+1) \oplus \text{Ker } (C|A;n+1)^{\perp} \oplus L_n \oplus L_n^{\circ}.$$

Since $M_{\tilde{\Sigma}}(n)$ is a unitary operator for each integer n, we have

$$A_{1,1}(n)^* A_{1,1}(n) + \begin{bmatrix} 0 & C_{2,1}(n)^* \end{bmatrix} \begin{bmatrix} 0 \\ C_{2,1}(n) \end{bmatrix} = I. \tag{38}$$

For each integer n define the Hilbert space

$$\mathcal{M}_n = \left\{ \begin{bmatrix} x \\ l \\ z \end{bmatrix} \in \text{Ker } (C|A;n) \oplus L_n \oplus L_n^{\circ} \mid A_{1,1}(n)^* x + C_{2,1}(n)^* z = 0 \right\},$$

and the operators

$$\beta_l(n): \quad \mathcal{M}_n \to \text{Ker } (C|A;n+1), \quad \beta_l(n) \begin{bmatrix} x \\ l \\ z \end{bmatrix} = x,$$

$$\delta_{1,l}(n): \quad \mathcal{M}_n \to L_n, \quad \delta_{1,l}(n) \begin{bmatrix} x \\ l \\ z \end{bmatrix} = l,$$

$$\delta_{2,l}(n): \quad \mathcal{M}_n \to L_n^{\circ}, \quad \delta_{2,l}(n) \begin{bmatrix} x \\ l \\ z \end{bmatrix} = z.$$

Then by Lemma 5.2 from (Kaashoek and Pik, 1998),

$$\Upsilon_l = \left(A_{1,1}(n), \beta_l(n), \begin{bmatrix} 0 \\ C_{2,1}(n) \end{bmatrix}, \begin{bmatrix} \delta_{1,l}(n) \\ \delta_{2,l}(n) \end{bmatrix}; \text{ Ker } (C|A;n), \mathcal{M}_n, L_n \oplus L_n^{\circ} \right)$$

is a unitary time-variant system. Notice that $\mathcal{M}_n = L_n \oplus V_n$, where

$$V_n = \left\{ \begin{bmatrix} x \\ z \end{bmatrix} \in \text{Ker } (C|A;n) \oplus L_n^\circ \mid A_{1,1}(n)^* x + C_{2,1}(n)^* z = 0 \right\}.$$

Relative to this decomposition of $\mathcal{M}_n$, the system Υ_l becomes

$$\begin{aligned} \Upsilon_l \;=\; & \left(A_1(n), \begin{bmatrix} 0 & B_l(n) \end{bmatrix}, \begin{bmatrix} 0 \\ C_{2,1}(n) \end{bmatrix}, \begin{bmatrix} I_{L_n} & 0 \\ 0 & \delta_{2,l}(n) \end{bmatrix}; \right. \\ & \left. \text{Ker } (C|A;n), L_n \oplus V_n, L_n \oplus L_n^\circ \right). \end{aligned} \tag{39}$$

Since both Υ_l and $\tilde{\Sigma}$ are unitary systems, we have

$$\begin{aligned} & \left[\begin{array}{cc|cc} A_{1,1}(n)^* & 0 & 0 & C_{2,1}(n)^* \\ 0 & I_{\text{Ker } (C|A;n)^\perp} & 0 & 0 \\ \hline 0 & 0 & I_{L_n} & 0 \\ B_l(n)^* & 0 & 0 & \delta_{2,l}(n)^* \end{array} \right] \left[\begin{array}{cc|cc} A_{1,1}(n) & A_{1,2}(n) & B_{1,1}(n) & B_{1,2}(n) \\ 0 & A_{2,2}(n) & B_{2,1}(n) & B_{2,2}(n) \\ \hline 0 & C_{1,2}(n) & D_{1,1}(n) & D(n) \\ C_{2,1}(n) & C_{2,2}(n) & D_{2,1}(n) & D_{2,2}(n) \end{array} \right] \\ & \qquad = \left[\begin{array}{cc|cc} I_{\text{Ker } (C|A;n)} & 0 & 0 & 0 \\ 0 & A_{2,2}(n) & B_{2,1}(n) & B_{2,2}(n) \\ \hline 0 & C_{1,2}(n) & D_{1,1}(n) & D(n) \\ 0 & \gamma_r(n) & \delta_{1,r}(n) & \delta_{2,r}(n) \end{array} \right], \end{aligned} \tag{40}$$

where

$$\begin{aligned} \gamma_r(n) &= \beta_l(n)^* A_{1,2}(n) + \delta_{2,l}(n)^* C_{2,2}(n), \\ \delta_{1,r}(n) &= \beta_l(n)^* B_{1,1}(n) + \delta_{2,l}(n)^* D_{2,1}(n), \\ \delta_{2,r}(n) &= \beta_l(n)^* B_{1,2}(n) + \delta_{2,l}(n)^* D_{2,2}(n). \end{aligned}$$

The system

$$\begin{aligned} \Upsilon_r \;=\; & \left(A_{2,2}(n), \begin{bmatrix} B_{2,1}(n) & B_{2,2}(n) \end{bmatrix}, \begin{bmatrix} C_{1,2}(n) \\ \gamma_r(n) \end{bmatrix}, \begin{bmatrix} D_{1,1}(n) & D(n) \\ \delta_{1,r}(n) & \delta_{2,r}(n) \end{bmatrix}; \right. \\ & \left. \text{Ker } (C|A;n)^\perp, K_n^\circ \oplus K_n, L_n \oplus V_n \right) \end{aligned} \tag{41}$$

is unitary because (40) is a product of two unitary operators. We conclude that $\tilde{\Sigma} = \Upsilon_l \Upsilon_r$, where Υ_l and Υ_r are unitary time-variant systems. The system $\tilde{\Sigma} =$

$\Upsilon_l \Upsilon_r$ is a controllable, observable and unitary realization of the unitary block lower triangular operator $W_{\mathcal{L},\mathcal{L}^\circ} U_T W_{\mathcal{K}^\circ,\mathcal{K}}$, where U_T is the $\mathcal{D}$-embedding (34) of T. By Theorem 3.1 from (Kaashoek and Pik, 1998), the unitary system $\tilde{\Sigma}$ is pointwise stable and pointwise star-stable. It follows that both the unitary systems Υ_l and Υ_r are pointwise stable and pointwise star-stable. By the same theorem their input-output maps T_{Υ_l} and T_{Υ_r} are unitary. From (39) we see that

$$W_{\mathcal{L},\mathcal{L}^\circ} T_{\Upsilon_l} W^*_{\mathcal{L},\mathcal{V}} = \begin{bmatrix} I_{L_n} & 0 \\ 0 & B_2 \end{bmatrix},$$

where B_2 is the input-output map of the system

$$\big(A_1(n), B_l(n), C_{2,1}(n), \delta_{2,l}(n); \text{Ker } (C|A;n), V_n, L_n^\circ\big),$$

and $W_{\mathcal{L},\mathcal{L}^\circ}$ and $W^*_{\mathcal{L},\mathcal{V}}$ are defined by (8). Since Υ_l is a pointwise stable and pointwise star-stable unitary system, it is controllable and observable by Theorem 3.1 from (Kaashoek and Pik, 1998). By assumption there exists an integer n such that Ker $(C|A;n) \neq \{0\}$. By Lemma 3 the operator B_2 is not diagonal. Finally, we will show that $W_{\mathcal{L},\mathcal{V}} T_{\Upsilon_r} W^*_{\mathcal{K}^\circ,\mathcal{K}}$ is a $\mathcal{D}$-embedding of T. Set

$$W_{\mathcal{L},\mathcal{V}} T_{\Upsilon_r} W^*_{\mathcal{K}^\circ,\mathcal{K}} = \begin{bmatrix} T_{r,1,1} & T_{r,1,2} \\ T_{r,2,1} & T_{r,2,2} \end{bmatrix}.$$

Notice that each of the operators $T_{r,i,j}$ $(i,j = 1,2)$ is a block lower triangular contraction. Then

$$\begin{bmatrix} T_{1,1} & T \\ T_{2,1} & T_{2,2} \end{bmatrix} = U_T = W_{\mathcal{L},\mathcal{L}^\circ} T_{\Upsilon_l} T_{\Upsilon_r} W^*_{\mathcal{K}^\circ,\mathcal{K}} = \begin{bmatrix} I & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} T_{r,1,1} & T_{r,1,2} \\ T_{r,2,1} & T_{r,2,2} \end{bmatrix}$$
$$= \begin{bmatrix} T_{r,1,1} & T_{r,1,2} \\ B_2 T_{r,2,1} & B_2 T_{r,2,2} \end{bmatrix}.$$

In particular, $T = T_{r,1,2}$ and $T_{1,1} = T_{r,1,1}$. Moreover, Im $T^*_{r,1,1} =$ Im $T^*_{1,1}$ is dense in $\ell^2(\mathcal{K}^\circ)$. Since B_2 is unitary, Im $T_{r,2,2} = B_2^*$Im $T_{2,2}$ is dense in $\ell^2(\mathcal{V})$. By Theorem 4 it follows that

$$T_{\Upsilon_r} = \begin{bmatrix} I & 0 \\ 0 & \hat{U}_2 \end{bmatrix} \begin{bmatrix} G & T \\ H & F \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \hat{U}_1 \end{bmatrix} \tag{42}$$

for some block lower triangular unitary operators $\hat{U}_2$ and $\hat{U}_1$. We have constructed a divisor $(\hat{U}_2, \hat{U}_1)$ of (U_2, U_1) such that $U_2 = B_2 \hat{U}_2$ with B_2 a block lower triangular unitary operator which is not diagonal.

Part (c). The second statement of the theorem can be obtained in a similar way, or by using adjoint systems. ■

8. Minimal $\mathcal{D}$-Embeddings and Optimal Systems

A special class of contractive systems is the class of optimal systems. A contractive realization

$$\Sigma_\circ = \big(A_\circ(n), B_\circ(n), C_\circ(n), D(n); H_{\circ,n}, K_n, L_n\big)$$

of a block lower triangular contraction T is called *optimal* if for each contractive realization $\Sigma = (A(n), B(n), C(n), D(n);\ H_n,\ K_n,\ L_n)$ of T, for each $n \in \mathbb{Z}$, and each input sequence $u_n,\ u_{n+1},\ u_{n+2},\ \ldots$, where $u_j \in K_j$, we have

$$\Big\| \sum_{j=n}^{n+k} \tau_{\mathcal{A}_\circ}(n+k+1, j+1) B_\circ(j) u_j \Big\| \leq \Big\| \sum_{j=n}^{n+k} \tau_{\mathcal{A}}(n+k+1, j+1) B(j) u_j \Big\|,$$

for each $k \geq 0$. In (Arov *et al.*, 1998) it is shown that each block lower triangular contraction appears as the input-output operator of a controllable, observable and optimal system, and that all controllable, observable and optimal realizations are unitarily equivalent.

An observable and contractive realization

$$\Sigma_\bullet = \big(A_\bullet(n), B_\bullet(n), C_\bullet(n), D(n); H_{\bullet,n}, K_n, L_n\big)$$

of T is called *star-optimal* if for each observable and contractive realization $\Sigma = (A(n), B(n),\ C(n),\ D(n);\ H_n, K_n, L_n)$ of T, and for each input sequence $u_n,\ u_{n+1},\ u_{n+2},\ \ldots$ with $u_j \in K_j$, we have

$$\Big\| \sum_{j=n}^{n+k} \tau_{\mathcal{A}}(n+k+1, j+1) B(j) u_j \Big\| \leq \Big\| \sum_{j=n}^{n+k} \tau_{\mathcal{A}_\bullet}(n+k+1, j+1) B_\bullet(j) u_j \Big\|$$

for each integer $k \geq 0$. (In the definition of a star-optimal system one has to restrict oneself to observable contractive systems for technical reasons, see (Arov *et al.*, 1998, Sec. 7). In the next theorem we will characterize the optimality of a realization induced by a $\mathcal{D}$-embedding by properties of the embedding U_T.

Theorem 6. *Let T be a block lower triangular operator which admits a minimal $\mathcal{D}$-embedding*

$$U_T = \begin{bmatrix} T_{1,1} & T \\ T_{2,1} & T_{2,2} \end{bmatrix}, \tag{43}$$

and let Σ be a realization of T induced by the $\mathcal{D}$-embedding U_T. Then Σ is observable and controllable. Moreover,

(i) $T_{2,2}$ is outer if and only if Σ is an optimal system,

(ii) $T_{1,1}$ is star-outer if and only if Σ is a star-optimal system.

Proof. Let $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ be a realization of T induced by the $\mathcal{D}$-embedding (43). So there exist contractive operators $B_1(n)$, $C_1(n)$, $D_{1,1}(n)$, $D_{2,1}(n)$, and $D_{2,2}(n)$ such that

$$\Sigma_{\mathcal{D}} = \Bigg(A(n), \begin{bmatrix} B_1(n) & B(n) \end{bmatrix}, \begin{bmatrix} C(n) \\ C_1(n) \end{bmatrix}, \begin{bmatrix} D_{1,1}(n) & D(n) \\ D_{2,1}(n) & D_{2,2}(n) \end{bmatrix}; \\ H_n, K_n^\circ \oplus K_n, L_n^\circ \oplus L_n \Bigg)$$

is a unitary realization of

$$\Xi = \left(\begin{bmatrix} t_{i,j}^{(1,1)} & t_{i,j} \\ t_{i,j}^{(2,1)} & t_{i,j}^{(2,2)} \end{bmatrix} \right)_{i,j=-\infty}^{\infty},$$

where $t_{i,j}^{(1,1)}$, $t_{i,j}^{(2,1)}$, $t_{i,j}^{(2,2)}$, and $t_{i,j}$ are defined by $T_{1,1} = (t_{i,j}^{(1,1)})_{i,j=-\infty}^{\infty}$, $T_{2,1} = (t_{i,j}^{(2,1)})_{i,j=-\infty}^{\infty}$, and $T_{2,2} = (t_{i,j}^{(2,2)})_{i,j=-\infty}^{\infty}$. By Theorem 5 (iii) the system Σ is controllable and observable since the $\mathcal{D}$-embedding (43) is minimal. Notice that the system

$$\Sigma_2 = \left(A(n), B(n), \begin{bmatrix} C(n) \\ C_1(n) \end{bmatrix}, \begin{bmatrix} D(n) \\ D_{2,2}(n) \end{bmatrix}; H_n, K_n, L_n \oplus L_n^\circ \right)$$

is an isometric observable realization of the block lower triangular isometry

$$\Xi_2 = \left(\begin{bmatrix} t_{i,j} \\ t_{i,j}^{(2,2)} \end{bmatrix} \right)_{i,j=-\infty}^{\infty}. \tag{44}$$

Let us first show the equivalence in statement (i). Assume that $T_{2,2}$ is an outer operator. Since U_T is unitary, in particular $I - T^*T = T_{2,2}^* T_{2,2}$, so $F = T_{2,2}$ is a maximal outer solution to the operator inequality $I - T^*T \geq F^*F$. By this we mean that if G is another block lower triangular contraction satisfying $I - T^*T \geq G^*G$, then $G = QF$ for some block lower triangular contraction Q. From Theorem 7.1 from (Arov *et al.*, 2000) we conclude that Σ is an optimal realization of T.

Let us now start with the assumption that Σ is optimal. We have to show that the block lower triangular operator $T_{2,2}$ is outer. Denote by $M_\Sigma(n)$ the system matrix of Σ at time n. Define the defect operator

$$D_{M_\Sigma(n)} = \big(I - M_\Sigma(n)^* M_\Sigma(n)\big)^{1/2} : H_n \oplus K_n \to H_n \oplus K_n,$$

and the defect space $\mathcal{D}_{M_\Sigma(n)} = \overline{\mathrm{Im}\ D_{M_\Sigma(n)}}$. Set

$$Y(n) = D_{M_\Sigma(n)} \tau_{H_n} : H_n \to \mathcal{D}_{M_\Sigma(n)}, \quad Z(n) = D_{M_\Sigma(n)} \tau_{K_n} : K_n \to \mathcal{D}_{M_\Sigma(n)}, \tag{45}$$

where τ_{H_n} and τ_{K_n} are the canonical embeddings of H_n and K_n into $H_n \oplus K_n$, respectively. Then the operator matrix

$$\begin{bmatrix} A(n) & B(n) \\ C(n) & D(n) \\ Y(n) & Z(n) \end{bmatrix} : H_n \oplus K_n \to H_{n+1} \oplus L_n \oplus \mathcal{D}_{M_\Sigma(n)} \tag{46}$$

is an isometry for each $n \in \mathbb{Z}$. Thus the system

$$\Phi = \Phi_\Sigma = \big(A(n), B(n), Y(n), Z(n); H_n, K_n, \mathcal{D}_{M_\Sigma(n)}\big) \tag{47}$$

is a contractive time-variant system. Let $T_\Phi = (t_{\Phi,i,j})_{i,j=-\infty}^{\infty}$ be its input-output map. By Theorem 2.1 from (Arov *et al.*, 2000) it follows that $I - T^*T = T_{2,2}^*T_{2,2} \leq T_\Phi^*T_\Phi = I - T^*T$, so $T_{2,2}^*T_{2,2} = T_\Phi^*T_\Phi$. Since the system Σ is controllable, observable and optimal, by the same theorem the operator T_Φ is outer. Since Σ_2 is an isometric realization of Ξ_2 in (44), from Lemma 4.1 (ii) from (Arov *et al.*, 2000) we obtain the identity

$$I - T(n,k)^*T(n,k) - T_{2,2}(n,k)^*T_{2,2}(n,k) = \Lambda_{n,k}(\Sigma)^*\Lambda_{n,k}(\Sigma)$$

for each pair of integers n, k with $n \leq k$. Here $\Lambda_{n,k}(\Sigma) : \bigoplus_{j=n}^{k} K_j \to H_{k+1}$ is the operator defined by

$$\Lambda_{n,k}(\Sigma)\vec{v} = \sum_{j=n}^{k} \tau_{\mathcal{A}}(k+1, j+1)B(j)v_j,$$

where $\vec{v} = (v_n, v_{n+1}, \ldots, v_{k-1}, v_k)$. By the same lemma it follows that

$$I - T(n,k)^*T(n,k) - T_\Phi(n,k)^*T_\Phi(n,k) = \Lambda_{n,k}(\Sigma)^*\Lambda_{n,k}(\Sigma)$$

for each n, k with $n \leq k$. We conclude that $T_{2,2}(n,k)^*T_{2,2}(n,k) = T_\Phi(n,k)^*T_\Phi(n,k)$ for each n, k with $n \leq k$. By Proposition 3.1 from (Arov *et al.*, 2000) we obtain the inclusion $\mathrm{Im}\ H_n(T_{2,2}) \subset \overline{\mathrm{Im}\ T_{2,2}(n,\infty)}$, for each $n \in \mathbb{Z}$. Here

$$H_n(T_{2,2}) = \begin{bmatrix} \cdots & t_{n,n-3}^{(2,2)} & t_{n,n-2}^{(2,2)} & t_{n,n-1}^{(2,2)} \\ \cdots & t_{n+1,n-3}^{(2,2)} & t_{n+1,n-2}^{(2,2)} & t_{n+1,n-1}^{(2,2)} \\ \cdots & t_{n+2,n-3}^{(2,2)} & t_{n+2,n-2}^{(2,2)} & t_{n+2,n-1}^{(2,2)} \\ & \vdots & \vdots & \vdots \end{bmatrix} : \bigoplus_{j=-\infty}^{n-1} K_j \to \bigoplus_{j=n}^{\infty} L_j^\circ. \tag{48}$$

It follows that

$$\mathrm{Im}\begin{bmatrix} H_n(T_{2,2}) & T_{2,2}(n,\infty) \end{bmatrix} \subset \overline{\mathrm{Im}\ T_{2,2}(n,\infty)} \subset \overline{\mathrm{Im}\begin{bmatrix} H_n(T_{2,2}) & T_{2,2}(n,\infty) \end{bmatrix}}.$$

Let τ be the canonical embedding of $\oplus_{j\geq n} L_j^\circ$ into $\ell^2(\mathcal{L})$. Then

$$\overline{\mathrm{Im}\ T_{2,2}(n,\infty)} = \overline{\mathrm{Im}\ \begin{bmatrix} H_n(T_{2,2}) & T_{2,2}(n,\infty) \end{bmatrix}} = \overline{\mathrm{Im}\ \tau T_{2,2}} = \tau \overline{\mathrm{Im}\ T_{2,2}} = \oplus_{j\geq n} L_j^\circ.$$

We have shown that $T_{2,2}$ is outer.

Part (ii). The second statement of the theorem can be shown in a similar way, or by using adjoint systems (Pik, 1999, Secs. 6.4, 6.6 and 6.7). ■

9. Exponential Stability of Systems

So far we have only considered the pointwise stability or the pointwise star-stability of time-variant contractive systems. We will now consider exponential stability, which is of special interest with respect to $\tilde{\mathcal{D}}$- and $\mathcal{D}$-embeddings. A system $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ is called *exponentially stable* if

$$\limsup_{\nu\to\infty} \left(\sup_j \|\tau_{\mathcal{A}}(j+\nu, j)\| \right)^{1/\nu} < 1.$$

The input-output map of an exponentially stable contractive system is exponentially decaying off the main diagonal. (A proof of this statement can be obtained by a non-essential generalization of the proof of Proposition 2.1 from (Kaashoek and Pik, 1998)). A block lower triangular unitary operator which is exponentially decaying off the main diagonal admits an exponentially stable contractive (in fact, unitary) realization (see Theorem 4.1 from (Kaashoek and Pik, 1998)). This does not hold true for general block lower triangular contractions. In this section we give necessary and sufficient conditions when a block lower triangular contraction admits an exponentially stable contractive realization.

Theorem 7. *A block lower triangular contraction T admits a $\tilde{D}$-embedding (13) such that each of the blocks T, $T_{1,1}$, $T_{2,1}$ and $T_{2,2}$ is exponentially decaying off the main diagonal, if and only if T admits an exponentially stable contractive realization.*

Proof. Assume that $\Sigma = (A(n), B(n), C(n), D(n); H_n, K_n, L_n)$ is an exponentially stable contractive realization of T. In Part (a) of the proof of Proposition 1 we have constructed a unitary system $\tilde{\Sigma} = (A(n), \tilde{B}(n), \tilde{C}(n), \tilde{D}(n); H_n, K_n \oplus \mathcal{D}_{M_\Sigma(n)^*}, L_n \oplus \mathcal{D}_{M_\Sigma(n)})$ with the same sequence of main operators $A(n)$. Let us denote by $T_{\tilde{\Sigma}}$ its input-output map. Since Σ is exponentially stable, the system $\tilde{\Sigma}$, having the same sequence of main operators, is exponentially stable, too. From Theorem 3.1 from (Kaashoek and Pik, 1998) it follows that $T_{\tilde{\Sigma}}$ is unitary. From Proposition 2.1 from (Kaashoek and Pik, 1998) it follows that $T_{\tilde{\Sigma}}$ is exponentially decaying off the main diagonal. Denote by $\mathcal{D}$ the doubly infinite sequence of Hilbert spaces $(\mathcal{D}_{M_\Sigma(n)})_{n\in\mathbb{Z}}$, and by $\mathcal{D}^*$ the doubly infinite sequence of Hilbert spaces $(\mathcal{D}_{M_\Sigma(n)^*})_{n\in\mathbb{Z}}$, where $\mathcal{D}_{M_\Sigma(n)}$ and $\mathcal{D}_{M_\Sigma(n)^*}$ are given in (14). Let the operators $W_{\mathcal{L},\mathcal{D}}$ and $Z_{\mathcal{K},\mathcal{D}^*}$ be defined by (8) and (9), respectively, and define $T_{1,1}$, $T_{2,1}$ and $T_{2,2}$ by

$$W_{\mathcal{L},\mathcal{D}} T_{\tilde{\Sigma}} Z^*_{\mathcal{K},\mathcal{D}^*} = \begin{bmatrix} T_{1,1} & T \\ T_{2,1} & T_{2,2} \end{bmatrix} : \ell^2(\mathcal{D}^*) \oplus \ell^2(\mathcal{K}) \to \ell^2(\mathcal{L}) \oplus \ell^2(\mathcal{D}).$$

The operators $T_{1,1}$, $T_{2,1}$ and $T_{2,2}$ are block lower triangular contractions, because $T_{\tilde{\Sigma}}$ is a block lower triangular operator. Since $T_{\tilde{\Sigma}}$ is exponentially decaying off the main diagonal, it follows that the operators $T_{1,1}$, $T_{2,1}$ and $T_{2,2}$ are exponentially decaying off the main diagonal, too. Hence the $\tilde{D}$-embedding is exponentially decaying off the main diagonal. Suppose now that the block lower triangular contraction T admits a $\tilde{D}$-embedding which is exponentially decaying off the main diagonal. Then, by Theorem 4.1 from (Kaashoek and Pik, 1998), the unitary realization $\tilde{\Sigma}$ constructed in the proof of Proposition 1 is exponentially stable. Hence the contractive realization Σ of T, having the same sequence of main operators, is also exponentially stable. ∎

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