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A NEW DEFINITION OF THE FUZZY SET

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The present fuzzy arithmetic based on Zadeh's possibilistic extension principle and on the classic definition of a fuzzy set has many essential drawbacks. Therefore its application to the solution of practical tasks is limited. In the paper a new definition of the fuzzy set is presented. The definition allows for a considerable fuzziness decrease in the number of arithmetic operations in comparison with the results produced by the present fuzzy arithmetic.

Keywords: fuzzy set theory, fuzzy arithmetic, possibility

1. Introduction

In the framework of fuzzy arithmetic (Kaufmann and Gupta, 1991) various operations as, e.g., addition, subtraction, etc., are realized. These operations are made with the use of Zadeh's possibilistic extension principle (Dubois and Prade, 1988) or its new, improved, and also possibilistic version proposed by Klir (1997), which takes into account the so-called *requisite constraints*. Arithmetic operations are also performed under the assumption which was introduced by Zadeh (1978) that the membership function of a fuzzy set is of a possibilistic character and that each element of the universal set, with a non-zero membership grade, belongs to a fuzzy set (Zadeh, 1965).

According to the author, all of the above factors are reasons for many known shortcomings of the present fuzzy arithmetic, which are often described by researchers in their publications. The shortcomings interfere with applications of fuzzy arithmetic in solving practical problems, cf. e.g., the contribution (Zadeh, 2002). Examples of the shortcomings include: large fuzziness of calculation results of arithmetic operations, especially of addition, subtraction and multiplication, paradoxes connected with some operations causing their uselessness, e.g., the insensitivity of the subtraction result to numbers successively subtracted from the minuend (Piegat, 2005b).

To eliminate these shortcomings, some researchers, e.g., Kosiński *et al.* (2003), try to develop new implementations of fuzzy arithmetic operations. In the author's opinion the main reason for the shortcomings of the present fuzzy arithmetic is inappropriate definition of a fuzzy set, which does not fully correspond to fuzzy sets used by people. Further on, several definitions of a fuzzy set used at present will be cited. The definition from (Klir and Folger, 1988) is as follows: "Let X denote a universal set. Then, the membership function μ_A by which a fuzzy set A is usually defined has the form

$$\mu_A : X \rightarrow [0,1],$$

where $[0,1]$ denotes the interval of real numbers from 0 to 1, inclusive. ... Such a function is called a membership function and the set defined by it a fuzzy set."

Zadeh's definition (1965), also accepted by Dubois and Prade (1988), states: "... a fuzzy set F is equivalent to giving a reference set Ω and a mapping μ_F, of Ω into $[0,1]$, the unit interval."

The definition from (Zimmermann, 1996) has the form: "If X is a collection of objects denoted generically by x then a fuzzy set A in X is a set of ordered pairs:

$$A = \big\{\big(x, \mu_A(x)\big) \mid x \in X\big\},$$

$\mu_A(x)$ is called the membership function or grade of membership (also degree of compatibility or degree of truth) of x in A which maps X to the membership space M. (When M contains only two points 0 and 1, A is non-fuzzy and $\mu_A(x)$ is identical to the characteristic function of a non-fuzzy set.)"

Fuzzy sets are similarly defined in (Bezdek, 1993; Driankov *et al.*, 1993; Yager and Filev, 1994). The above classic definitions of fuzzy sets are, in the author's opinion, insufficient. This can be illustrated by a simple example.

Example 1. Let us consider membership functions of two fuzzy sets: $A = \textit{water}$ and $B = \textit{wine}$, cf. Fig. 1.

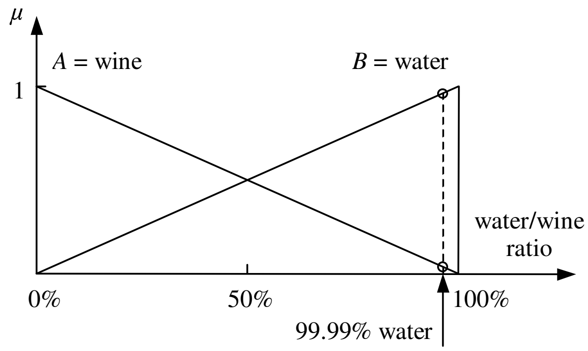


Fig. 1. Membership functions of the fuzzy sets $A = \textit{water}$ and $B = \textit{wine}$ characterizing the mixture of water and wine.

According to the classic definitions of a fuzzy set, the wine/water mixture with 0.01% of wine and 99.99% of water is qualified in the fuzzy set *wine*. However, would we (or other people) really classify such a mixture, after tasting it, as wine? No!

Thus, why does the present definition of a fuzzy set order such a qualification? ♦

2. Proposed Definition of a Fuzzy Set

Let X be a universe of elements denoted by x. A **fuzzy set** A of the elements x is a collection of the elements $x \mid x \in X$, which possess a specific property p_A of the set and which were qualified in the set by a qualifier Q_A using a qualification algorithm $QAlg_A$. At least one element of a fuzzy set must possess the specific property p_A of the set in an amount less than 1. If all elements x qualified in a set possess the specific property in a full amount, equal to 1, then the set is a **crisp set**.

The decision $m_A(x) \mid m_A(x) \in \{0, 1\}$ of the **qualifier** Q_A about the qualification of the element x in the set A depends, in the general case, on the minimal amount $p_{A\min}$ of the required, specific set property, on the type T_{QA} of the qualifier and on one or more conditions C_1, $\dots$ determined by the qualifier or an outside **definer**. It can be expressed as

$$m_A(x) = QAlg(x) = f(p_{A\min}, T_{QA}, C_1, \dots). \tag{1}$$

If the qualification decision of the qualifier is positive $(m_A(x) = 1)$, then the element x acquires a membership in the set A; otherwise $(m_A(x) = 0)$ the element is not in the set.

The qualification algorithm $QAlg(x)$ is generally a procedure consisting of formulas and IF–THEN conditions. Its output takes a value from the set $\{0, 1\}$. This is information whether or not a given element x has been qualified in the set. Thus the output of $QAlg(x)$ is the value of the membership $m_A(x)$ in a set. The notation $f(p_{A\min}, T_{QA}, C_1, \dots)$ implies that the output of $QAlg(x)$ in the general case depends on $p_{A\min}$, T_{QA}, and various conditions C_1, ..., which can exist in the analysed problem. The type of qualifier T_{QA} in (1) can take linguistic values from the set {deterministic, probabilistic, possibilistic, ...}. It should be noticed that in the present definition of a fuzzy set the notions of the qualifier and the qualification algorithm do not appear at all. However, qualification is always realized in one and the same way.

In the next sections a new approach to the notion of the fuzzy set will be explained.

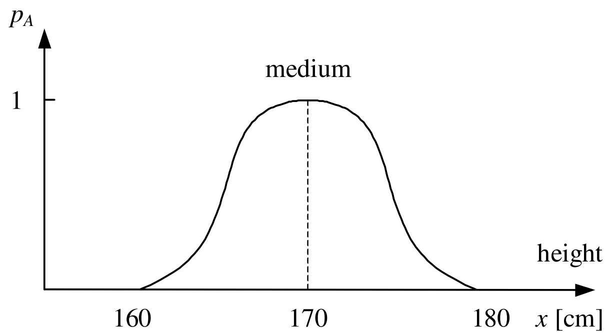
3. Explanations Referring to the Property Function $p_A(x)$ of a Set

In the present fuzzy set theory, the membership of an element x in a fuzzy set A, usually denoted by $\mu_A(x)$, fulfils two tasks simultaneously. It expresses both the grade of the membership of the element x in the set A and informs about the amount of the specific property of the set A possessed by the element x. This specific property distinguishes the elements of the set A from other elements of the universal set X.

The specific property of a set is a primary notion. The New Oxford Dictionary of English (Pearsal, 1999) explains property as “an attribute, quality or characteristic of something: *the property of heat to expand metal at uniform rates*.” The specific property of a set is defined by a set definer according to what he or she is interested in. A specific property can take linguistic values, e.g., *quite tall*, *vehicle*. It can also be a fuzzy number, e.g., *close to* 7. The amount of specific property can take real values in the interval $[0, 1]$.

In the new definition of a fuzzy set these two notions are separated, because such a separation is made by people creating sets in real problems. The mere possession of a set-specific property p_A in an amount greater than zero is not always sufficient for an element x to be qualified in a fuzzy set A. For example, the set of *beautiful* girls in a class is not a set of all girls who possess the property *beauty* to any grade, e.g., 0.001 as it defines the present fuzzy set theory, but the set of girls who have the property *beauty* at least in a sufficiently high (according to the qualifier) amount. An example of the property function $p_A(x)$ is depicted in Fig. 2.

The property function maps x into [0,1]. Sometimes a set A of elements x can be chosen by a qualifier Q_A fully at random from the elements of the universal set X (e.g., a set of samples for testing a neural network chosen from among all samples being at disposal for modeling a



$$p_{medium}(x) = \begin{cases} e^{\frac{(180-x)(x-160)-100}{(180-x)(x-160)}} & \text{for } 160 \le x \le 180 \\ 0 & \text{else} \end{cases}$$

Fig. 2. Property function $p_{medium}(x)$ that determines the amount of the property *medium* possessed by a person of the height x [cm].

system, a set of soldiers chosen by a sergeant for carrying out some task). In this case the specific property p_A required from elements x is that they belong to the universal set X. The qualification in to the set A is a result of a random qualification algorithm $QAlg_A(x)$.

Dubois and Prade (1996; 1997) give three interpretations of the traditional notion of the membership degree $\mu_A(x)$ in a fuzzy set. This degree can (according to the definition of the problem) be understood as a *degree of similarity* (the degree of proximity of x to prototype elements of A), a *degree of preference* (A represents a set of more or less preferred objects or values of a decision variable X and $\mu_A(x)$ represents an intensity of preference in favour of the object x, or the feasibility of selecting x as a value of X) and a *degree of uncertainty* (the quantity $\mu_A(x)$ is then the degree of possibility, …, that "x is A"). The degree of the specific property $p_A(x)$ introduced in the new definition seems to be more connected with the degree of similarity and the degree of preference. The degree of uncertainty (of possibility that "x is A") seems to be connected with the qualifier type and the qualification algorithm (possibilistic type).

4. Explanations Referring to the Qualifier Q_A

A qualifier Q_A can be **Person** 1 qualifying Person 2 in a set of persons of *short*, *medium* or *tall* height on the basis of the visual evaluation of height. The qualifier may act as an academic teacher who qualifies students into sets of *good*, *medium* or *weak* students (based on the evaluation of their knowledge). It can also be a **group of persons** who make the decision about the admission of candidates for studies at a faculty of computer science (based on marks from the chosen subjects and on fulfillment of the required conditions). The qualifier can be a **computer program** qualifying elements x from the universe X in assumed and mathematically formulated sets, e.g., *small* X, *medium* X, *large* X, on the basis of numerical values of elements x. It can also be a **technical device** qualifying elements into some sets, e.g., the sorting machine which sorts bottles according to their colors on the basis of the spectrum analysis of the light transmitted by the bottle glass.

Generally, a qualifier can be of various structures. It can be a simple, one-person, one-program, or one-device (machine) qualifier. It can also be a complex qualifier, which is composed, e.g., of many sub-qualifiers and of one super-qualifier as is in the case of the qualification of candidates for the full professorship in Poland (three reviewers make evaluations of the scientific, educational, and organizational achievements of a candidate and, next, a secret super-reviewer makes the final qualification decision based on the prior reviews of the open reviewers and on his or her own evaluation of the candidate's achievements).

The **qualifier Q_A always exists** (explicitly or implicitly) in each process of set creation from elements of the universal set X. First, the qualifier determines a specific feature p_A of the set A he or she wants to create, then formulates qualification conditions, and next carries out the qualification of elements to create the set A. If the qualifier is not a person, but a machine/device/computer program, then the qualification algorithm is created outside by a human **definer** and introduced into it. The qualifier can also be the nature itself (natural selection), but who/what is then the definer of the qualification algorithm?

Qualifiers are all measuring instruments. Let us analyze, for simplicity, a discrete measuring device of temperature, which can indicate temperature values with the accuracy of $0.1°$C (0, 0.1, 0.2, …, 15.0, 15.1, 15.2, etc). Then, if the real atmospheric temperature is, e.g., equal to $15.145739\ldots°$C, the device must qualify it in only one possible indication, i.e., $15.1°$C or $15.2°$C. The measuring instrument makes a similar qualification of temperature as a man qualifying it in his or her possible indications as, e.g., *low*, *medium*, and *high* temperature. The difference consists only in the width of qualification distributions. The instruments have distributions of smaller width (higher accuracy) and the man's distributions are of larger width (lower accuracy). The qualifier can always use the same qualification algorithm, but it also can change the algorithm in time.

5. Explanations Referring to the Qualification Conditions C_i

In simple cases people qualify elements x of the universal set X in a set A, e.g., when the elements possess a specific property p_A of the set at least to a certain minimal grade $p_{A\min}$. In this case the necessary qualification condition is expressed by

$$p_A(x) \geq p_{A\min}. \tag{2}$$

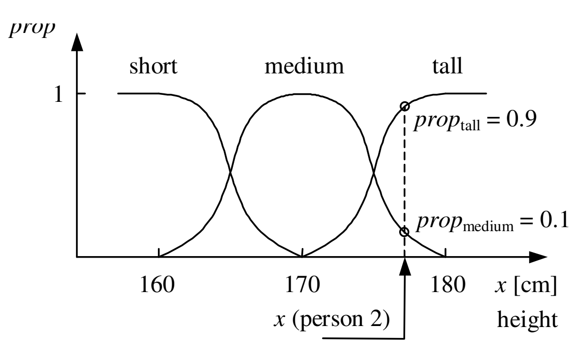
The minimal amount of a feature which is required for qualification can, e.g., be equal to 0.5. It can also be equal to, e.g., 0.9 if the qualifier is an especially exacting one. The minimal requirement for the set membership can also be as low as in classic fuzzy sets, i.e.,

$$p_A(x) > 0. \tag{3}$$

This means that **classic fuzzy sets are a special case of generalized fuzzy sets** determined by the new definition. People frequently use the following qualification condition: "an element x belongs to the set A_i, $i \in \{1, \dots, m\}$, whose specific property p_{Ai} it possesses at most." This condition is expressed by

$$\text{IF } \left[p_{Ai}(x) = \max\left\{p_{A1}(x), p_{A2}(x), \dots, p_{Am}(x)\right\}\right] \text{ THEN } (x \in A_i). \tag{4}$$

For example, a qualifier qualifies a person in the set of *tall* people if the person is (according to the qualifier) more *tall* than *medium*. This means that the person has more property *tall* than the property *medium* or *short*, cf. Fig. 3.



$$prop_{tall}(x) > prop_{medium}(x) > prop_{short}(x)$$

Fig. 3. Example of the qualification of an element x in a set A_i whose property the element has to the highest degree.

However, the mere possession of a specific property p_A by an element x in the grade higher than the minimal grade $p_{A\min}$ or in the grade $p_{Ai}(x)$, which is higher for the set A_i than for other sets A_j, $j \neq i$, $j \in \{1, \dots, m\}$, is not always sufficient for the qualification of the element x in the set A_i. For example, if a person possesses the property *tall* to the degree $p_{tall}(x) = 0.6$ and the property *medium* to the degree $p_{medium}(x) = 0.4$, then the person does not necessarily have to be qualified in the set *tall*. The person can sometimes be qualified in the set *medium*. Why? It will be explained in Section 6.2.

In this section only the simplest qualification conditions were described. In real tasks, the total condition can be complex, multidimensional and composed of many sub-conditions, as was shown through the example of professorship in Section 4.

6. Explanation Referring to the Qualifier Type T_{QA}

A qualifier Q_A making decisions about the qualification of an element x in a set A can be of various type, e.g., deterministic, probabilistic, possibilistic one, etc.

6.1. Deterministic Qualifier

A deterministic qualifier is a qualifier which qualifies identical elements x of the universal set X always in one and the same set A_i. The deterministic qualifier uses a deterministic qualification algorithm $QAlg_A$. An example of the deterministic qualifier is a person who exactly knows the qualification algorithm, is able to describe this algorithm, and makes the qualification thoroughly consciously without using sub-consciousness. To make deterministic qualifications, the value of the qualified element x must be exactly known. Figure 4 depicts a simple example of deterministic property functions of height evaluations, arbitrarily constructed by an expert.

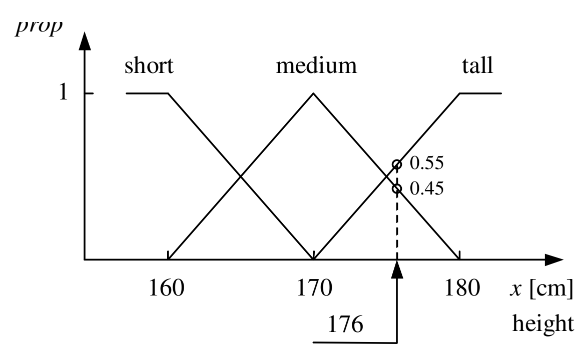


Fig. 4. Deterministic property functions $prop_{short}(x)$, $prop_{medium}(x)$, $prop_{tall}(x)$ of linguistic evaluations of height, arbitrarily defined by an expert.

If the property functions and the height of a given person are exactly known (e.g., 176 cm), then we or a computer can exactly calculate how much of the property *short*, *medium*, or *tall* the person's height has (e.g., $prop_{short}(176) = 0$, $prop_{medium}(176) = 0.45$, $prop_{tall}(176) = 0.55$). If the qualification algorithm is: "The element x belongs to the set whose property it has most of all", then the person's height 176 cm is deterministically qualified to the set *tall*. However, in the next section we will see that such a qualification does not always occur if the qualifier is a probabilistic one.

Now, let us consider the *inverse qualification problem* (*dequalification problem*), i.e., the identification of the element x-value, which was qualified in a fuzzy set A. The problem is solved under the assumption that the **only information** we have at our disposal is the information below.

Information

An element x, whose value is unknown to us, was qualified in the set A by a deterministic qualifier Q_A.

Query

What is the probable value of this element (what is the probability density distribution of x)?

Solution

To solve this problem, we can use the opinion by Klir and Folger (1988): "Within all probability measures, total ignorance is expressed by the uniform probability distribution

$$p(x) = \frac{1}{|X|} \quad \text{for all} \quad x \in X,$$

where $|X|$ is the cardinality of X." A solution of the dequalification problem will be illustrated by Example 2.

Example 2.

Information

A person of height x, which is unknown to us but exactly known to a deterministic qualifier Q_{Ai}, was qualified in the set *medium*. The qualifier uses only three evaluations (linguistic indications) of height: *short*$= A_1$, *medium*$= A_2$, and *tall*$= A_3$. The corresponding property functions are depicted in Fig. 5. The qualifier uses a deterministic qualification algorithm $QAlg_{Ai}(x)$: "an element x is qualified in the set A_i whose property $prop_{Ai}$ it has at most",

$$x \in A_i \mid A_i : i = 1, \ldots, m,$$

$$m_{Ai} = \begin{cases} 1 \text{ if } prop_{Ai}(x) = \\ \qquad \max\{prop_{A1}(x), \ldots, prop_{Am}(x)\}, \\ 0 \text{ otherwise.} \end{cases} \quad (5)$$

Query

What are the probable values of height x?

Solution

In the case of a deterministic qualifier, we can easily determine the distribution function of qualification probability in a set $qprob_{Ai}(x)$ and the distribution function of probability density $deqprobd_{Ai}(x)$ depicted in Fig. 5.

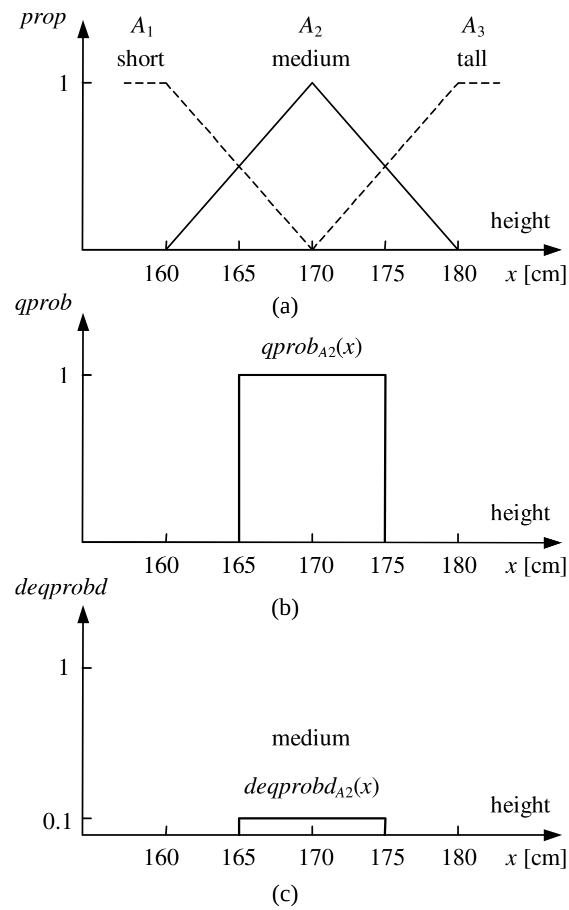


Fig. 5. Property functions $prop_{A1}(x)$, $prop_{A2}(x)$, $prop_{A3}(x)$ of height evaluations *short*, *medium*, *tall* (a), the distribution of the qualification probability $qprob_{A2}(x)$ of the elements x in the set $A_2 =$ *medium* height (b), and the distribution of the dequalification probability density $deqprobd_{A2}(x)$ that the element qualified in the set $A_2 =$ *medium* has the value x (c).

The rectangular distribution of the qualification probability $qprob_{A2}(x)$ in the set $A_2 =$ *medium* results from the deterministic qualification algorithm, cf. (5). Because all heights x satisfying the condition $165 < x \leq 175$ (Fig. 5(a)) have more property *medium* than the properties *short* or *tall*, they are always, with probability 1 (certainty), qualified in the set *medium*, cf. Fig. 5(b). As can be seen, there exists (in the case of the deterministic qualifier) no relation between the shape of the property function $prop_A(x)$ and the shape of the qualification function $qprob_A(x)$, which is always rectangular, independently of the shape of the property function. If our only information is that height x, whose value is unknown to us (but known to the qualifier) was qualified in the set $A_2 = \textit{medium}$, then the density distribution $\textit{deqprobd}_A(x)$ of the probability that the evaluated height had the value x (Fig. 5(c)) can be determined by a transformation of the distribution of qualification probability $\textit{qprob}_{A2}(x)$ from Fig. 5(b) such that the achieved distribution has the area normalized to 1 (the total probability of all possible x-values must be equal to 1). To this end, the area a of the function $\textit{qprob}_A(x)$ should be calculated in accordance with

$$a = \int_{X_{\min}}^{X_{\max}} \textit{qprob}_A(x) \geq 1. \tag{6}$$

Next, the transformation coefficient $\alpha = 1/a$ should be determined. To make the transformation $\textit{qprob}_A(x) \rightarrow \textit{deqprobd}_A(x)$, we use

$$\textit{deqprobd}_A(x) = \alpha \cdot \textit{qprob}_A(x), \tag{7}$$

Once more the basic difference between the qualification probability distribution $\textit{qprob}_{A2}(x)$ and the density distribution of dequalification probability $\textit{deqprobd}_{A2}(x)$ should be underlined. Both distributions give answers to contrary questions: The function $\textit{qprob}_{A2}(x)$ answers the question "What is the probability that the deterministic qualifier Q_{A2} will qualify height x in the set $A_2 = \textit{medium}$?". The probability of a single element x can be equal to 1 and the integral of the distribution (area) is greater than 1. The dequalification function $\textit{deqprobd}_{A2}(x)$ gives an answer to the question "What is the probable value of height x, which was qualified in the set $A_2 = \textit{medium}$?". The maximal value of density is lower than 1 (apart from a singleton case) and the integral of dequalification probability density distribution (area) equals 1. ♦

Remark 1. One should differentiate the set *medium* height shown in Fig. 5(c) from the set of heights which possess the property *medium* in an amount greater than zero, cf. Fig. 6(b). The set *medium* height contains only heights which have more property *medium* than any other property (*short* or *tall*). Therefore its support $[165, 175]$, cf. Fig. 5(c), is narrower than the support $[160, 180]$, cf. Fig. 6(b), of the set of heights which possess the feature *medium*. The last set is a fuzzy set in the classical sense. One can also notice in Figs. 5 and 6 that in the case of deterministic fuzzy sets the property functions and the qualification probability functions are of different shapes. The relation between the two functions is very weak. In the next section we will see whether the same takes place for probabilistic qualifiers.

In the short form, a set being a result of deterministic qualification can be presented as a set of ordered pairs

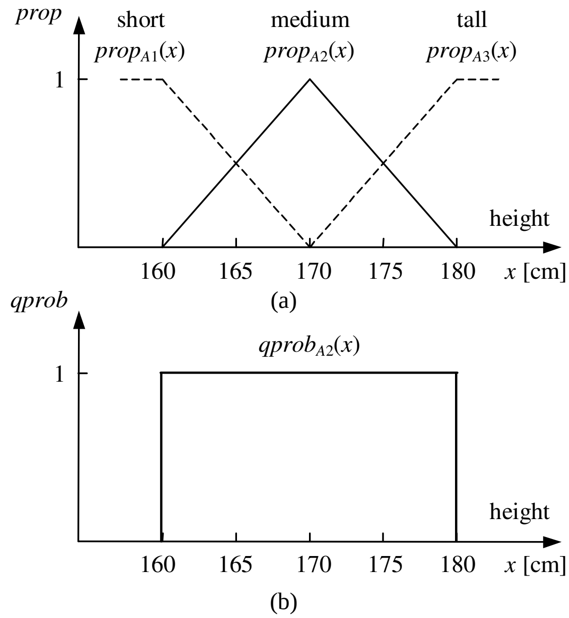


Fig. 6. Deterministic property function $\textit{prop}_{A2}(x)$ of the height set *medium* (a) and the probability distribution $\textit{qprob}_{A2}(x)$ qualifying height x in the set of the heights which have the property *medium* in an amount greater than zero (b).

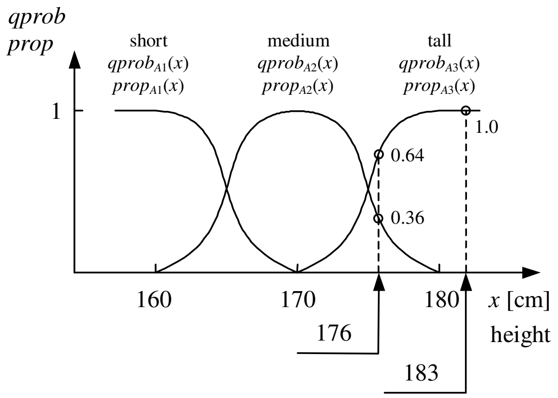
referring to elements x which were qualified in the set

$$A = \big\{\big(x, \textit{prop}_A(x)\big), \textit{QAlg}_A(x) \mid \forall x : \big(m_A(x) = 1\big) \wedge (x \in X)\big\}.$$

6.2. Probabilistic Qualifier

The distribution functions of the qualification probability $\textit{qprob}_{Ai}(x)$, which are declared by people in inquires differ, often considerably, from functions which are really used by them. The reason for that is that people qualify elements in sets not always fully consciously, but mostly more or less subconsciously (sometimes fully subconsciously), and they are not able to precisely express and describe the qualification (Piegat, 2001). Therefore, instead of identifying qualification functions from spoken inquires of people, it is better to identify them experimentally, more objectively (Piegat, 2004).

Experimental investigations show that different persons of the same height, e.g., $x = 176$ cm, can sometimes be qualified as *medium*, and sometimes as *tall* people. If a qualified person of the height 176 cm is slim or stands near a person of short height, e.g., 150 cm, then we will rather qualify the person as *tall*. If the person (176 cm) is corpulent or stands near a tall person, e.g., of the height 2 m, we will rather qualify he or she as *medium*. This means that qualification algorithms really used by people are often not of a deterministic but of a probabilistic type and that they contain distribution functions $qprob_{Ai}(x)$ of qualification probability as, e.g., the one depicted in Fig. 7 for height.



$$\sum_{i=1}^{3} qprob_{Ai}(x) = 1$$

Fig. 7. Examples of the distribution functions $qprob_{Ai}(x)$ of the qualification probability of the height x in the sets *short, medium* and *tall*. In this case the qualification functions are numerically equal to the property functions $prop_{Ai}(x)$ of particular sets.

A person of the height 185 cm will, with probability 1, (certainty) be qualified in the set *tall*, cf. Fig. 7.

Remark 2. In the case of probabilistic qualifiers, qualification probability functions $qprob_{Ai}(x)$ inform us about the probability that an element x of the universal set X will be qualified in the set A_i. The sum of qualification probabilities in all sets A_i equals 1. A given element x can be qualified only in one set A_i.

Technical measuring instruments are mostly probabilistic qualifiers. Each of the instruments qualifies the measured quantity x into one of its possible indications x_{ind} with some probability. For example, if a measuring instrument of temperature indicates temperature with the accuracy of $0.1°$C, then its indication $x_{ind} = 19.7°$C means that the real temperature x of neighborhood is about $19.7°$C. The qualification function $qprob_{19.7}(x)$ of that indication is the probability distribution of qualification of the real temperature x in the indication set about $19.7°$C. Each possible indication of the measuring instrument, e.g., 0.0, 0.1, 0.2, …, 19.0, 19.1, 19.2, …, 99.8, 99.9, 100.0°C, is characterized by its own qualification function $qprob_{ind}(x)$, e.g., $qprob_{0.0}(x)$, $qprob_{0.1}(x)$, …, $qprob_{99.9}(x)$, $qprob_{100.0}(x)$, etc. People also make measurements (evaluations) of various quantities. In the case of height we observe the height x of a given person and then qualify it in one of our possible linguistic indications as *short*, *medium* or *tall*. More exactly, we qualify the observed height into one of the indication sets we use. People make the qualification of the observed height x subconsciously with the use of probabilistic qualification functions $qprob_{short}(x)$, $qprob_{medium}(x)$ and $qprob_{tall}(x)$ which exist in their brains.

If the qualification in a set is probabilistic, then it may happen that an element x which has the less specific property $prop_{Ai}(x)$ of the set A_i than the property $prop_{A(i+1)}$ or $prop_{A(i-1)}$ of other neighboring sets A_{i+1} or A_{i-1} ($prop_{Ai}(x) < prop_{A(i+1)}(x)$ or $prop_{Ai}(x) < prop_{A(i-1)}$) will be qualified in the set A_i and not in the set A_{i+1} or A_{i-1}. For example, a person of the height $x = 176$ cm can be qualified by a probabilistic qualifier (another person) not in the set *tall* whose property his or her height has to the degree 0.64 ($prop_{tall}(x) = 0.64$) but in the set *medium* whose property the height has to the degree 0.36 ($prop_{medium}(x) = 0.36$), cf. Fig. 7.

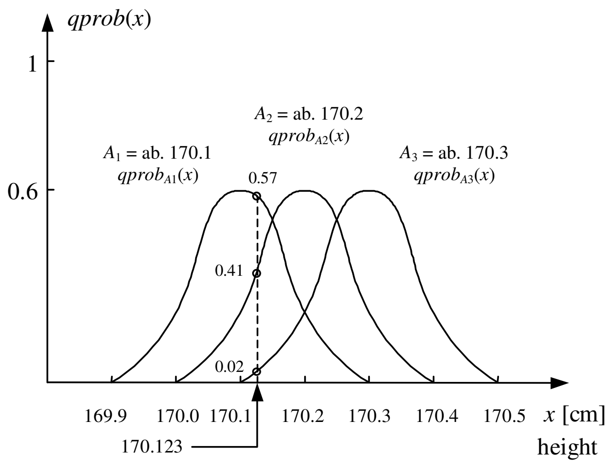
An interesting issue is **the difference between the meanings of the qualification function $qprob_{Ai}(x)$, the property function $prop_{Ai}(x)$, and the dequalification function $deqprobd_{Ai}(x)$** in the case of a probabilistic qualifier. The qualification function $qprob_{Ai}(x)$ informs us about the level of the probability of qualifying an element x in the set A_i, e.g., the probability of qualifying the height 170.23 cm for the indication 170.1 cm of a technical instrument of height measurement. If "the measuring instrument" is a man, than the qualification function $qprob_{medium}(x)$ informs us about the probability of the height, e.g., 170.23 cm, to be qualified in the indication set *medium*. In Fig. 8 three exemplary qualification functions $qprob_{Ai}(x)$ of a discrete measuring instrument of height, which gives indications with the accuracy 0.1 cm, are presented.

It should be noticed that, since the qualification functions $qprob_{Ai}(x)$ inform us about the qualification probability of an element x in particular indication sets, the sum of qualification probabilities of the element into all indication sets must be equal to 1,

$$\sum_{i=1}^{n} qprob_{Ai}(x) = 1. \tag{8}$$

The maximal values of the qualification functions cannot be higher than 1 ($\max qprob_{Ai}(x) \le 1$). In the example shown in Fig. 8 the maximal values of the qualification functions are smaller than 1.

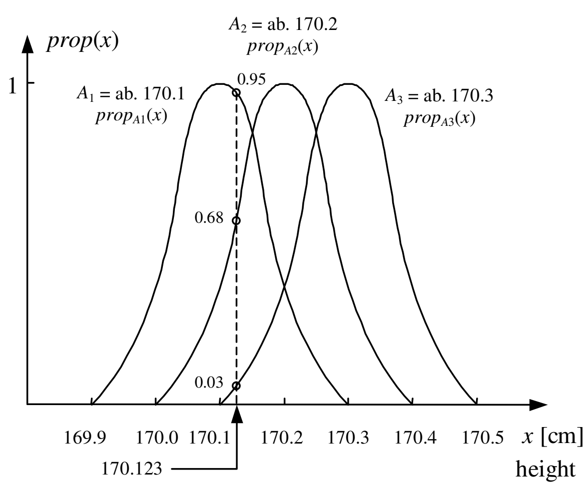
Property functions $prop_{Ai}(x)$ inform us to what degree the element x possesses the specific property of the set A_i. It is obvious that a typical element x of the set A_i must have a full amount of the set property, e.g., $prop_{170.1}(170.1) = 1$, in Fig. 9 (the height 170.1 cm has the property of being about 170.1 cm to the degree 1). In the case of probabilistic qualifiers, property functions $prop_{Ai}(x)$ are achieved by normalizing



$$qprob_{A1}(170.123) + qprob_{A2}(170.123) + qprob_{A3}(170.123) = 1$$

$$\forall i,\ i = 1, \ldots, n: \quad \int_{-\infty}^{\infty} qprob_{Ai}(x)\,\mathrm{d}x \geq 1$$

Fig. 8. Exemplary functions $qprob_{Ai}(x)$ qualifying the measured height x [cm] into three of many possible indication sets $A_1 =$ *about* 170.1 cm, $A_2 =$ *about* 170.2 cm, $A_3 =$ *about* 170.3 cm of a technical measuring instrument.



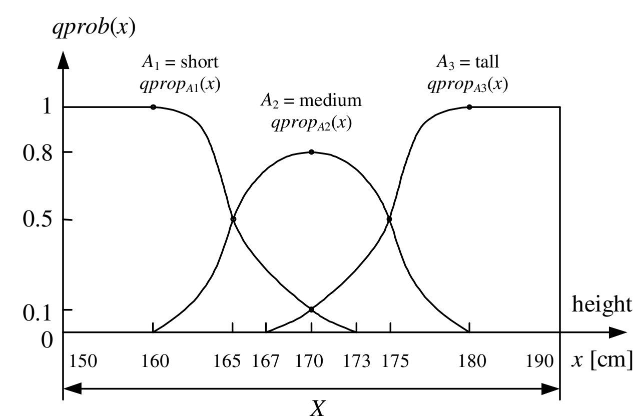
$$prop_{A1}(170.123) + prop_{A2}(170.123) + prop_{A3}(170.123) \geq 1$$

$$\forall i,\ i = 1, \ldots, n: \quad \int_{-\infty}^{\infty} prop_{Ai}(x)\,\mathrm{d}x \geq 1$$

Fig. 9. Property functions $prop_{Ai}(x)$ of three indication sets A_i of a technical measuring instrument of height, achieved by normalizing the qualification functions $qprob_{Ai}(x)$ of the instrument from Fig. 8.

qualification functions $qprob_{Ai}(x)$ to the interval [0,1]. In Fig. 9 exemplary property functions of three indication sets of a technical measuring instrument of height achieved by normalizing qualification functions of the instrument from Fig. 8 are depicted.

Noticeably, the summarized amount of properties $prop_{Ai}(x)$ a given element x has must not be equal to 1. It can be greater than 1, as takes place in the case shown in Fig. 9. The property function $prop_{Ai}(x)$ itself is not of a probabilistic but of a deterministic type, though it refers to a probabilistic qualifier. So in the example in Fig. 9 the property functions inform us that the element $x = 170.123$ cm has the property A_1 (*about* 170.1 cm) to the degree 0.95, the property of the set A_2 (*about* 170.2 cm) to the degree 0.68 and the property of the set A_3 (*about* 170.3 cm) to the degree 0.03. However, the mere possessing of the full amount of the specific property of set A_i (to the degree 1) by an element x does not necessarily forejudge that the element will be qualified by a probabilistic qualifier in the set A_i. It depends on the qualification algorithm, which is probabilistic in this case. Figure 10 depicts exemplary, experimentally identified qualification functions used by a person in visual height evaluation of adults, under the assumption that the person uses only three linguistic indications of height: $A_1 =$ *short*, $A_2 =$ *medium*, $A_3 =$ *tall*, and that the person qualifies the perceived height in only one set. The assumed height universe X is confined to the interval [150 cm, 190 cm].



$$\forall x: \ qprob_{A1}(x) + qprob_{A2}(x) + qprob_{A3}(x) = 1$$

Fig. 10. Exemplary qualification functions $qprob_{Ai}(x)$ of a person qualifying the observed heights to the linguistic indications *short*, *medium* and *tall* height.

The qualification function $qprob_{A2}(x)$ qualifying in the *medium* height has the maximal value equal to 0.8 and not to 1, because 10% of persons of the height 170 cm are qualified by the qualifier-person in the *short* height (under the influence of the corpulence, clothes, and height of the previously seen person), and 10% of evaluated persons of the height 170 cm the qualifier qualifies as *tall* persons for the same reasons. The property functions $prop_{A1}(x)$, $prop_{A2}(x)$, $prop_{A3}(x)$, being numerically equal to the qualification functions $qprob_{Ai}(x)$ (Fig. 10) normalized to the interval $[0, 1]$, are depicted in Fig. 11.

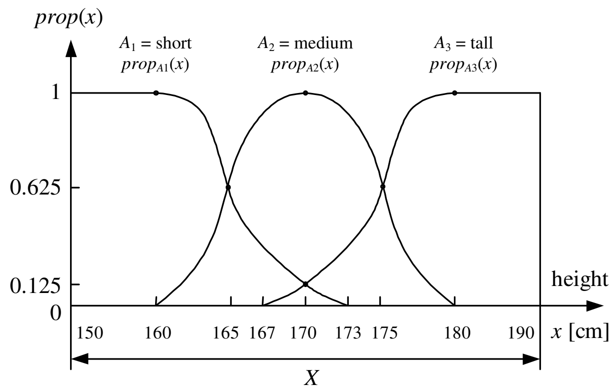


Fig. 11. Property functions $prop_{Ai}(x)$ defining the amount of specific properties of the sets A_1, A_2, A_3 (*short*, *medium*, *tall* height) possessed by an element x achieved by normalizing the qualification functions $qprob_{Ai}(x)$ from Fig. 10.

Obviously, the height 170 cm has a full amount (to the degree 1) of the property *medium* = *about* 170 = A_2 but also, according to the qualifier, it partly (to the degree 0.125) possesses the properties *short* $= A_1$ and *tall* $= A_3$. The author claims that the membership functions $\mu_{Ai}(x)$ used in fuzzy set theory correspond to the property functions $prop_{Ai}(x)$ of fuzzy sets.

In a short form, the set A being a result of probabilistic qualification can be presented as a set of ordered pairs referring to elements x qualified in the set:

$$A = \big\{\big(x, qprob_A(x)\big), QAlg_A(x) \mid \forall x : \big(m_A(x) = 1\big) \wedge (x \in X)\big\}.$$

6.3. Possibilistic Qualifier

A qualifier Q_A is possibilistic if it uses in its qualification algorithm $QAlg_A(x)$ a possibility distribution $\pi_A(x)$ of the qualification of an element x in a set A. The possibility distribution must be used to model the qualification process of a real qualifier, when it is not possible to determine the precise distribution $qprob_A(x)$ of the qualification probability of the qualifier, because we do not have precise information about the way of qualification but only inaccurate, nested information. This problem will be explained further on. The notion of a possibility measure $\Pi(A)$ and a necessity measure $N(A)$ of the event occurrence $(x \in A)$ is described in the literature (Dubois and Prade, 1988; Klir and Folger, 1988; Zimmermann, 1996). The notions of the possibility and the necessity measure are dual. They are characterized by the formulas (9), (Dubois and Prade, 1988). We have

$$\Pi(A) = 1 - N(\bar{A}) = \sup\{\pi(x) \mid x \in A\},$$
$$N(A) = 1 - \Pi(\bar{A}) = \inf\{1 - \pi(x) \mid x \notin A\},$$
$$\pi(x) = \Pi(\{x\}).$$

Additionally, possibility and necessity measures are connected by relations (9). We have

$$\Pi(A) \geq N(A),$$
$$N(A) > 0 \quad \Rightarrow \quad \Pi(A) = 1,$$
$$\Pi(A) < 1 \quad \Rightarrow \quad N(A) = 0.$$

As Dubois and Prade state in their monograph (Dubois and Prade, 1988), the occurrence possibility of an event A means the maximal probability $P^*(A)$ of this event, whereas the occurrence necessity $N(A)$ of the event A means the minimal, but sure probability $P_*(A)$ of the event occurrence, see also (Piegat, 2005a).

These two notions are used when we have only uncertain, nested evidence information about a given problem. It will be illustrated by Example 3.

Example 3. Let us assume that **we do not have** precise information about the way of qualification of height x [cm] in the set $A_2 =$ *medium* such as the exemplary information given below:

- 80% of persons of the height 167 cm are qualified by the qualifier in the set $A_2 =$ *medium*,
- 86% of persons of the height 168 cm are qualified in the set $A_2 =$ *medium*,
- etceteras.

Instead, **we have** the inaccurate evidence information E_i as below.

The qualifier made qualifications of persons from three groups in the set A_2 of the *medium* height.

- Evidence information E_1: five persons of the height confined to the interval [167 cm, 173 cm] from the first group were qualified to be of the *medium* height.
- Evidence information E_2: ten persons of the height confined to the interval [164 cm, 176 cm] from the second group were qualified to be of the *medium* height.
- Evidence information E_3: fifteen persons of the height confined to the interval [160 cm, 180 cm] from the third group were qualified to be of the *medium* height.

The evidence information about the way of qualification can be presented visually as in Fig. 12.

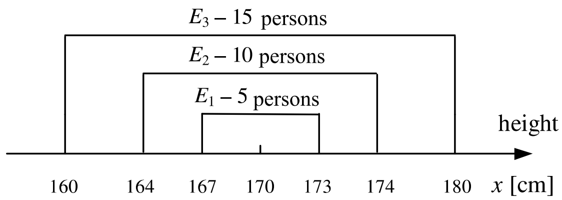


Fig. 12. Visual presentation of the inaccurate, nested information E_i about the way of qualification of height x [cm] in the set $A_2 = \textit{medium}$ height realized by the qualifier Q_{A2}.

Because the information about the way of qualification is not precise, it is not possible to determine the precise distribution of the qualification probability $\textit{qprob}_{A2}(x)$ in the set $A_2 = \textit{medium}$ height. However, using the formula (9) from (Dubois and Prade, 1988), the possibility distribution $\pi_{A2}(x)$ of height qualification in the set *medium* can be determined,

$$\forall x,\ \pi_{A2}(x) = P^*_{A2}(\{x\}) = \begin{cases} \sum_{j=i}^{p} m(E_j) & \text{if } x \in E_i,\ x \notin E_{i-1}, \\ 0 & \text{if } x \in X - E_p, \end{cases} \tag{9}$$

where $m(E_i)$ denotes the *probability mass* corresponding to the evidence information E_i (Dubois and Prade, 1988), $m(E_1) = 1/6,\ m(E_2) = 2/6,\ m(E_3) = 3/6$.

The possibility distribution $\pi_{A2}(x)$ determined with (9) and the dual necessity distribution $\eta_{A2}(x)$ of the qualifying height x in the set $A_2 = \textit{medium}$ is shown in Fig. 13.

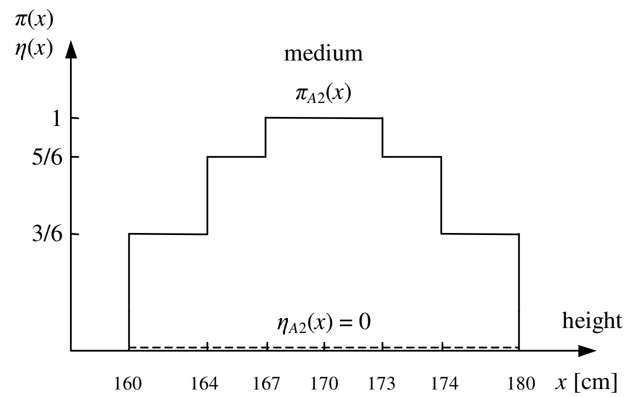


Fig. 13. Possibility distribution $\pi_{A2}(x)$ and necessity distribution $\eta_{A2}(x)$ of qualifying the height x in the set $A_2 = \textit{medium}$ determined on the basis of the inaccurate evidence information E_1, E_2 and E_3 (Fig. 12) about the way of qualification used by the qualifier Q_{A2}.

Information uncertainty results in the impossibility of determining the precise probability distribution $\textit{qprob}_{A2}(x)$ of qualification. We can only determine the upper probability constraint $\pi_{A2}(x)$ (a possibility distribution of qualification), and the lower probability constraint $\eta_{A2}(x)$ (a necessity distribution of qualification). The possibility distribution and the necessity distribution are only two of many possible probability distributions of qualification, which may result from the evidence information E_i (which could be used by the qualifier). In the case when the variable x is a continuous one, the number of possible distributions of qualification probability is infinite! Therefore, the probability that the qualifier Q_{A2} used in the qualification process has a qualification probability distribution $\textit{qprob}_{A2}(x)$ just identical to the possibility distribution $\pi_{A2}(x)$ or to the necessity distribution $\eta_{A2}(x)$ is very small (in the case of discrete variables) or infinitesimal (in the case of continuous variables). In this situation, instead of using a very little probable possibility or necessity distribution, it is reasonable to determine the "probable, average" probability distribution of qualification $\textit{qprob}_{A2aver}(x)$ in the set $A_2 = \textit{medium}$. If we have at our disposal the previously determined possibility distribution $\pi_A(x)$ of qualifying the element x in a set A, then the "average" probability distribution $\textit{qprob}_{Aaver}(x)$ of qualification can be determined using the formula (10) from (Dubois and Prade, 1988),

$$\textit{qprob}_{Aaver}(x) = \sum_{i=j}^{n} \frac{1}{j} \{\pi_A(x_j) - \pi_A(x_{j+1})\}, \tag{10}$$

where x_i is the i-th discrete value of the variable x. The numeration of the discrete values x_i satisfies

$$\pi_A(x_i) = 1 \geq \pi_A(x_2) \geq \cdots \geq \pi_A(x_{n+1}). \tag{11}$$

Here x_{n+1} is a dummy value of the variable x, whose universe was divided into n elements. Using (10), the average probability distribution $\textit{qprob}_{A2}(x)$ of qualification in set $A_2 = \textit{medium}$ was determined, cf. Fig. 14.

The possibilistic qualifier can be a computer, which qualifies the elements x of the universe X in a set A with the use of a possibilistic distribution $\pi_A(x)$ instead of the unknown distribution of the qualification probability $\textit{qprob}_A(x)$. In this case possibility distribution (only very approximately) models the way of qualification of a real probabilistic qualifier, e.g., of a man. For a given x-value, the possibilistic qualifier determines, similarly to the probabilistic one, the possibility grade $\pi_A(x)$, which means the maximal possible probability of qualifying the element x in the set A (Piegat, 2005). Next, with a probability $p_A(x)$ determined at random, such that

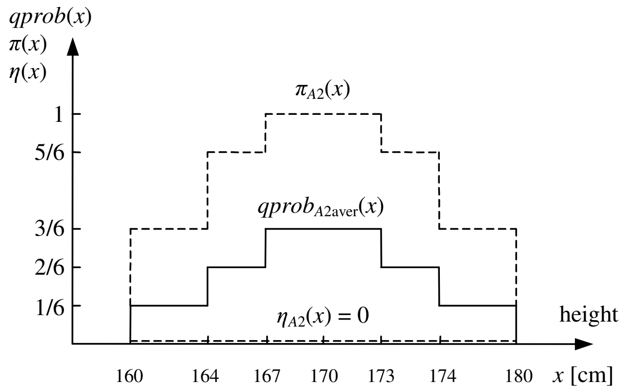


Fig. 14. Average probability distribution $qprob_{A2aver}(x)$ of height qualification in the set A_2 = *medium* determined on the basis of the possibility distribution $\pi_{A2}(x)$ of the way of qualification, which was obtained from the inaccurate evidence information E_i about qualification realized by the qualifier Q_{A2}.

$\eta_A(x) \leq p_A(x) \leq \pi_A(x)$, it generates 1 or 0. Generating 1 means the qualification of the element x in the set A. Otherwise, the element is not qualified in the set. The possibilistic qualifier is a very inaccurate model of the probabilistic one, whose way of qualification was not precisely identified because of the lack of precise information about qualification results (only the inaccurate, nested information E_i about the qualified elements x is at our disposal). Therefore the author does not recommend using possibilistic qualifiers. When we have only inaccurate, nested information, first the possibility distribution $\pi_A(x)$ and next the average distribution $qprob_{Aaver}(x)$ of qualification probability should be determined according to the formula (10). Thus the possibilistic model of the qualifier is transformed into a probabilistic one, which can further be used according to the remarks contained in Section 6.2.

In the short form, a set A being a result of possibilistic qualification (that was not transformed into the probabilistic one) can be presented as a set of ordered triplets referring to the elements x, which were qualified in the set:

$$A = \big\{\big(x, \pi_A(x), \eta_A(x)\big),\ QAlg_A(x) \mid \forall x : \big(m_A(x) = 1\big) \wedge (x \in X)\big\}.$$

♦

7. Impact of the New Definition of the Fuzzy Set on Fuzzy Arithmetic

7.1. Deterministic Qualifier Case

A deterministic qualifier uses a deterministic qualification algorithm with deterministic property functions. Further on, from among many operations of fuzzy arithmetic, addition of two fuzzy numbers will be considered as an exemplary operation. Example 4 will show how this operation is realized with the methods of the present fuzzy arithmetic based on the classical definition of a fuzzy set. Example 5 will show the influence of our new definition of the fuzzy set on the results of the addition.

Example 4. (*Classical approach to the addition of fuzzy numbers*) Assume that we have information about the incomes of two firms A and B as below:

I_1: Income of the firm A is *medium* (*about* 4 million euro).

I_2: Income of the firm B is *medium* (*about* 4 million euro).

The membership functions of the *low*, *medium* and *high* income are depicted in Fig. 15.

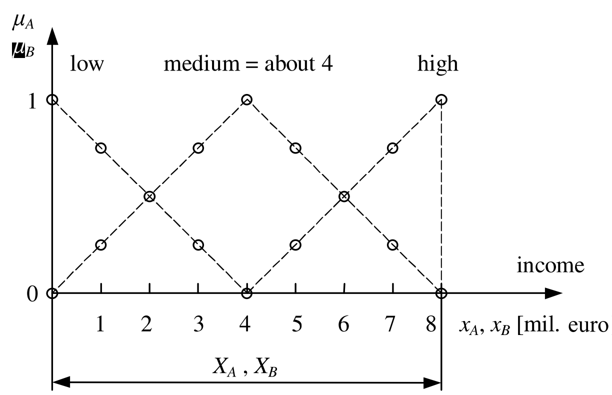


Fig. 15. Membership functions of *low*, *medium* and *high* income of the firms A and B.

Query

What is the sum of both the incomes (*medium* + *medium*)?

Solution

In the present fuzzy arithmetic, addition can be implemented with the use of Zadeh's extension principle expressed by

$$\forall (x_A, x_B) \mid x_A + x_B = y \quad \mu_{A+B}(y) = \max\big\{\min\big[\mu_A(x_A), \mu_B(x_B)\big]\big\}. \tag{12}$$

The result of the addition is presented in Fig. 16.

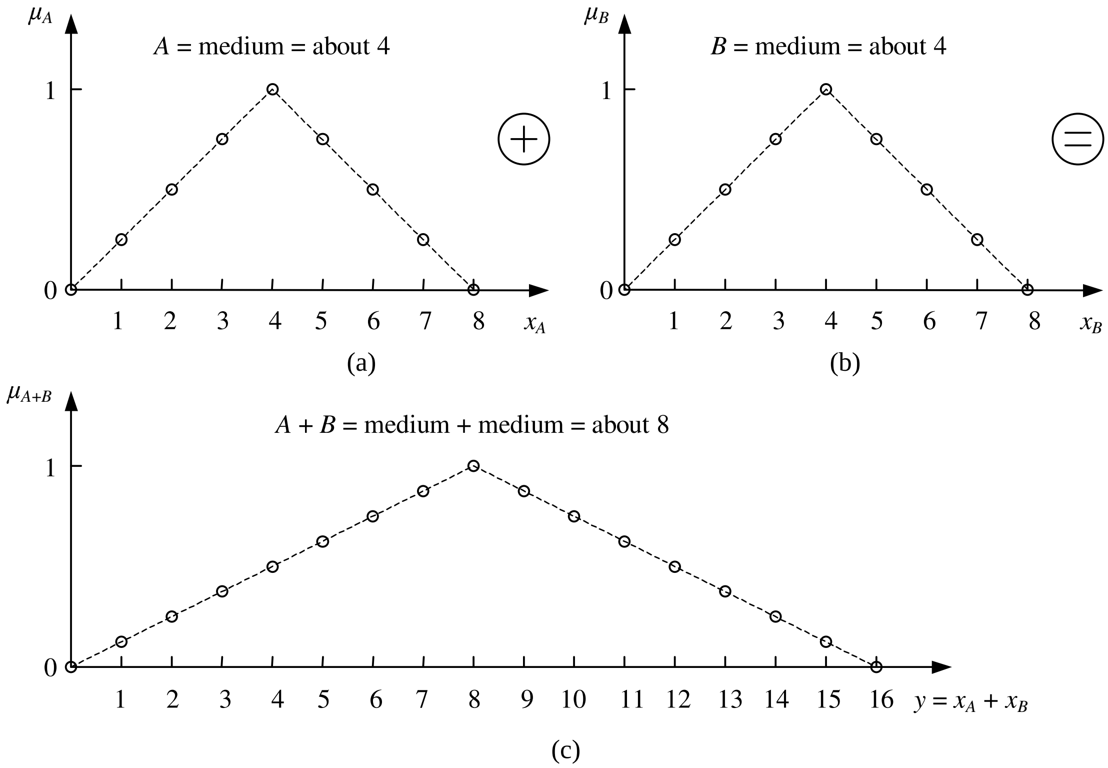


Fig. 16. Result (c) of the addition of two fuzzy numbers (a) and (b), (*about* $4 +$ *about* 4).

As can be seen in Fig. 16, the addition result of two fuzzy numbers A and B has a support (16) which is equal to the sum of the supports (8+8) of both fuzzy numbers. Thus the fuzziness of the sum is very large and therefore its practical usefulness is small. For this reason fuzzy arithmetic is rather seldom used in practice. In Example 5, the addition of two fuzzy numbers will be shown with the use of the new definition of a fuzzy set. ♦

Example 5. (*New approach to the addition of fuzzy numbers*) A deterministic qualifier evaluated the incomes of the firms A and B as below:

I_1: Income of the firm A is *medium* (*about* 4 million euro).

I_2: Income of the firm B is *medium* (*about* 4 million euro).

Let us notice that each of the incomes can take only one crisp value from all values being in the set *medium* $=$ *about* 4. The qualifier used an algorithm which qualifies the income x in the set whose property the given x-value has at most. In Fig. 17, the property functions of the sets *low*, *medium* and *high* income are shown.

Query

What is the sum of the incomes of the firms A and B?

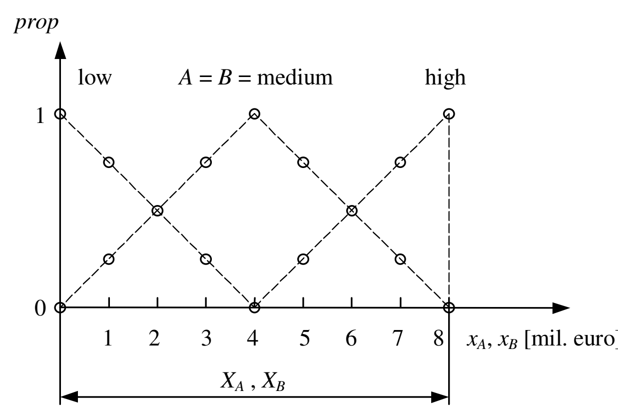


Fig. 17. Property functions $prop_{low}(x)$, $prop_{medium}(x)$ and $prop_{high}(x)$ of the fuzzy sets *low*, *medium* and *high* income of the firms A and B.

Solution

Although the qualification algorithm is deterministic, the same problem of fuzzy number addition is not deterministic but probabilistic. It follows from the qualification algorithm and from the property functions in Fig. 17 that only the values {2,3,4,5,6} could be qualified as a *medium* income. Both the income of the firm A and that of the firm B can be equal to one of these values with the same probability. The probability distributions $deqprob_A(x_A)$ and $deqprob_B(x_B)$ of both firms are shown in Fig. 18.

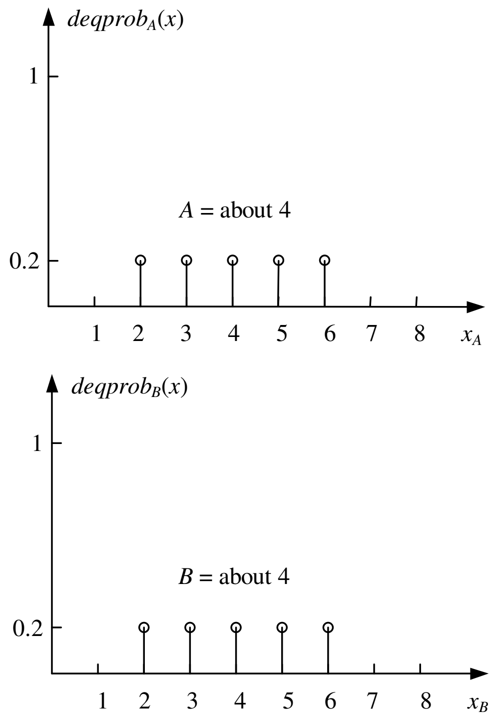


Fig. 18. Distributions of the dequalification probability $deqprob_A(x_A)$ and $deqprob_B(x_B)$ of the income of the firms A and B.

With the use of dequalification probability distributions of single incomes, the distribution $deqprob_{A+B}(y)$ of the income sum can be calculated as follows:

$$deqprob_{A+B}(y) = \mathrm{card}\big[R(x_A + x_B = y)\big] = \sum_{(x_A,x_B)\,|\,x_A+x_B=y} deqprob_A(x_A)\cdot deqprob_B(x_B). \quad (13)$$

Figure 19 illustrates the calculation process.

As can be seen in Fig. 19, only one event $x_A+x_B = 4$ is possible. It occurs when the income of the firm A, $x_A = 2$, and the income of the firm B, $x_B = 2$. The probability of such an event is equal to $1/25$. However, 5 events $x_A + x_B = 8$ are possible, e.g., when ($x_A = 2$ and $x_B = 6$), ($x_A = 3$ and $x_B = 5$), etc. Thus, the probability that the income sum will be equal to 8 equals $5/25$. The dequalification probability distribution of the sum income of both the firms A and B is shown in Fig. 20.

After the normalization of the resulting dequalification probability distribution $deqprob_{A+B}(x)$ to the interval $[0, 1]$, the property function $prop_{A+B}(y)$ of the income sum was obtained, cf. Fig. 21(b).

As can be seen in Fig. 21, the addition result of two fuzzy numbers A and B achieved with the use of the new definition of a fuzzy set is considerably less fuzzy than the result achieved with use of the classic definition of a fuzzy set. Therefore the new definition has greater practical usefulness than the classic one. Less fuzzified results are also achieved in other operations of fuzzy arithmetic.

It should also be mentioned that the property function as a representation of a fuzzy number has small informative meaning. For example, in the case of the fuzzy number *about* 8 in Fig. 21, its property function $prop_{A+B}(y)$ informs us only how much of the property of the set *about* 8 a given y-value has. However, we do not know what practical meaning the information that, e.g., $y = 5$ possesses the specific property of the set *about* 8 to the degree 0.4 has. Considerably greater practical meaning is assigned to the dequalification probability distribution $deqprob_{A+B}(y)$ form in Fig. 20c. The information that "the sum $y = x_A + x_B$ of the firm incomes can be equal to 5 million euro with probability 2/5" is understandable to everyone and is of the great meaning for the user.

It seems that the application of property functions is useful only in the phase of the qualification of the elements x in a fuzzy set. In the phase of the interpretation of calculation results are of practical meaning only dequalification probability distributions. ♦

7.2. Probabilistic Qualifier Case

In the case of a probabilistic qualifier, the most important function characterizing a fuzzy set A is the qualification probability distribution $qprob_A(x)$. By the normalization of the function abscissas to the interval $[0, 1]$, the property function $prop_A(x)$ of a fuzzy set is achieved. By the normalization of the area of the qualification probability distribution $qprob_A(x)$ to the value 1, the distribution of the dequalification probability density $deqprob_A(x)$ is achieved (for continuous variables). Thus, there exists a strict relation between the qualification probability distribution $qprob_A(x)$ and both functions $prop_A(x)$ and $deqprob_A(x)$, which fully depend on it.

It seems that the most advantageous implementation of fuzzy arithmetic operations is calculation with dequalification probability distributions (for discrete variables) or with distributions of the dequalification probability density $deqprobd_A(x)$ in the case of continuous variables. An example of such calculations was shown in Section 7.1.

7.3. Possibilistic Qualifier Case

In the case of arithmetic operations with fuzzy sets characterized by the possibility distributions $\pi_{Ai}(x)$ used by a possibilistic qualifier, the operations can be realized with Zadeh's extension principle. However, the results

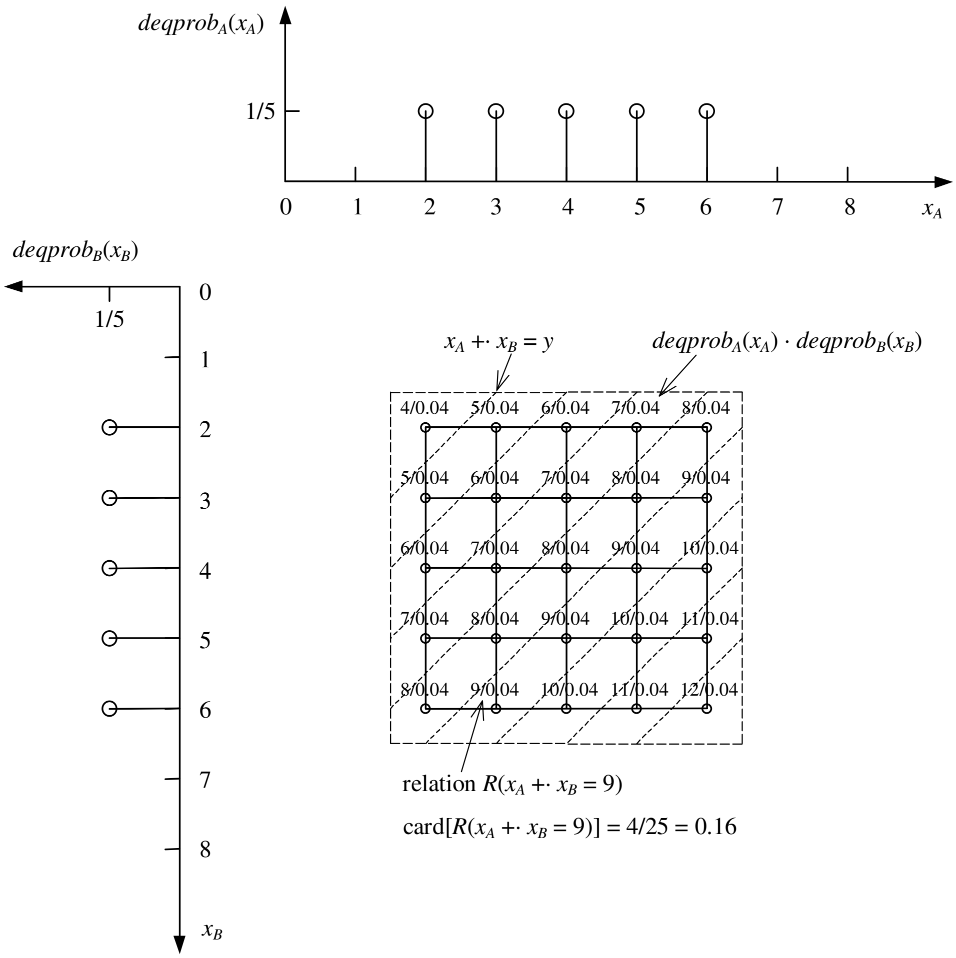


Fig. 19. Illustration of the calculation of the dequalification probability distribution *deqprob*$_{A+B}(y)$ of the sum [*deqprob*$_A(x_A)$ + *deqprob*$_B(x_B)$] in the addition of the incomes of the firms A and B.

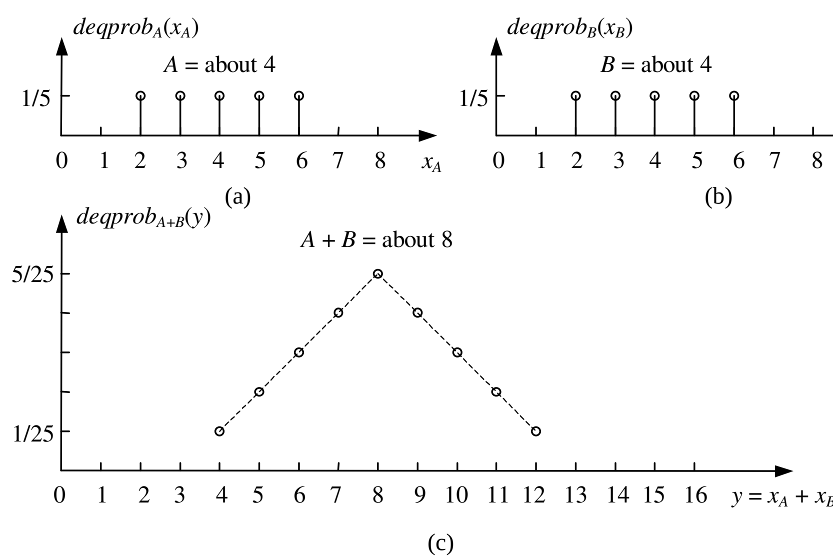


Fig. 20. Result (c) of the addition of the dequalification probability distributions (a) and (b) of the incomes of two firms A and B.

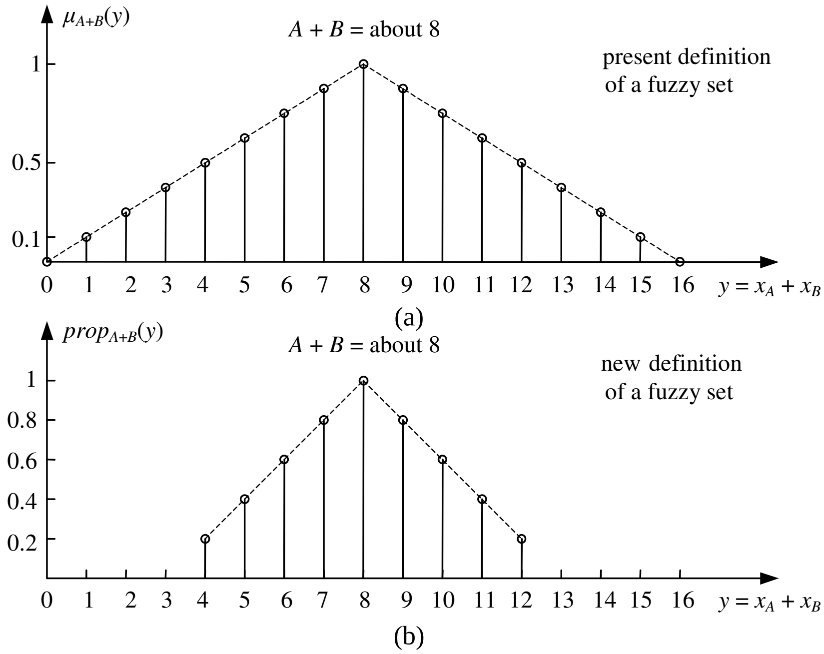


Fig. 21. Membership function $\mu_{A+B}(y)$ representing the addition result of two fuzzy numbers A and B with the use of the classical definition of the fuzzy set (a), and the property function $prop_{A+B}(y)$ representing the addition result achieved with the use of our new definition of a fuzzy set (b).

of such operations will also be possibility distributions, which are of small practical meaning (see explanations in Section 6.3). Therefore, the author recommends the transformation of the possibility distributions $\pi_{Ai}(x)$ into the corresponding average probability distributions of qualification $qprob_{Aiaver}(x)$ and then the a realization of arithmetic operations, similarly to the case of the probabilistic qualifier.

8. Conclusions

In the paper a new definition of a fuzzy (and crisp) set was presented. Compared with the present definition, the definition introduces new notions such as the qualifier, qualification algorithm, and property function of a set. The new definition is more useful than the present definition in solving practical problems and allows achieving less fuzzified results of arithmetic operations than the present definition of a fuzzy set.

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