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DELAY-DEPENDENT ASYMPTOTIC STABILITZATION FOR UNCERTAIN TIME-DELAY SYSTEMS WITH SATURATING ACTUATORS

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This paper concerns the issue of robust asymptotic stabilization for uncertain time-delay systems with saturating actuators. Delay-dependent criteria for robust stabilization via linear memoryless state feedback have been obtained. The resulting upper bound on the delay time is given in terms of the solution to a Riccati equation subject to model transformation. Finally, examples are presented to show the effectiveness of our result.

Keywords: stability, delay-dependency, time delay system, Riccati equation

1. Introduction

For engineering systems, uncertainty and time delays are two important issues that designers must confront today (Kolmanovskii and Nosov, 1986; Su and Huang, 1992). Uncertainty is often encountered in various dynamical systems due to modeling misfits, measurement errors, and linearization and approximations (Liu and Su, 1998; Su *et al.*, 1991). All actuation and measurement devices are subject to time delays. Specifically, time delays arise in control actuation devices (e.g., a transport lag), as well as computation delays in sensor measurement processing. On the other hand, time delays often occur in systems such as transformation and communication ones, chemical and metallurgical processes, environmental models and power networks (Tsay and Liu, 1996). Time delays have always been among the most difficult problems encountered in process control. In practical applications of feedback control, time delay arise frequently and can severely degrade closed-loop system performance and, in some cases, drive the system to instability (Cao *et al.*, 1998; Liu *et al.*, 2001; Su and Chu, 1999; Su and Liu, 1996).

Stabilization analysis and synthesis of uncertain time delay systems with saturating actuators is an important issue addressed by many authors and for which surveys can be found in several monographs (Han and Mehdi, 1998; Su *et al.*, 1991; 2001; 2002). Recently, one of the important issues is to maximize the allowable delay size for robust stabilization of uncertain time delay systems (Liu *et al.*, 2001; Su *et al.*, 2001; 2002). Upper bounds on time delays which guarantee asymptotic stability of saturating actuator systems via a state feedback control law are given (Su *et al.*, 1991). A Riccati equation-based global and local static, output feedback control design framework for time-delay systems with saturating actuators was developed (Tsay and Liu, 1996). Based on a matrix measure, a matrix norm, and a decomposition technique, stability criteria are derived by Goubet *et al.* (1997).

In this paper, we analyze the stabilization and domain of attraction for linear time delay systems with actuator saturation. A less conservative estimate of the domain of attraction will be derived based on a Lyapunov-Razumikhin and Riccati equation (Su and Liu, 1996; Su *et al.*, 1991). We emphasize that our Riccati equation design approaches with the relevant decomposition technique are constructive in nature, rather than existential. The effectiveness of the approach is illustrated by numerical examples. However, the results of this paper indeed give us one more choice for the stabilization examination of time delay systems with actuator saturation. In this paper the following notation is adopted:

$\mathbb{R}$	the real number field,
$\mathbb{R}^n$	the n-dimensional real vector space,
x	a vector, $x = [x_1\ \ x_2\ \ \dots\ \ x_n]^T,\ \ x_i \in \mathbb{R}$,
A^T	the transpose of a matrix A,
$\lambda_i(A)$	the i-th eigenvalue of a matrix A,
$\lambda_{\max}(A)$	the maximum eigenvalue of a matrix A,
$\lambda_{\min}(A)$	the minimum eigenvalue of a matrix A,
$\|A\|$	the norm of a matrix A, defined as $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$.

2. Main Result

Consider a perturbed time-delay system described by the following differential-difference equation:

$$\dot{x}(t) = (A + \Delta A)x(t) + (A_1 + \Delta A_1)x(t-\tau) + (B + \Delta B)u(t), \tag{1a}$$

$$y(t) = Cx(t), \tag{1b}$$

$$x(t) = \phi(t), \quad t \in [-\tau, 0], \tag{1c}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, τ is the time delay of the system in the state. A, A_1, and B are real constant matrices of appropriate dimensions. Furthermore, $\phi(t)$ is a smooth continuous vector-valued initial function. Besides, ΔA, ΔA_1 and ΔB are linear parameter uncertainties in the system model with appropriate dimensions. A different idea exploited in the literature (Goubet *et al.*, 1997) is to find some decomposition of the "delayed" term A_1 of the form $A_1 = A_{11} + A_{12}$ in order to improve the delay bounds. In this paper, the admissible uncertainties are assumed to be of the form

$$\|\Delta A\| \le \alpha, \tag{2a}$$

$$\|\Delta A_1\| \le \alpha_1, \tag{2b}$$

$$\|\Delta B\| \le \beta. \tag{2c}$$

We let

$$\sigma = \frac{\lambda_{\min}(D)}{2\lambda_{\max}(P)}, \tag{3a}$$

$$\delta = \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}, \tag{3b}$$

where $D = P^T B (R^{-1})^T B^T P + Q$, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalue of the matrix A, respectively. P and R are symmetric positive-definite matrices and Q is a symmetric positive-semidefinite matrix, involved in the following Riccati equation:

$$(A+A_{11})^T P + P(A+A_{11}) - PBR^{-1}B^T P + Q = 0 \tag{4}$$

for some square matrix A_{11} of appropriate dimensions.

It is assumed that $(A + A_{11}, B)$ is stabilizable. Our problem is to design the state feedback controller

$$u(t) = -Kx(t), \tag{5}$$

where $K = R^{-1}B^T P$, such that the closed-loop system

$$\dot{x}(t) = (A_k + \Delta A_k)x(t) + (A_1 + \Delta A_1)x(t-\tau) \tag{6}$$

results, where $A_k = A - BR^{-1}B^T P$ and $\Delta A_k = \Delta A - \Delta B R^{-1} B^T P$.

Without loss of generality, we consider the case where the initial time is zero and let $x(t)$, where $t \ge 0$, be the solution of (6) through $(0, \phi)$. Since $x(t)$ is continuously differentiable for $t \ge 0$, one can write

$$\begin{aligned} x(t-\tau) &= x(t) - \int_{t-\tau}^{t} \dot{x}(\theta)\,\mathrm{d}\theta \\ &= x(t) - \int_{t-\tau}^{t} \big[(A_k + \Delta A_k)x(\theta) \\ &\quad + (A_1 + \Delta A_1)x(\theta - \tau)\big]\,\mathrm{d}\theta. \end{aligned} \tag{7}$$

Applying the decomposition "delayed term" A_1 as $A_1 = A_{11} + A_{12}$ and substituting (7) into (6), we have

$$\begin{aligned} \dot{x}(t) &= (A_k + A_{11})x(t) + A_{12}x(t-\tau) + \Delta A_k x(t) \\ &\quad + \Delta A_1 x(t-\tau) - A_{11}\big[x(t) - x(t-\tau)\big] \\ &= (A_k + A_{11})x(t) + A_{12}x(t-\tau) + \Delta A_k x(t) \\ &\quad + \Delta A_1 x(t-\tau) \\ &\quad - A_{11}\int_{t-\tau}^{t} \big[(A_k + \Delta A_k)x(\theta) \\ &\quad + (A_1 + \Delta A_1)x(\theta-\tau)\big]\,\mathrm{d}\theta. \end{aligned} \tag{8}$$

Our result is summarized in the following theorem:

Theorem 1. *Consider the uncertain time delay system (1). Suppose that $(A + A_{11}, B)$ is stabilizable and there exists a positive number $q > 1$ such that the system (8) is asymptotically stable if*

$$\tau \le \frac{\sigma - \big[\alpha + \beta\|K\| + q\delta(\|A_{12}\| + \alpha_1)\big]}{q\delta\big[\,\|A_{11}A_k\| + \|A_{11}A_1\| + \|A_{11}\|\,(\alpha + \beta\|K\| + \alpha_1)\big]} \tag{9}$$

is satisfied for $\tau > 0$. Then the uncertain time delay system (1) is asymptotically stable, that is, the uncertain parts of the nominal system can be tolerated.

Proof. We consider the system (8) and take the following positive definite function as our Lyapunov function:

$$V\big[x(t)\big] = x^T(t)Px(t). \tag{10}$$

Thus

$$\dot{V}\big[x(t)\big] = \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t). \tag{11}$$

Substituting (8) into (11), we obtain

$$\begin{aligned}
\dot{V}\big[x(t)\big] &= \Big\{(A_k + A_{11})x(t) + A_{12}x(t-\tau) \\
&\quad + \Delta A_k x(t) + \Delta A_1 x(t-\tau) \\
&\quad - A_{11}\int_{t-\tau}^{t} \big[(A_k + \Delta A_k)x(\theta) \\
&\quad + (A_1 + \Delta A_1)x(\theta-\tau)\big]\,\mathrm{d}\theta\Big\}^T P x(t) \\
&\quad + x^T(t)P\Big\{(A_k + A_{11})x(t) \\
&\quad + A_{12}x(t-\tau) + \Delta A_k x(t) + \Delta A_1 x(t-\tau) \\
&\quad - A_{11}\int_{t-\tau}^{t} \big[(A_k + \Delta A_k)x(\theta) \\
&\quad + (A_1 + \Delta A_1)x(\theta-\tau)\big]\,\mathrm{d}\theta\Big\} \\
&\le x^T(t)\big[(A_k + A_{11})^T P + P(A_k + A_{11})\big]x(t) \\
&\quad + x^T(t)(\Delta A_k^T P + P\Delta A_k)x(t) \\
&\quad + x^T(t-\tau)A_{12}^T P x(t) + x^T(t)PA_{12}x(t-\tau) \\
&\quad + x^T(t-\tau)\Delta A_1^T P x(t) \\
&\quad + x^T(t)P\Delta A_1 x(t-\tau) \\
&\quad - 2x^T(t)PA_{11}\int_{t-\tau}^{t} \big[(A_k + \Delta A_k)x(\theta) \\
&\quad + (A_1 + \Delta A_1)x(\theta-\tau)\big]\,\mathrm{d}\theta \\
&\le x^T(t)\big[(A_k + A_{11})^T P + P(A_k + A_{11})\big]x(t) \\
&\quad + \big\|x^T(t)(\Delta A_k^T P + P\Delta A_k)x(t)\big\| \\
&\quad + \big\|x^T(t-\tau)A_{12}^T P x(t)\big\| \\
&\quad + \big\|x^T(t)PA_{12}x(t-\tau)\big\| \\
&\quad + \big\|x^T(t-\tau)\Delta A_1^T P x(t)\big\| \\
&\quad + \big\|x^T(t)P\Delta A_1 x(t-\tau)\big\| \\
&\quad + 2\Big\|x^T(t)PA_{11}\int_{t-\tau}^{t} \big[(A_k + \Delta A_k)x(\theta) \\
&\quad + (A_1 + \Delta A_1)x(\theta-\tau)\big]\,\mathrm{d}\theta\Big\|. \qquad (12)
\end{aligned}$$

Following the Razumikhin-type theorem (Kolmanovskii and Nosov, 1986), assume that there exists a constant $q > 1$ such that

$$V\big(x(t-\tau)\big) < q^2 V\big(x(t)\big). \qquad (13)$$

Then we have

$$\|x(t-\tau)\| < q\delta\,\|x(t)\|\,, \qquad (14)$$

where δ is defined in (3b).

Substituting (13) and (14) into (12), we have the following inequality:

$$\dot{V}\big[x(t)\big] \le -\omega\,\|x(t)\|^2\,, \quad \omega \in \mathbb{R}, \qquad (15)$$

where

$$\begin{aligned}
\omega &= \lambda_{\min}(D) - 2\Big\{\Big[\alpha + \beta\,\|K\| + q\delta\big(\,\|A_{12}\| + \alpha_1\big)\Big] \\
&\quad + q\delta\tau\Big[\,\|A_{11}A_k\| + \|A_{11}A_1\| \\
&\quad + \|A_{11}\|\,\big(\alpha + \beta\,\|K\| + \alpha_1\big)\Big]\Big\}\lambda_{\max}(P).
\end{aligned}$$

Consequently, we have $\dot{V}[x(t)] \le -\omega\,\|x(t)\|^2$ for sufficiently small $\omega > 0$. But $\omega > 0$ if and only if (15) is satisfied. Hence the system (8) is asymptotically stable, and therefore (1) yields asymptotical stabilization. ■

3. Extension to the Stabilitzation of Time–Delay Systems with Saturating Actuators

We consider the linear uncertain time-delay saturating-actuator systems described by the differential difference equation of the form

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + \Delta Ax(t) + A_1x(t-\tau) \\
&\quad + \Delta A_1x(t-\tau) + Bu_s(t) + \Delta Bu_s(t), \qquad (16a)\\
y(t) &= Cx(t) + \Delta Cx(t), \qquad (16b)\\
u_s(t) &= \mathrm{Sat}\,\big[u(t)\big]. \qquad (16c)
\end{aligned}$$

The saturation function is defined as follows (Fig. 1):

$$\mathrm{Sat}\,\big[u(t)\big] = \big[\,\mathrm{Sat}(u_1(t))\,\mathrm{Sat}(u_2(t))\ldots\mathrm{Sat}(u_m(t))\big]^{\mathrm{T}}, \qquad (17)$$

and

$$\mathrm{Sat}\,\big(u_i(t)\big) = \begin{cases} u_{iL} & \text{if} \quad u_i < u_{iL} < 0, \\ u_i & \text{if} \quad u_{iL} \le u_i \le u_{iH}, \\ u_{iH} & \text{if} \quad 0 < u_{iH} < u_i. \end{cases} \qquad (18)$$

For any saturating actuator $\mathrm{Sat}(u_i(t))$, which saturates at u_{iH} or u_{iL}, the following inequality is satisfied (Su *et al.*, 1991):

$$\left\|\mathrm{Sat}(u(t)) - \frac{u(t)}{2}\right\| \le \frac{\|u(t)\|}{2}. \qquad (19)$$

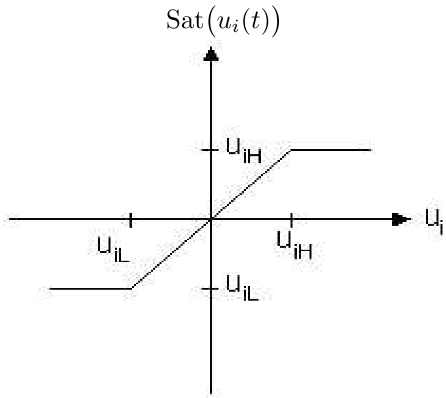

Fig. 1. Saturation function.

In this control system, (A, B) is controllable, i.e., the process state $x(t)$ can be determined on the basis of the control input $u(s)$ for $s \leq t$.

Substituting (5) into the system of (16), we obtain the following closed-loop equations:

$$\dot{x}(t) = A_s x(t) + A_1 x(t-\tau) + \Delta A_s x(t) + \Delta A_1 x(t-\tau) + (B + \Delta B)\Big(u_s(t) - \frac{u(t)}{2}\Big), \quad (20a)$$

$$y(t) = Cx(t), \quad (20b)$$

where $A_s = A - BR^{-1}B^T P/2$ and $\Delta A_s = \Delta A - \Delta BR^{-1}B^T P/2$.

From (8), we have

$$\begin{aligned}\dot{x}(t) = {} & (A_s + A_{11})x(t) + A_{12}x(t-\tau) + \Delta A_s x(t) \\ & + \Delta A_1 x(t-\tau) + (B+\Delta B)\Big(u_s(t) - \frac{u(t)}{2}\Big) \\ & - A_{11}\int_{t-\tau}^{t} \Big[(A_s + \Delta A_s)x(\theta) \\ & + (A_1 + \Delta A_1)x(\theta - \tau) \\ & + (B+\Delta B)\Big(u_s(\theta) - \frac{u(\theta)}{2}\Big)\Big]\,\mathrm{d}\theta \end{aligned} \quad (21)$$

for some square matrix A_{11} of appropriate dimensions. Then the problem is how to choose the control parameters R, P and Q involved in the following Riccati equation:

$$(A + A_{11})^T P + P(A + A_{11}) - PBR^{-1}B^T P + Q = 0 \quad (22)$$

such that the closed-loop equation (21) is asymptotically stable. In other words, parametrical uncertainties can be tolerated.

Theorem 2. *Consider the system (21) and assume that* $A + A_{11}$ *is a Hurwitz stable matrix satisfying*

$$\tau < \frac{\sigma_s - \alpha_s - 0.5(\|B\| + \beta)\|K\| - q\delta(\|A_{12}\| + \alpha_1)}{q\delta_s\varsigma}, \quad (23)$$

where $\varsigma = \|A_{11}A_s\| + \|A_{11}A_1\| + \|A_{11}\|(\alpha_s + \alpha_1 + 0.5(\|B\| + \beta)\|K\|)$, $\sigma_s = \lambda_{\min}(D)/2\lambda_{\max}(P)$, $\alpha_s = \alpha + 0.5\beta\,\|K\|$ *and* $D = P^T B(R^{-1})^T B^T P + Q$, $\tau \geq 0$. *Then the uncertain time-delay saturating actuator system (16) is asymptotically stable for any positive number* $q > 1$, *i.e., the uncertain and saturating actuator parts of the nominal system can be tolerated.*

Proof. We consider (21) and take the following positive definite function as our Lyapunov function (10). Substituting (18) into (11), we obtain

$$\begin{aligned}\dot{V}[x(t)] = {} & \Big\{(A_s + A_{11})x(t) + A_{12}x(t-\tau) \\ & + \Delta A_s x(t) + \Delta A_1 x(t-\tau) \\ & + (B+\Delta B)(u_s(t) - \frac{u(t)}{2}) \\ & - A_{11}\int_{t-\tau}^{t} \Big[(A_s + \Delta A_s)x(\theta) \\ & + (A_1 + \Delta A_1)x(\theta-\tau) \\ & + (B+\Delta B)\Big(u_s(\theta) - \frac{u(\theta)}{2}\Big)\Big]\,\mathrm{d}\theta\Big\}^T Px(t) \\ & + x^T(t)P\Big\{(A_s + A_{11})x(t) + A_{12}x(t-\tau) \\ & + \Delta A_s x(t) + \Delta A_1 x(t-\tau) \\ & + (B+\Delta B)\Big(u_s(t) - \frac{u(t)}{2}\Big) \\ & - A_{11}\int_{t-\tau}^{t} \Big[(A_s + \Delta A_s)x(\theta) \\ & + (A_1 + \Delta A_1)x(\theta-\tau) \\ & + (B+\Delta B)\Big(u_s(\theta) - \frac{u(\theta)}{2}\Big)\Big]\,\mathrm{d}\theta\Big\} \\ \leq {} & x^T(t)\big[(A_s + A_{11})^T P + P(A_s + A_{11})\big]x(t) \\ & + x^T(t)\Delta A_s^T Px(t) + x^T(t)P\Delta A_s x(t) \\ & + x^T(t-\tau)A_{12}^T Px(t) + x^T(t)PA_{12}x(t-\tau) \\ & + x^T(t-\tau)\Delta A_1^T Px(t)\end{aligned}$$

$$
\begin{aligned}
&+ x^T(t)P\Delta A_1 x(t-\tau)\\
&+ \Big[(B+\Delta B)(u_s(t) - \frac{u(t)}{2}\Big)\Big]^T Px(t)\\
&+ x^T(t)P\Big[(B+\Delta B)\Big(u_s(t) - \frac{u(t)}{2}\Big)\Big]\\
&- 2x^T(t)PA_{11}\int_{t-\tau}^{t}\Big[(A_s+\Delta A_s)x(\theta)\\
&+ (A_1+\Delta A_1)x(\theta-\tau)\\
&+ (B+\Delta B)\Big(u_s(\theta) - \frac{u(\theta)}{2}\Big)\Big]\,\mathrm{d}\theta. \qquad (24)
\end{aligned}
$$

Applying the Razumikhin-type theorem, we assume that for any positive number $q > 1$, the following inequality holds:

$$V\big(x(t-\tau)\big) < q^2 V\big(x(t)\big). \qquad (25)$$

Thus

$$\|x(t-\tau)\| < q\delta_s \|x(t)\|. \qquad (26)$$

Substituting (25) and (26) into (24), we have

$$\dot V\big[x(t)\big] \le -\omega_s \|x(t)\|^2, \quad \omega \in \mathbb{R}, \qquad (27)$$

where

$$
\begin{aligned}
\omega_s = \lambda_{\min}(D) - 2\Big\{&\alpha + 0.5\big(\|B\| + \beta\big)\|K\|\\
&- q\delta_s\big(\|A_{12}\| + \alpha_1\big)\\
&+ q\delta_s\tau\Big[\|A_{11}A_s\| + \|A_{11}A_1\| + \|A_{11}\|\Big(\alpha_s\\
&+ \alpha_1 + 0.5\big(\|B\| + \beta\big)\|K\|\Big)\Big]\Big\}\lambda_{\max}(P).
\end{aligned}
$$

Based on the results obtained in the proof of Theorem 2, we have $\dot V[x(t)] < 0$. Then $x(t) \to 0$ as $t \to \infty$ if $\omega_s > 0$. But $\omega_s > 0$ if and only if (27) holds. This will guarantee asymptotic stability of the time delay system (21). Therefore, the system (21) is asymptotically stable. ■

4. Examples

To ilustrate the previous results, we give three examples.

Example 1. We consider the following linear uncertain time-delay system:

$$
\begin{aligned}
\dot x(t) = &(A+\Delta A)x(t) + (A_1+\Delta A_1)x(t-\tau)\\
&+ \big(B + \Delta B(t)\big)u(t), \qquad (28)
\end{aligned}
$$

where

$$A = \begin{bmatrix} 0 & 1\\ -1 & -2\end{bmatrix}, \quad A_1 = \begin{bmatrix} -2 & -1\\ 1 & 0\end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}, \quad \Delta A = \Delta A_1 = \Delta B = \begin{bmatrix} 0.1 & 0\\ 0 & 0.1\end{bmatrix}.$$

We now find the range of the time delay τ with the state feedback controller (5) to guarantee that the above system is asymptotically stable.

Solution. We set

$$Q = \begin{bmatrix} 5 & 0\\ 0 & 5\end{bmatrix}, \quad R = \begin{bmatrix} 0.1 & 0\\ 0 & 0.1\end{bmatrix},$$

$$A_{11} = \begin{bmatrix} -1.95 & -0.9\\ 0.9 & 0\end{bmatrix}.$$

From the Riccati equation, cf. (4),

$$(A + A_{11})^T P + P(A + A_{11}) - PBR^{-1}B^T P + Q = 0,$$

we get

$$P = \begin{bmatrix} 0.5385 & 0\\ 0 & 0.5348\end{bmatrix}.$$

We then find the state feedback controller

$$K = \begin{bmatrix} 5.3850 & 0.0002\\ 0.0002 & 5.3485\end{bmatrix},$$

$$A_k = \begin{bmatrix} -5.3850 & 0.9998\\ -1.0002 & -7.3485\end{bmatrix},$$

$$
\begin{aligned}
&\alpha = 0.1, && \lambda_{\min}(P) = 0.5348, && \|A_{11}A_k\| = 13.0257,\\
&\beta = 0.1, && \lambda_{\max}(P) = 0.5385, && \|A_{11}A_1\| = 3.9679,\\
&\sigma = 7.2986, && \lambda_{\min}(D) = 7.8606, && \|A_{12}\| = 0.1281,\\
&\delta = 0.9966, && \|K\| = 5.385.
\end{aligned}
$$

From (9) of Theorem 1, we obtain

$$
\begin{aligned}
0 &< \tau\\
&\le \frac{\sigma - [\alpha + \beta\|K\| + (\|A_{12}\| + \alpha_1)q\delta]}{q\delta[\|A_{11}A_k\| + \|A_{11}A_1\| + \|A_{11}\|(\alpha + \beta\|K\| + \alpha_1)]}\\
&= 0.3429.
\end{aligned}
$$

For this example asymptotic stability of the system (28) is guaranteed for $0 \le \tau \le 0.3429$. We note that the result in (Su and Liu, 1996) guarantees robust stabilization of (28) when $0 \le \tau \le 0.32$. This example shows that the method of this paper is an improvement of this previous result.

Table 1. Comparison between the result in this paper and a previous result.

K	τ (Su *et al.*, 2002)	τ (by our result)
[5.1926, 4.7212]	4.0813	6.3833
[0.2209, 1.3031]	4.4206	8.4388

Example 2. Consider the uncertain time-delay system with a saturating actuator

$$\dot{x}(t) = (A + \Delta A)x(t) + (A_1 + \Delta A_1)x(t-\tau) + B\,\mathrm{Sat}\left[u(t)\right], \tag{29}$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -0.8 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\Delta A = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad \Delta A_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

Find the range of the delay time τ by using a state feedback controller K to guarantee that the above system is asymptotically stable.

Solution. We set

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} -1 & 0 \\ -0.8 & -0.8 \end{bmatrix}.$$

We choose a tolerance coefficient of saturation $\mathrm{Sat}[u(t)]$. From the Riccati equation (4), we get

$$P = \begin{bmatrix} 0.1625 & 0.0036 \\ 0.0036 & 0.1294 \end{bmatrix}$$

and the state feedback controller

$$K = \begin{bmatrix} 0.1625 & 0.0036 \\ 0.0036 & 0.1294 \end{bmatrix}.$$

For this example asymptotic stability of the system (15) is guaranteed for $\tau < 0.5522$. On the other hand, the stability criterion in (Liu *et al.*, 2001) gives a bound for the time delay of 0.3781. On the other hand, the delay bound for guaranteeing the asymptotic stability of the system (29) is $\tau < 0.2841$ (Su and Chu, 1999; Su *et al.*, 2001). Applying Theorem 1 to this uncertain time-delay system (29), the maximum time delay for stability, τ, is found and compared with the result by Su *et al.* (2002), cf. Table 1.

As Table 1 indicates, the maximum time delay τ for stability obtained by our approach is less conservative than the other result. Hence, for this example, the robust stability criterion of this paper is less conservative than the existing results (Su and Chu, 1999; Su *et al.*, 2001; 2002).

Example 3. Consider the following linear time-delay system, which suffers from the following parameter perturbations (Cao *et al.*, 1998; Liu and Su, 1998; Su and Huang, 1992):

$$\dot{x}(t) = (A + \Delta A)x(t) + (A_1 + \Delta A_1)x(t-\tau), \tag{30}$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

$$\Delta A = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.2 \end{bmatrix}, \qquad \Delta A_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

We can find τ to guarantee that the system (30) is asymptotically stable.

Solution. Let $K = 0$ and

$$B_1 = \begin{bmatrix} -0.9 & 0 \\ 0.9 & -0.9 \end{bmatrix}.$$

Applying Theorem 1 to the uncertain time delay system (30), it is found that $\tau < 0.2836$. The maximum time delay for the stability τ as estimated by the criteria of (Cao *et al.*, 1998; Liu and Su, 1998; Su and Huang, 1992) and the approach in this note is listed in Table 2.

Table 2. Comparison between the proposed and other methods.

Method	τ
Su and Huang, 1992	0.1575
Liu and Su, 1998	0.2130
Cao *et al.*, 1998	0.2558
This paper	0.2836

From Table 2, the proposed criteria are less conservative than those in (Cao *et al.*, 1998; Liu and Su, 1998; Su and Huang, 1992). Hence, our result gives a less conservative bound than those obtained by using a delay-dependent stability criterion (Cao *et al.*, 1998; Liu and Su, 1998; Su and Huang, 1992).

5. Conclusion

In this paper, the delay-dependent robust stabilization problem for a class of uncertain linear time-delay systems containing saturating actuators is considered. The objective of this paper is to guarantee an allowable bound on a delay time τ such that if the time delay is less than the obtained constant delay bound, the constrained system with time delay can be tolerated. The analysis and synthesis problems addressed are used to obtain a delay-dependent stability criterion and design a memoryless state feedback control law such that the closed-loop system is asymptotically stable, along with a sufficient condition for the existence of such a control law presented in terms of a Riccati equation with a decomposition technique. Compared with several existing stability criteria, the allowable bound on the delay time is significantly improved. The significance of the obtained results is demonstrated by three illustrative examples.

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