REALIZATION PROBLEM FOR POSITIVE MULTIVARIABLE DISCRETE–TIME LINEAR SYSTEMS WITH DELAYS IN THE STATE VECTOR AND INPUTS

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The realization problem for positive multivariable discrete-time systems with delays in the state and inputs is formulated and solved. Conditions for its solvability and the existence of a minimal positive realization are established. A procedure for the computation of a positive realization of a proper rational matrix is presented and illustrated with examples.

Keywords: positive realization, discrete-time system, time delay, existence, computation

1. Introduction

In positive systems, inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of the state of the art in positive systems theory is given in the monographs (Farina and Rinaldi, 2000; Kaczorek, 2002). Recent developments and some new results are given in (Kaczorek, 2003). Realization problems of positive linear systems without time delays were considered in many papers and books (Benvenuti and Farina, 2004; Farina and Rinaldi, 2000; Kaczorek, 2002). Explicit solution of equations describing discrete-time systems with time delay was discussed in (Busłowicz, 1982). Recently, reachability, controllability and minimum energy control of positive linear discrete-time systems with delay were considered in (Busłowicz and Kaczorek, 2004; Xie and Wang, 2003).

In this paper, the realization problem for positive multivariable discrete-time systems with delays in the state and inputs will be formulated and solved. Conditions for the solvability of the realization problem and the existence of a minimal positive realization will be established. A procedure for the computation of a positive realization of a proper rational matrix will be presented. To the best of the author's knowledge, the realization problem for positive linear systems with delays in the state vector and inputs has not been considered yet.

2. Problem Formulation

Consider the multivariable discrete time linear system with one time delay in the state and input:

$$x_{i+1} = A_0 x_i + A_1 x_{i-1} + B_0 u_i + B_1 u_{i-1}, \quad i \in \mathbb{Z}_+ = \{0, 1, \dots\}, \tag{1a}$$

$$y_i = C x_i + D u_i, \tag{1b}$$

where $x_i \in \mathbb{R}^n, u_i \in \mathbb{R}^m, y_i \in \mathbb{R}^p$ are the state, input and output vectors, respectively, and $A_k \in \mathbb{R}^{n\times n}, B_k \in \mathbb{R}^{n\times m},\ k = 0, 1, C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m}$. The initial conditions for (1a) are given by

$$x_{-1},\, x_0 \in \mathbb{R}^n, \quad u_{-1} \in \mathbb{R}^m. \tag{2}$$

Let $\mathbb{R}_+^{n\times m}$ be the set of $n \times m$ real matrices with nonnegative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

Definition 1. (Busłowicz and Kaczorek, 2004) The system (1) is called *(internally) positive* if for every $x_{-1},\, x_0 \in \mathbb{R}_+^n, u_{-1} \in \mathbb{R}_+^m$ and all inputs $u_i \in \mathbb{R}_+^m,\ i \in Z_+$, we have $x_i \in \mathbb{R}_+^n$ and $y_i \in \mathbb{R}_+^p$ for $i \in \mathbb{Z}_+$.

Theorem 1. (Busłowicz and Kaczorek, 2004) *The system (1) is positive if and only if*

$$A_k \in \mathbb{R}_+^{n\times n},\ B_k \in \mathbb{R}_+^{n\times m},\ k = 0, 1,$$
$$C \in \mathbb{R}_+^{p\times n},\ D \in \mathbb{R}_+^{p\times m}. \tag{3}$$

The transfer matrix of (1) is given by

$$T(z) = C\left[I_n z - A_0 - A_1 z^{-1}\right]^{-1} \times \left(B_0 + B_1 z^{-1}\right) + D. \tag{4}$$

Definition 2. The matrices (3) are called a *positive realization* of a given proper rational matrix $T(z)$ if and only if they satisfy (4). The realization (3) is called *minimal* if and only if the dimension $n \times n$ of A_0 and A_1 is minimal among all positive realizations of $T(z)$.

The positive realization problem can be formulated as follows: Given a proper matrix transfer function $T(z)$, find the positive minimal realization (3) of $T(z)$. Conditions for its solvability will be established and a procedure for computation of a positive realization will be presented.

3. Problem Solution

The transfer matrix (4) can be rewritten in the form

$$\begin{aligned} T(z) &= C\,[z^{-1}(I_n z^2 - A_0 z - A_1)]^{-1}(B_0 + B_1 z^{-1}) + D \\ &= \frac{C\,\mathrm{Adj}\,[I_n z^2 - A_0 z - A_1](B_0 z + B_1)}{\det\,[I_n z^2 - A_0 z - A_1]} + D \\ &= \frac{N(z)}{d(z)} + D, \end{aligned} \tag{5}$$

where

$$\begin{aligned} N(z) &= C\,\mathrm{Adj}\,[I_n z^2 - A_0 z - A_1](B_0 z + B_1), \\ d(z) &= \det\,[I_n z^2 - A_0 z - A_1], \end{aligned} \tag{6}$$

and $\mathrm{Adj}\,[I_n z^2 - A_0 z - A_1]$ denotes the adjoint matrix for $[I_n z^2 - A_0 z - A_1]$.

From (5), we have

$$D = \lim_{z\to\infty} T(z) \tag{7}$$

since

$$\lim_{z\to\infty}\,[z^{-1}(I_n z^2 - A_0 z - A_1)]^{-1} = 0.$$

The strictly proper part of $T(z)$ is given by

$$T_{sp}(z) = T(z) - D = \frac{N(z)}{d(z)}. \tag{8}$$

Therefore, the positive realization problem has been reduced to finding the matrices

$$A_k \in \mathbb{R}_+^{n\times n},\;\; B_k \in \mathbb{R}_+^{n\times m},\; k = 0, 1,\;\; C \in \mathbb{R}_+^{p\times n} \tag{9}$$

for the given strictly proper rational matrix (8).

Lemma 1. *If the matrices A_0 and A_1 have the following forms:*

$$A_0 = \begin{bmatrix} 0 & \cdots & 0 & a_1 \\ 0 & \cdots & 0 & a_3 \\ 0 & \cdots & 0 & a_5 \\ \dots & \dots & \dots & \dots \\ 0 & \cdots & 0 & a_{2n-1} \end{bmatrix} \in \mathbb{R}^{n\times n},$$

$$A_1 = \begin{bmatrix} 0 & 0 & \dots & 0 & a_0 \\ 1 & 0 & \dots & 0 & a_2 \\ 0 & 1 & \dots & 0 & a_4 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & a_{2(n-1)} \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{10}$$

then

$$\begin{aligned} d(z) &= \det[I_n z^2 - A_0 z - A_1] \\ &= z^{2n} - a_{2n-1} z^{2n-1} - \cdots - a_1 z - a_0. \end{aligned} \tag{11}$$

Proof. The expansion of the determinant with respect to the n-th column yields

$$d(z) = \det\left[I_n z^2 - A_0 z - A_1\right]$$

$$= \begin{vmatrix} z^2 & 0 & \cdots & 0 & -a_1 z - a_0 \\ -1 & z^2 & \cdots & 0 & -a_3 z - a_2 \\ 0 & -1 & \cdots & 0 & -a_5 z - a_4 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & z^2 & -a_{2n-3} z - a_{2(n-2)} \\ 0 & 0 & \cdots & -1 & z^2 - a_{2n-1} z - a_{2(n-1)} \end{vmatrix}$$

$$= z^{2n} - a_{2n-1} z^{2n-1} - \cdots - a_1 z + a_0.$$

■

Remark 1. There may exist many different pairs of the matrices $A_0,\ A_1$ giving the desired polynomial $d(z)$ (Kaczorek, 2004; 2005).

Remark 2. The matrices (10) have nonnegative entries if and only if the coefficients $a_k,\ k = 0, 1, \dots, 2n-1$ of the polynomial (11) are nonnegative.

Remark 3. The dimension $n \times n$ of the matrices (10) is the smallest possible for (8).

Lemma 2. *If the matrices $A_0,\ A_1$ have the forms (10), then the n-th row of the adjoint matrix $\mathrm{Adj}\,[I_n z^2 - A_0 z - A_1]$ has the form*

$$R_n(z) = [\;1 \quad z^2 \quad z^4 \quad \dots \quad z^{2(n-1)}\;]. \tag{12}$$

Proof. Taking into account the fact that

$$(\mathrm{Adj}\,[I_n z^2 - A_0 z - A_1])[I_n z^2 - A_0 z - A_1] = I_n d(z),$$

it is easy to verify that

$$R_n(z)\,[I_n z^2 - A_0 z - A_1] = [\ 0 \quad \dots \quad 0 \quad 1\ ]\, d(z).$$

■

A strictly proper $T_{sp}(z)$ can always be written in the form

$$T_{sp}(z) = \begin{bmatrix} \dfrac{N_1(z)}{d_1(z)} \\ \vdots \\ \dfrac{N_p(z)}{d_p(z)} \end{bmatrix}, \tag{13}$$

where

$$d_i(z) = z^{2q_i} - a_{i2q_1-1}z^{2q_i-1} - \dots - a_{i1}z - a_{i0}, \qquad i = 1,\dots,p, \tag{14}$$

is the least common denominator of the i-th row of $T_{sp}(z)$ and

$$\begin{aligned} N_i(z) &= [\ n_{i1}(z) \quad n_{i2}(z) \quad \dots \quad n_{im}(z)\ ], \\ & \qquad i = 1,\dots,p, \\ n_{ij}(z) &= n_{ij}^{2q_i-1}z^{2q_i-1} + \dots + n_{ij}^1 z + n_{ij}^0, \\ & \qquad j = 1,\dots,m. \end{aligned} \tag{15}$$

Note that by Lemma 1 we may associate to any polynomial (14) the pair of the matrices

$$A_{0i} = \begin{bmatrix} 0 & \cdots & 0 & a_{i1} \\ 0 & \cdots & 0 & a_{i3} \\ 0 & \cdots & 0 & a_{i5} \\ \dots & \dots & \dots & \dots \\ 0 & \cdots & 0 & a_{i2q_i-1} \end{bmatrix} \in \mathbb{R}^{q_i \times q_i},$$

$$A_{1i} = \begin{bmatrix} 0 & 0 & \dots & 0 & a_{i0} \\ 1 & 0 & \dots & 0 & a_{i2} \\ 0 & 1 & \dots & 0 & a_{i4} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & a_{i2(q_i-1)} \end{bmatrix} \in \mathbb{R}^{q_i \times q_i}, \tag{16}$$

$i = 1,\dots,p$, satisfying the condition

$$d_i(z) = \det\,[I_{q_i}z^2 - A_{0i}z - A_{1i}], \quad i = 1,\dots,p. \tag{17}$$

Let

$$\begin{aligned} C &= \text{block diag}\,[\ C_1 \quad C_2 \quad \dots \quad C_p\ ], \\ C_i &= [\ 0 \quad 0 \quad \dots \quad 1\ ] \in \mathbb{R}^{1\times q_i}, \quad i = 1,\dots,p \end{aligned} \tag{18}$$

and

$$\begin{bmatrix} b_{1j}^k \\ b_{2j}^k \\ \vdots \\ b_{pj}^k \end{bmatrix}, \qquad b_{ij}^k = \begin{bmatrix} b_{ij}^{k1} \\ b_{ij}^{k2} \\ \vdots \\ b_{ij}^{kq_i} \end{bmatrix},$$

$$k = 0,1; \quad i = 1,\dots,p; \quad j = 1,\dots, \tag{19}$$

be the j-th column of the matrix B_k.

Then from (6), (12), (18) and (19) we obtain Eqn. (20) for the j-th column of $B_0 z + B_1$, in which the polynomials $n_{ij}(z)$ are given by (15).

The comparison of the coefficients at the same powers of z yields Eqn. (21).

Theorem 2. *The positive realization (3) of $T(z)$ exists if*

(i) $T(\infty) = \lim_{z\to\infty}(T(z)) \in \mathbb{R}_+^{p\times m}$,

(ii) the coefficients of $d_i(z)$, $i = 1,\dots,p$, are nonnegative, i.e.,

$$a_{ij} \ge 0 \ \ for \ \ i = 1,\dots,p \text{ and } j = 0,1,\dots,2q_i-1, \tag{22}$$

(iii) the coefficients of $N_i(z)$, $i = 1,\dots,p$ are nonnegative, i.e.,

$$n_{ij}^k \ge 0 \quad for \qquad i = 0,1,\dots,p;\ j = 1,\dots,m; \qquad k = 0,1,\dots,2q_i-1. \tag{23}$$

Proof. Condition (i) implies $D \in \mathbb{R}_+^{p\times m}$. If (22) is satisfied, then the matrices (16) have nonnegative entries and

$$\begin{aligned} A_0 &= \text{block diag}\,[A_{01},\dots,A_{0p}] \in \mathbb{R}_+^{n\times n}, \\ A_1 &= \text{block diag}\,[A_{11},\dots,A_{1p}] \in \mathbb{R}_+^{n\times n}, \\ n &= q_1 + \dots + q_p. \end{aligned} \tag{24}$$

If, additionally, the conditions (23) are satisfied, then from (21) it follows that $B_0 \in \mathbb{R}_+^{n\times m}$ and $B_1 \in \mathbb{R}_+^{n\times m}$. The matrix C of the form (18) is independent of $T(z)$ and has always nonnegative entries. ■

Theorem 3. *The realization (3) of $T(z)$ is minimal if the denominators $d_1(z), d_2(z),\dots,d_p(z)$ are relatively prime (coprime).*

Proof. If the polynomials $d_1(z),\dots,d_p(z)$ are relatively prime, then

$$d(z) = d_1(z)\,d_2(z)\,\dots\,d_p(z) \tag{25}$$

and, by Remark 3, the matrices (24) have minimal possible dimensions. ■

$$
\begin{aligned}
&C\,[I_n z^2 - A_0 z - A_1]^{-1}(B_0 z + B_1)_j \\
&= \text{block diag}\,[\ C_1 \quad C_2 \quad \dots \quad C_p](\text{block diag}\,\{[I_{q_1}z^2 - A_{01}z - A_{11}]^{-1}, \dots, [I_{q_p}z^2 - A_{0p}z - A_{1p}]^{-1}\ \}) \\
&\quad \times \begin{bmatrix} b_{1j}^0 z + b_{1j}^1 \\ \vdots \\ b_{pj}^0 z + b_{pj}^1 \end{bmatrix} \\
&= \text{block diag}\left\{\frac{1}{d_1(z)}[\ 1 \quad z^2 \quad \dots \quad z^{2(q_1-1)}\ ], \dots, \frac{1}{d_p(z)}[\ 1 \quad z^2 \quad \dots \quad z^{2(q_p-1)}\ ]\right\}\begin{bmatrix} b_{1j}^0 z + b_{1j}^1 \\ \vdots \\ b_{pj}^0 z + b_{pj}^1 \end{bmatrix} \qquad (20) \\
&= \begin{bmatrix} \dfrac{b_{1j}^{11} + b_{1j}^{01}z + b_{1j}^{12}z^2 + b_{1j}^{02}z^3 + \dots + b_{pj}^{1q_1}z^{2(q_1-1)} + b_{pj}^{0q_1}z^{2q_1-1}}{d_1(z)} \\ \vdots \\ \dfrac{b_{pj}^{11} + b_{pj}^{01}z + b_{pj}^{12}z^2 + b_{pj}^{02}z^3 + \dots + b_{pj}^{1q_p}z^{2(q_p-1)} + b_{pj}^{0q_p}z^{2q_p-1}}{d_1(z)} \end{bmatrix} = \begin{bmatrix} \dfrac{n_{1j}(z)}{d_1(z)} \\ \vdots \\ \dfrac{n_{pj}(z)}{d_p(z)} \end{bmatrix}, \quad j = 1, \dots, m
\end{aligned}
$$

$$
\begin{aligned}
&b_{1j}^{0q_1} = n_{1j}^{2q_1-1},\ b_{1j}^{1q_1} = n_{1j}^{2(q_1-1)}, \dots, b_{1j}^{01} = n_{1j}^1,\ b_{1j}^{11} = n_{1j}^0, \quad j = 1, \dots, m, \\
&\vdots \\
&b_{pj}^{0q_p} = n_{pj}^{2q_p-1},\ b_{pj}^{1q_p} = n_{pj}^{2(q_p-1)}, \dots, b_{pj}^{01} = n_{pj}^1,\ b_{pj}^{11} = n_{pj}^0.
\end{aligned} \tag{21}
$$

If the conditions of Theorem 2 are satisfied, then the positive realization (3) of $T(z)$ can be found using the following procedure:

Procedure

Step 1. Using (7) and (8), find $D \in \mathbb{R}_+^{p\times m}$ and the strictly proper matrix $T_{sp}(z)$.

Step 2. Knowing the coefficients a_{ij} $(i = 1, \dots, p;$ and $j = 0, 1, \dots, 2q_i - 1)$ of $d_i(z)$, $i = 1, \dots, p$, find the matrices (16) and (24).

Step 3. Knowing the coefficients n_{ij}^k $(i = 1, \dots, p;\ j = 1, \dots, m;\ k = 0, 1, \dots, 2q_i - 1)$ of $N_i(z)$ $(i = 1, \dots, p)$ and using (21), find B_k for $k = 0, 1$ and the matrix C of the form (18).

Example 1. Using the above procedure, we wish to find the positive realization (3) of the matrix transfer function

$$
T(z) = \begin{bmatrix} \dfrac{z^4 - 2z^3 + z^2 - 2}{z^4 - 2z^3 - z^2 - z - 2} & \dfrac{2z^4 - 3z^3 - 2z^2 - 2z - 3}{z^4 - 2z^3 - z^2 - z - 2} \\[2ex] \dfrac{z+1}{z^2 - 2z - 1} & \dfrac{z^2 - z - 1}{z^2 - 2z - 1} \end{bmatrix}. \tag{26}
$$

It is easy to verify that the assumptions of Theorem 2 are satisfied.

Step 1. From (7) and (26), we have

$$
D = \lim_{z\to\infty} T(z) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \tag{27}
$$

and

$$
\begin{aligned}
T_{sp}(z) &= T(z) - D \\
&= \begin{bmatrix} \dfrac{2z^2 + z}{z^4 - 2z^3 - z^2 - z - 2} & \dfrac{z^3 + 1}{z^4 - 2z^3 - z^2 - z - 2} \\[2ex] \dfrac{z+1}{z^2 - 2z - 1} & \dfrac{z}{z^2 - 2z - 1} \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{n_{11}(z)}{d_1(z)} & \dfrac{n_{12}(z)}{d_1(z)} \\[2ex] \dfrac{n_{21}(z)}{d_2(z)} & \dfrac{n_{22}(z)}{d_2(z)} \end{bmatrix}.
\end{aligned} \tag{28}
$$

Step 2. Taking into account the fact that in this case $d_1(z) = z^4 - 2z^3 - z^2 - z - 2,\ d_2(z) = z^2 - 2z - 1$ and using (16) and (24), we obtain

$$A_0 = \begin{bmatrix} A_{01} & 0 \\ 0 & A_{02} \end{bmatrix} = \left[\begin{array}{cc:c} 0 & 1 & 0 \\ 0 & 2 & 0 \\ \hdashline 0 & 0 & 2 \end{array}\right],$$

$$A_1 = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{12} \end{bmatrix} = \left[\begin{array}{cc:c} 0 & 2 & 0 \\ 1 & 1 & 0 \\ \hdashline 0 & 0 & 1 \end{array}\right]. \tag{29}$$

Step 3. In this case, $n_{11}(z) = 2z^2 + z, n_{12}(z) = z^3 + 1, n_{21}(z) = z + 1, n_{22}(z) = z$.

Using (21), we obtain

$$B_0 = \begin{bmatrix} b_{11}^{01} & b_{12}^{01} \\ b_{11}^{02} & b_{12}^{02} \\ b_{21}^{01} & b_{22}^{01} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} b_{11}^{11} & b_{12}^{11} \\ b_{11}^{12} & b_{12}^{12} \\ b_{21}^{11} & b_{22}^{11} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{30}$$

The desired positive minimal realization (3) of (26) is given by (27), (29) and (30). Note that the realization is minimal since the polynomials $d_1(z) = z^4 - 2z^3 - z^2 - z - 2$ and $d_2(z) = z^2 - 2z - 1$ are relatively prime. ♦

Lemma 3. *If* $d_i(z),\ i = 1, \dots, p$ *is an odd-degree polynomial and the rational function* $n_{ij}(z)/d_i(z)$ $(j = 1, \dots, m)$ *is strictly proper, then*

$$a_{i0} = 0 \quad \textit{for} \quad i = 1, \dots, p \tag{31}$$

and

$$b_{1i}^1 = 0 \quad \textit{for} \quad i = 1, \dots, p. \tag{32}$$

Proof. From (14) and (20) if follows that $\deg d_i(z) = 2q_i - 1$ and $n_{ij}(z)/d_i(z)$ is strictly proper if the conditions (31) and (32) are satisfied since $n_{ij}(z)/d_i(z) = zn'_{ij}(z)/zd'_i(z) = n'_{ij}(z)/d'_i(z)$ for $i = 1, \dots, p$ and $j = 1, \dots, m$. ■

Remark 4. From (16) it follows that if (31) holds, then $\det A_{1i} = 0,\ i = 1, \dots, p$.

Using Lemma 3 and the Procedure, we may find the positive realization (3) of $T(z)$ with $d_i(z),\ i = 1, \dots, p$ of odd degrees.

Example 2. We wish to find the positive realization (3) of the strictly proper matrix transfer function

$$T(z) = \begin{bmatrix} \dfrac{z^2 + 2z + 1}{z^3 - 2z^2 - z - 1} & \dfrac{2z^2 + 2}{z^3 - 2z^2 - z - 1} \\[2ex] \dfrac{z + 1}{z^2 - 2z - 1} & \dfrac{z}{z^2 - 2z - 1} \end{bmatrix}. \tag{33}$$

Step 1. In this case,

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{34}$$

and

$$T_{sp}(z) = T(z) = \begin{bmatrix} \dfrac{n_{11}(z)}{d_1(z)} & \dfrac{n_{12}(z)}{d_1(z)} \\[2ex] \dfrac{n_{21}(z)}{d_2(z)} & \dfrac{n_{22}(z)}{d_2(z)} \end{bmatrix}.$$

Step 2. Using (16) and (24) and taking into account the fact that $d_1(z) = z^3 - 2z^2 - z - 1$ and $d_2(z) = z^2 - 2z - 1$, we obtain

$$A_0 = \begin{bmatrix} A_{01} & 0 \\ 0 & A_{02} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{35}$$

Step 3. Taking into account the fact that $n_{11}(z) = z^2 + 2z + 1, n_{12}(z) = 2z^2 + 2, n_{21}(z) = z + 1, n_{22}(z) = z$ and using (21) and (32), we obtain

$$B_0 = \begin{bmatrix} b_{11}^{01} & b_{12}^{01} \\ b_{11}^{02} & b_{12}^{02} \\ b_{21}^{01} & b_{22}^{01} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} b_{11}^{11} & b_{12}^{11} \\ b_{11}^{12} & b_{12}^{12} \\ b_{21}^{11} & b_{22}^{11} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{36}$$

The desired positive realization of (33) is given by (34)–(36).

4. Concluding Remarks

The realization problem for positive multivariable discrete-time systems with delays in the state vector and inputs was formulated and solved. The special forms (10) and (24) of the matrices A_0 and A_1 were introduced. Sufficient conditions for the existence of the positive realization (3) of a proper matrix transfer function $T(z)$ were established. It was shown that the positive realization (3) exists for any proper matrix transfer function $T(z)$ if its strictly proper part (8) satisfies the conditions (22) and (23) and the realization is minimal if the denominators $d_i(z),\ \ i = 1,\dots,p$ are relatively prime. A procedure for the computation of the positive realization (3) of the proper matrix transfer function $T(z)$ was presented and illustrated by examples. The presented method can be extended to positive multivariable continuous-time systems. An extension to two-dimensional linear systems (Kaczorek, 2002; Klamka, 1991) is also possible.

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References

Benvenuti L. and Farina L. (2004): *A tutorial on the positive realization problem*. — IEEE Trans. Automat. Contr., Vol. 49, No. 5, pp. 651–664.

Busłowicz M. (1982): *Explicit solution of discrete-delay equations*. — Found. Contr. Eng., Vol. 7, No. 2, pp. 67–71.

Busłowicz M. and Kaczorek T. (2004): *Reachability and minimum energy control of positive linear discrete-time systems with one delay*. — Proc. *12th Mediterranean Conference Control and Automation*, Kasadasi, Izmir, Turkey, (on CD-ROM).

Farina L. and Rinaldi S. (2000): *Positive Linear Systems — Theory and Applications*. — New York: Wiley.

Kaczorek T. (2002): *Positive 1D and 2D Systems*. — London: Springer.

Kaczorek T. (2003): *Some recent developments in positive systems*. — Proc. 7-th Conf. *Dynamical Systems Theory and Applications*, Łódź, Poland, pp. 25–35.

Kaczorek T. (2004): *Realization problem for positive discrete-time systems with delay*. — Syst. Sci., Vol. 30, No. 4, pp. 117–130.

Kaczorek T. (2005): *Realization problem for positive continuous-time systems with delays*. — Proc. Int. Conf. *Systems Engineering*, Las Vegas, USA, (on CD-ROM).

Kaczorek T. and Busłowicz M. (2004): *Minimal realization for positive multivariable linear systems with delay*. — Int. J. Appl. Math. Comput. Sci., Vol. 14, No. 2, pp. 181–187.

Klamka J. (1991): *Controllability of Dynamical Systems*. — Dordrecht: Kluwer.

Xie G. and Wang L. (2003): *Reachability and controllability of positive linear discrete-time systems with time-delays*, In: Positive Systems (L. Benvenuti, A. De Santis and L. Farina, Eds.). — Berlin: Springer, pp. 377–384.

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