HOW TO SECURE A HIGH QUALITY KNOWLEDGE BASE IN A RULE-BASED SYSTEM WITH UNCERTAINTY?

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Although the first rule-based systems were created as early as thirty years ago, this methodology of expert systems designing still proves to be useful. It becomes especially important in medical applications, while treating evidence given in an electronic format. Constructing the knowledge base of a rule-based system and, especially, of a system with uncertainty is a difficult task because of the size of this base as well as its heterogeneous character. The base consists of facts, ordinary rules and meta-rules, which differ from each other regarding both the syntax structure and the semantics. Having no tool to aid designing and maintaining the knowledge base of a rule-based system with uncertainty, we propose the algebra of rules with uncertainty which gives us theoretical foundations to build such a tool. Using the tool, it will be possible to indicate the facts and rules of a redundant character, as well as the pairs of facts and the pairs of rules which are contradictory to each other. The above tool is used in designing and maintaining the knowledge base of a system intended to prognosticate the effects of a medical treatment of the bronchial asthma disease.

Keywords: rule-based systems, uncertainty, knowledge base, truth maintenance module

1. Introduction

The amount of knowledge available in different fields of science is increasing systematically. Its assimilation by an individual person is becoming more and more difficult. Therefore, the obvious result is the increasing importance of expert systems (Giarratano and Riley, 2004); they help us to evaluate different phenomena and situations, to make decisions and prognosticate elements of the future.

On the other hand, the acquired knowledge often has a relative, contextual character. Its particular significance can be conditioned by time and place, as well as other less obvious factors. For this reason, expert systems with uncertainty have nowadays a very special role to play.

Generally speaking, what we mean by the term of uncertainty is the lack of information which is precise enough to make a decision. The uncertainty is the subject of many formal theories, e.g., the following ones:

- Pascal-Fermat's theory, introduced in the 18th century and considered to be a classical theory of probability;
- Carnap's theory (Carnap, 1945), pointed at a new type of probability, also called an epistemic probability;
- Dempster-Shaffer's theory (Shaffer, 1976), developed in the 1960s and 1970s in accordance with Carnap's theory;
- Zadeh's theory (Zadeh, 1965), the most general theory of uncertainty that has been formulated so far;
- belief networks (Pearl, 1988), introduced in the 1980s and being developed intensively up to now, based on using the Bayes theory of conditional probabilities.

Let us have a closer look at some of those.

Both the Pascal-Fermat and Bayes theories are different from the others. What makes the main difference between them is the way how the notion of ignorance is used. Namely, the classical theories claim that the evidence not supporting a hypothesis H is an evidence for the refutation of H. There is no place for ignorance here (it follows from the axiom $P(H) + P(H') = 1$). Unlike here, in the Carnap and Dempster-Shaffer theories, while lacking the knowledge about H, we do not have to assign any belief to H or to its negation H'. Instead, we may assign the remaining belief to the environment (the set of all possible hypotheses) θ. In order to deal with the idea of ignorance, Dempster and Shaffer consider the following certainty factors in their theory:

- the certainty factor cf: $2^\theta \to [0; 1]$, fulfilling the conditions $\mathrm{cf}(\emptyset) = 0$ and $\Sigma(X \in 2^\theta)\,\mathrm{cf}(X) = 1$;
- the global certainty factor Bel: $2^\theta \to [0, 1]$, such that $\mathrm{Bel}(H) = \Sigma(X \subseteq H)\,\mathrm{cf}(X)$;

- the plausibility factor Pls: $2^\theta \rightarrow [0,1]$, such that $\mathrm{Pls}(H) = 1 - \mathrm{Bel}(H') = 1 - \Sigma(X \subseteq H')\,\mathrm{cf}(X)$;
- the ignorance factor Igr: $2^\theta \rightarrow [0,1]$, such that $\mathrm{Igr}(H) = \mathrm{Pls}(H) - \mathrm{Bel}(H)$.

By means of certainty factors, we are also able to estimate how much credit should be granted to the conclusions obtained as the results of processing the evidence coming from different sources. In such cases, it is sufficient to apply the following Dempster rule of combination:

$$\mathrm{cf}_1 \oplus \mathrm{cf}_2(Z) = \Sigma(X \cap Y = Z)\,\mathrm{cf}_1(X) * \mathrm{cf}_2(Y),$$

in which X and Y are any input hypotheses, Z is a result hypothesis, $\mathrm{cf}_1(X)$, $\mathrm{cf}_2(Y)$ and $\mathrm{cf}_1 \oplus \mathrm{cf}_2(Z)$ stand for the certainty factors of X, Y and Z, respectively.

Having rigorous mathematical foundations, the Dempster-Shaffer theory has been widely implemented in expert systems, particularly in rule-based systems, where the uncertainty of knowledge is expressed through certainty factors attributed to facts and rules (Duda *et al.*, 1979; Lucas *et al.*, 1989; Shortliffe, 1976). The course and final result of the reasoning process depend on calculating the certainty factors of conclusions, and also on the algorithm of conflict resolution in the set of active rules. This algorithm, in many implementations, takes as the most important criterion the one of how detailed a rule is, which is measured by the number and the internal complexity of its premises. Then, it usually considers the rule newness, measured by the moment of introducing its premises into the knowledge base. In other cases, a decisive factor to select the rule is its priority. It is given statically in the process of designing the knowledge base, or determined dynamically on the basis of certainty factors of the rule and its premises.

The notion of ignorance is widely discussed in Zadeh's theory, in which all hypotheses from the environment are characterized not by means of numerical certainty factors, but with the so-called membership functions.

Consider the following definitions founded on the proposal of certainty factors:

```
(fact-A) CF 0.8                         fact
(defrule example
   (fact-A) CF 0.3
⇒                                       rule
   (assert (fact-B) CF 1.0)
   (assert (∼fact-C) CF 0.5)).
```

In Zadeh's fuzzy logic, they take the following, equivalent form:

```
(fact-A, almost-certain)                fact
(defrule example
  (fact-A, rather-uncertain)
⇒                                       rule
  (assert (fact-B) certain)
  (assert (∼fact-C) medium-certain)).
```

Here, the notions `almost-certain`, `rather-certain`, `certain`, and `medium-certain` are all fuzzy notions used to determine the frequency of the events. Without going into details, note that in order to represent the first three notions, we shall use the membership S-function and, to represent `medium-certain`, the membership Π-function.

Clearly, the conclusions deduced in high quality fuzzy systems usually meet our requirements much better than those deduced with the use of Dempster's rule of combination. Nevertheless, we shall point out that a fuzzy system will perform well if and only if the numerical evidence, obtained as an experiment result, is correctly transformed to a set of membership functions. Such a coding (and decoding) is not a trivial task.

Since the late 1980s, the designers started focusing more on the methods of representing uncertain knowledge in the form of probabilistic graphical models and, in particular, belief networks. As it has been recently proved, the certainty factors used in rule-based systems are in fact closely related to the uncertainty model used in belief networks (Lucas, 2001). Because of their character, each method is useful in different applications. Therefore, in the cases where knowledge has a character of simple, linear cause-and-effect dependencies, the belief networks prove to be the most convenient. However, in the cases where knowledge has a character of complex implications, which reflect the course of expert thinking and combine a multi-parameter input data with a multi-parameter output data, the rule-based systems are better. For example, in medical applications, knowledge representation in the form of belief networks turns out to be useful in diagnostic systems, whereas the knowledge in the form of rules is good in systems assigned to plan and prognosticate the effects of pharmacological treatment. All in all, although rule-based systems have often been criticized (Heckerman, 1990), only a balanced judgement seems to be fair (Oniśko *et al.*, 2001).

Undoubtedly, the credibility of an expert system increases as soon as it is equipped with a truth maintenance module, namely, the module which is able not only to construct justifications of conclusions, but also to maintain the knowledge base in the state of internal consistency.

Being inspired to build an expert system which could have an advisory function in both the diagnostics and treatment of the bronchial asthma disease, we agreed to model it in the form of a rule-based expert system with uncertainty. An additional argument in favour of the rule-based system is the ease of acquiring knowledge from electronic sources (the Internet), automatically or semi-automatically (Jankowska and Szymkowiak, 2005). To improve the quality of our system, we decided that it will be equipped with a truth maintenance module. Having no tools to aid designing the truth maintenance module for a rule-based system with uncertainty, we set our mind on defining a formal system which might be helpful in building such a tool.

We shall notice that the knowledge base of a rule-based system with the truth maintenance module consists of three types of elements: facts—which play the role of statements with an axiomatic nature, rules—the implications which enable reasoning in the system, and meta-rules—responsible for keeping the knowledge base in the state of internal consistency. Defining this heterogeneous knowledge base often involves the necessity of using various methods and various formal languages.

The heterogeneous character mentioned above is also an ever-present feature of rule-based systems with uncertainty. In these systems, all elements of the knowledge base have some additional attributes usually in the form of certainty factors. Their role is to designate the probability of the occurrence of particular facts or dependencies in reality. If particular elements of the knowledge base of the system with uncertainty are similar thanks to their attributes, it suggests the idea of perceiving the knowledge base as a certain whole.

2. Syntax of Rules with Uncertainty

At the beginning, let us define the concept of a rule with uncertainty. We intend to make use of this concept for subsequent modelling of both ordinary rules and meta-rules kept in the knowledge base of a rule-based system with uncertainty. Since each fact from this knowledge base is actually a particular case of the rule (with a universal fact as a premise and with the analyzed fact as a new conclusion), we aim at obtaining a formal system which would provide a basis to model the whole knowledge base in a homogeneous way.

Such a formal system will be defined in an increasing manner, applying in each following step the concepts defined in the previous steps.

2.1. Uncertain Facts. Let us consider an infinite set $\mathbb{T}$ of facts. This set will be settled axiomatically, together with the relations $=_0$ and $\leq_0$. The elements of $\mathbb{T}$ will be denoted by T_1, T_2, T_3, etc. As a result, we obtain

$$\mathbb{T} = \{T_1, T_2, T_3, \dots\}.$$

The relation $=_0$: $\mathbb{T} \times \mathbb{T}$ meaning “the same as” is an equivalence relation. Let us assume that T_i, T_j and T_k are any facts from the set $\mathbb{T}$. Then, the relation $\leq_0$: $\mathbb{T} \times \mathbb{T}$ meaning “subsumed by” must fulfil the following conditions:

- $T_i =_0 T_j \Rightarrow T_i \leq_0 T_j$,
- $T_i \leq_0 T_j \wedge T_j \leq_0 T_i \Rightarrow T_i =_0 T_j$,
- $T_i \leq_0 T_j \wedge T_j \leq_0 T_k \Rightarrow T_i \leq_0 T_k$.

Obviously, the relation $\leq_0$ is a partial ordering. An exemplary model of $\mathbb{T}$ may be a set of facts composing a medical evidence. Here are two such facts, specified by means of the FuzzyCLIPS (Orchard, 1998) notation. These facts characterize Mr Smith’s state of health, who is suffering from the bronchial asthma disease (BAD, 2002):

`(assert (SMITH cough-before lev-c(3))),` (1)

`(assert (SMITH cough-before lev-c(2 3))).` (2)

They show that the frequency of Mr Smith’s morning cough is no less than a few episodes a week for (1), and a few episodes a month or a few episodes a week for (2). The above facts are related to each other in the following way: `if-then0((2),(1))`, where `if-then0` is the two-argument relation, corresponding to $\leq_0$ in the model under discussion.

The determination of the relations `if-then0` and `if-and-only-if0` (equivalent to $=_0$ in the model) may not be an easy task. In the above example, we refer to the dependencies which hold between the ranges of frequency. However, it does not imply that using numerical ranges is indispensable to determine these relations. If we consider any field of knowledge which has a well-defined ontology, then `if-and-only-if0` and `if-then0` are obtained by adapting the relations of equality and subsumption which hold between the concepts of this field.

For example, in the field of genealogy there are some obvious dependencies, which might be expressed by means of an extended FuzzyCLIPS notation (the `assertmf` construct) as follows:

```
(assertmf if-and-only-if0((WIFE X Y),
                    (HUSBAND Y X))),        (3)

(assertmf if-then0((PARENT X Y),
                    (FATHER X Y))).         (4)
```

All the comments concerning the semantics of the subsequently defined concepts will be included in Section 3.

The uncertain fact will be each fact which has the form:

- $\perp$ (an empty fact—always false),
- $\top$ (a universal fact—always true),

- $\langle T_i, p_i \rangle$, where $T_i \in \mathbb{T}$, and $p_i \in [0,1]$ denotes the probability of the occurrence of T_i.

The infinite set of uncertain facts will be denoted by $\mathbb{F}$ and its elements by $F_1,\ F_2,\ F_3$, etc. The result is

$$\mathbb{F} = \{F_1, F_2, F_3, \dots\}.$$

In the next stage, let us define the relations $=_1$ and $\leq_1$ on the set of uncertain facts $\mathbb{F}$ which are extensions of the previously introduced relations $=_0$ and $\leq_0$, respectively. We shall assume that the uncertain facts F_1 and F_2 have the forms $F_1 = \langle T_1, p_1 \rangle$ and $F_2 = \langle T_2, p_2 \rangle$, respectively. Such assumptions lead to

$$F_1 =_1 F_2 \Leftrightarrow (T_1 =_0 T_2) \wedge (p_1 = p_2),$$

$$F_1 \leq_1 F_2 \Leftrightarrow (T_1 \leq_0 T_2) \wedge (p_1 \leq p_2).$$

2.2. Conjunction of Uncertain Facts. Having introduced the concept of uncertain facts, we can formulate a recursive definition of the set $\mathbb{FC}$ of conjunctions of uncertain facts:

- if $F_i \in \mathbb{F}$, then $F_i \in \mathbb{FC}$,
- if $FC_i,\ FC_j \in \mathbb{FC}$, then $FC_i \wedge_2 FC_j \in \mathbb{FC}$ (the $\wedge_2$ symbol is the connective of the conjunction of uncertain facts),
- no other element belongs to the set $\mathbb{FC}$.

Denoting particular elements of the infinite set $\mathbb{FC}$ by $FC_1,\ FC_2,\ FC_3$, etc., we obtain

$$\mathbb{FC} = \{FC_1, FC_2, FC_3, \dots\}.$$

Let us extend the relations $=_1$ and $\leq_1$ to the set $\mathbb{FC}$. To this end, we shall assume that $FC_i = F_{i1} \wedge_2 F_{i2} \wedge_2 \cdots \wedge_2 F_{ik}$ and $FC_m = F_{m1} \wedge_2 F_{m2} \wedge_2 \cdots \wedge_2 F_{mp}$. Then the relation $\leq_2$: $\mathbb{FC} \times \mathbb{FC}$ will be given the following meaning:

$$FC_i \leq_2 FC_m \Leftrightarrow \big(\forall(1 \leq j \leq k)\exists(1 \leq n \leq p)(F_{ij} \leq_1 F_{mn})\big).$$

In turn, the relation $=_2$: $\mathbb{FC} \times \mathbb{FC}$ will be defined in two succeeding steps. First, we will set *a priori* a number of pairs $\langle FC_i,\ FC_m \rangle$ such that the dependency $FC_i =_2 FC_m$ holds. Finally, we will define $=_2$ as the least equivalence relation comprising all the pairs mentioned above and fulfilling the condition

$$FC_i \leq_2 FC_m \wedge FC_m \leq_2 FC_i \Rightarrow FC_i =_2 FC_m.$$

The last definition enables us to put the constraint on the uncertain facts $F_{i1}, F_{i2}, \dots, F_{ik}$ so as they cannot hold simultaneously. It is sufficient to include $\langle F_{i1} \wedge_2 F_{i2} \wedge_2 \cdots \wedge_2 F_{ik}, \bot \rangle$ in the axiomatically formulated set of pairs.

We will say that a conjunction of uncertain facts $FC_i = F_{i1} \wedge_2 F_{i2} \wedge_2 \cdots \wedge_2 F_{ik}$ has a normal form if and only if

$$\forall(1 \leq j1 \leq k)\big(\big(\exists(1 \leq j2 \leq k)(F_{ij2} \leq_1 F_{ij1})\big) \Rightarrow (j1 = j2)\big).$$

Obviously, every conjunction of uncertain facts FC_i can be assigned a conjunction of uncertain facts FC_m in a normal form, which complies with the dependency $FC_i =_2 FC_m$. The set of all conjunctions of uncertain facts in normal form will be denoted by $\mathbb{FC}_{\mathrm{norm}}$, and the appropriate normalization function by fnorm_2: $\mathbb{FC} \rightarrow \mathbb{FC}_{\mathrm{norm}}$.

2.3. Disjunction of Uncertain Facts. In the following step, we will perform the extension of the set of conjunctions of uncertain facts $\mathbb{FC}$ to the set of disjunctions of uncertain facts $\mathbb{FD}$. This new set is defined recurrently as follows:

- if $FC_i \in \mathbb{FC}_{\mathrm{norm}}$, then $FC_i \in \mathbb{FD}$,
- if $FD_i, FD_j \in \mathbb{FD}$, then $FD_i \vee_3 FD_j \in \mathbb{FD}$ (the symbol $\vee_3$ is the connective of the disjunction of uncertain facts),
- no other element belongs to the set $\mathbb{FD}$.

Let FD_1, FD_2, FD_3, etc. denote the particular elements of the infinite set $\mathbb{FD}$. We obtain

$$\mathbb{FD} = \{FD_1, FD_2, FD_3, \dots\}.$$

Consistently, let us define "the same as" relation $=_3$ and the "subsumed by" relation $\leq_3$ on the set $\mathbb{FD}$. Let us make the assumption that the uncertain facts disjunctions $FD_i,\ FD_m$ have, respectively, the following forms: $FD_i = FC_{i1} \vee_3 FC_{i2} \vee_3 \cdots \vee_3 FC_{ik}$ and $FD_m = FC_{m1} \vee_3 FC_{m2} \vee_3 \cdots \vee_3 FC_{mp}$. The relation $\leq_3$: $\mathbb{FD} \times \mathbb{FD}$ will be given the following meaning:

$$FD_i \leq_3 FD_m \Leftrightarrow \big(\forall(1 \leq n \leq p)\exists(1 \leq j \leq k)(FC_{ij} \leq_2 FC_{mn})\big).$$

In order to define the relation $=_3$: $\mathbb{FD} \times \mathbb{FD}$, first we will arbitrarily set a number of pairs $\langle FD_i, FD_m \rangle$ such that $FD_i =_3 FD_m$ holds. Next, we will appoint $=_3$ as the least equivalence relation comprising all the pairs mentioned above and fulfilling the condition

$$FD_i \leq_3 FD_m \wedge FD_m \leq_3 FD_i \Rightarrow FD_i =_3 FD_m.$$

In the light of this definition, we can easily impose the constraint on the uncertain facts conjunctions $FC_{i1}, FC_{i2}, \dots, FC_{ik}$ so as they exhaust the "space of solutions". It is sufficient to include $\langle FC_{i1} \vee_3 FC_{i2} \vee_3 \cdots \vee_3 FC_{ik}, \top \rangle$ in the axiomatically formulated set of pairs.

We can say that the disjunction of uncertain facts $FD_i = FC_{i1} \vee_3 FC_{i2} \vee_3 \cdots \vee_3 FC_{ik}$ has a normal form if and only if

$$\forall(1 \leq j1 \leq k)\big(\big(\exists(1 \leq j2 \leq k)(FC_{ij1} \leq_2 FC_{ij2})\big) \Rightarrow (j1 = j2)\big).$$

Just as in the case of any conjunction of uncertain facts FC_i, each disjunction of uncertain facts FD_i can be assigned a disjunction of uncertain facts FD_m in a normal form which complies with the dependence $FD_i =_3 FD_m$. The set of all uncertain facts disjunctions will be denoted by $\mathbb{FD}_{\text{norm}}$, and the appropriate normalization function by fnorm$_3$: $\mathbb{FD} \rightarrow \mathbb{FD}_{\text{norm}}$.

Since "the same as" relations $=_0$, $=_1$, $=_2$ and $=_3$ are equivalence relations, it can be easily proved that the "subsumed by" relations $\leq_1$, $\leq_2$ and $\leq_3$ are partial order relations like the axiomatically given relation $\leq_0$.

At last, let us define the auxiliary functions of the sum $\cup_3$ and the intersection $\cap_3$ on the set $\mathbb{FD}_{\text{norm}}$. Meeting the former assumptions about the form of the disjunctions of uncertain facts FD_i and FD_m, the sum function $\cup_3$: $\mathbb{FD}_{\text{norm}} \times \mathbb{FD}_{\text{norm}} \rightarrow \mathbb{FD}_{\text{norm}}$ is defined as follows:

$$FD_i \cup_3 FD_m = \text{fnorm}_3(FC_{i1} \vee_3 FC_{i2} \vee_3 \cdots \vee_3 FC_{ik} \vee_3 FC_{m1} \vee_3 FC_{m2} \vee_3 \cdots \vee_3 FC_{mp}),$$

and the intersection function $\cap_3$: $\mathbb{FD}_{\text{norm}} \times \mathbb{FD}_{\text{norm}} \rightarrow \mathbb{FD}_{\text{norm}}$ is defined as

$$\begin{aligned} FD_i \cap_3 FD_m &= \text{fnorm}_3\big(\text{fnorm}_2(FC_{i1} \wedge_2 FC_{m1}) \\ &\quad \vee_3 \cdots \vee_3 \text{fnorm}_2(FC_{i1} \wedge_2 FC_{mp}) \\ &\quad \vee_3 \text{fnorm}_2(FC_{i2} \wedge_2 FC_{m1}) \\ &\quad \vee_3 \cdots \vee_3 \text{fnorm}_2(FC_{i2} \wedge_2 FC_{mp}) \\ &\quad \vee_3 \ldots \text{fnorm}_2(FC_{ik} \wedge_2 FC_{m1}) \\ &\quad \vee_3 \cdots \vee_3 \text{fnorm}_2(FC_{ik} \wedge_2 FC_{mp})\big). \end{aligned}$$

2.4. Rules with Uncertainty. Finally, we shall introduce the main concept of the set $\mathbb{R}$ of rules with uncertainty. To this end, we will use the following recursive definition:

- $\top, \bot \in \mathbb{R}$,
- if $FD_{i1}, FD_{i2}, FD_{i3} \in \mathbb{FD}_{\text{norm}}$, then $(FD_{i1}) \Rightarrow_4 (\bullet FD_{i2}, FD_{i3}) \in \mathbb{R}$,
- if $R_{i1}, R_{i2}, \ldots, R_{ik}, R_{i(k+1)}, R_{i(k+2)} \in \mathbb{R}$, then $(R_{i1}, R_{i2}, \ldots, R_{ik}) \Rightarrow_4 (\bullet R_{i(k+1)}, R_{i(k+2)}) \in \mathbb{R}$ (symbol the $\Rightarrow_4$ stands for the connective of the implication which means the relationship between a set of premises and a pair of conclusions),
- no other element belongs to the set $\mathbb{R}$.

Applying the symbols R_1, R_2, R_3, etc. to denote the uncertain rules, we will obtain the following form of the infinite set $\mathbb{R}$ of rules with uncertainty:

$$\mathbb{R} = \{R_1, R_2, R_3, \ldots\}.$$

What is the semantics of the rules $(FD_{i1}) \Rightarrow_4 (\bullet FD_{i2}, FD_{i3})$ and $(R_{i1}, R_{i2}, \ldots, R_{ik}) \Rightarrow_4 (\bullet R_{i(k+1)}, R_{i(k+2)})$? They tell us, respectively, that

- if the premise FD_{i1} is in the knowledge base, then the old conclusion FD_{i2} should be removed from it (if FD_{i2} exists), and the new conclusion FD_{i3} should be added to this knowledge base,
- if all of the rules $R_{i1}, R_{i2}, \ldots, R_{ik}$ are in the knowledge base, then the rule $R_{i(k+1)}$ should be removed from it (if $R_{i(k+1)}$ exists), and the rule $R_{i(k+2)}$ should be added to this knowledge base.

Let us notice that in the hierarchical process of defining rules we did not use a negation operator. As a consequence, we refer here to the special type of the closed world assumption, called "negation by absence". Such an approach clearly decreases the expressive power of the rules being proposed. On the other hand, however, it enables us to easily express incomplete information. If we assume that a rule's premise has the following form:

$$\langle T_{i1}, p_1\rangle \wedge \langle T_{i2}, p_2\rangle \wedge \cdots \wedge \langle T_{in}, p_n\rangle,$$

then, except for the knowledge about the truth of the facts $T_{i1}, T_{i2}, \ldots, T_{in}$, it expresses the lack of our knowledge about the truthfulness of the facts from the set $\mathbb{T} - \{T_{i1}, T_{i2}, \ldots, T_{in}\}$. Such a situation occurs in many experimental fields, where knowledge is acquired step by step through collecting positive evidence. If necessary, we are able to express the falseness of the fact T_{i0} by completing the premise into the form:

$$\langle T_{i1}, p_1\rangle \wedge \langle T_{i2}, p_2\rangle \wedge \cdots \wedge \langle T_{in}, p_n\rangle \wedge \langle T_{i0}, 0\rangle.$$

Now, on the set $\mathbb{R}$ of rules with uncertainty, we will determine the "the same as" relation $=_4$ and the "subsumed by" relation $\leq_4$. First, the relation $\leq_4$: $\mathbb{R} \times \mathbb{R}$ can be defined recurrently as follows:

- for any rule $R_i \in \mathbb{R}$ there holds $\top \leq_4 R_i$ and $R_i \leq_4 \bot$,
- if $R_i = (FD_{i1}) \Rightarrow_4 (\bullet FD_{i2}, FD_{i3})$ and $R_m = (FD_{m1}) \Rightarrow_4 (\bullet FD_{m2}, FD_{m3})$, then $R_i \leq_4 R_m \Leftrightarrow (FD_{m1} \leq_3 FD_{i1}) \wedge (FD_{m2} \leq_3 FD_{i2}) \wedge (FD_{i3} \leq_3 FD_{m3})$,
- if $R_i = (R_{i1}, R_{i2}, \ldots, R_{ik}) \Rightarrow_4 (\bullet R_{i(k+1)}, R_{i(k+2)})$ and $R_m = (R_{m1}, R_{m2}, \ldots, R_{mp}) \Rightarrow_4 (\bullet R_{m(p+1)}, R_{m(p+2)})$, then $R_i \leq_4 R_m \Leftrightarrow (\forall(1 \leq n \leq p)\exists(1 \leq j \leq k)(R_{mn} \leq_4 R_{ij})) \wedge (R_{m(p+1)} \leq_4 R_{i(k+1)}) \wedge (R_{i(k+2)} \leq_4 R_{m(p+2)})$,

- no other pair of rules R_i, $R_m \in \mathbb{R}$ can be in relation $\leq_4$.

It can be easily proved that $\leq_4$ is a partial ordering.

Assume that the former notation remains valid. The relation $=_4$: $\mathbb{R} \times \mathbb{R}$ is defined as the least equivalence relation, which fulfils the following conditions:

- if $R_i = (FD_{i1}) \Rightarrow_4 (\bullet FD_{i2}, FD_{i3})$ and $FD_{i1} =_3 \bot$, then $R_i =_4 \top$,
- if $R_i = (R_{i1}, R_{i2}, \ldots, R_{ik}) \Rightarrow_4 (\bullet R_{i(k+1)}, R_{i(k+2)})$ and $\exists(1 \leq j \leq k)(R_{ij} =_4 \bot)$, then $R_i =_4 \top$,
- if $R_i = (FD_{i1}) \Rightarrow_4 (\bullet FD_{i2}, FD_{i3})$ and $FD_{i1} =_3 \top$ and $FD_{i3} =_3 \bot$, then $R_i =_4 \bot$,
- if $R_i = (R_{i1}, R_{i2}, \ldots, R_{ik}) \Rightarrow_4 (\bullet R_{i(k+1)}, R_{i(k+2)})$ and $\forall(1 \leq j \leq k)(R_{ij} =_4 \top)$ and $R_{i(k+2)} =_4 \bot$, then $R_i =_4 \bot$,
- if $R_i = (FD_{i1}) \Rightarrow_4 (\bullet FD_{i2}, FD_{i3})$ and $\neg(FD_{i1} =_3 \bot)$
  and $(FD_{i1} \leq_3 FD_{i2})$ and $(FD_{i3} \leq_3 FD_{i1})$,
  then $R_i =_4 \bot$,
- if $R_i = (R_{i1}, R_{i2}, \ldots, R_{ik}) \Rightarrow_4 (\bullet R_{i(k+1)}, R_{i(k+2)})$
  and $\neg\exists(1 \leq j \leq k)(R_{ij} =_3 \bot)$
  and $\forall(1 \leq j \leq k)(R_{ij} \leq_4 R_{i(k+1)})$,
  and $\exists(1 \leq j \leq k)(R_{i(k+2)} \leq_4 R_{ij})$, then $R_i =_4 \bot$,
- if $R_i \leq_4 R_j$ and $R_j \leq_4 R_i$, then $R_i =_4 R_j$.

We say that a rule $R_i \in \mathbb{R}$ has a normal form if and only if:

- $R_i = \top$ or
- $R_i = \bot$ or
- $R_i = (FD_{i1}) \Rightarrow_4 (\bullet FD_{i2}, FD_{i3})$, where FD_{i1}, $FD_{i2}, FD_{i3} \in \mathbb{FD}$, and the following condition is fulfilled:

$$\neg\big((R_i = \top) \vee (R_i = \bot)\big) \wedge$$

$$\neg\big((FD_{i1} \leq_3 FD_{i2}) \wedge (FD_{i3} \leq_3 FD_{i1})\big),$$

  or

- $R_i = (R_{i1}, R_{i2}, \ldots, R_{ik}) \Rightarrow_4 (\bullet R_{i(k+1)}, R_{i(k+2)})$, and the following condition is fulfilled:

$$\neg\big((R_i = \top) \vee (R_i = \bot)\big) \wedge$$

$$\forall(1 \leq j1 \leq k)\big(\big(\exists(1 \leq j2 \leq k)(R_{ij2} \leq_4 R_{ij1})\big)$$

$$\Rightarrow (j1 = j2)\big) \wedge$$

$$\big(\big(\exists(1 \leq j \leq k)\neg(R_{ij} \leq_4 R_{i(k+1)})\big) \vee$$

$$\big(\forall(1 \leq j \leq k)\neg(R_{i(k+2)} \leq_4 R_{ij})\big)\big).$$

Let $\mathbb{R}_{\text{norm}}$ denote the appropriate subset $\mathbb{R}$ containing all the rules which have a normal form. We shall notice that each rule $R_i \in \mathbb{R}$ can be assigned a rule $R_m \in \mathbb{R}_{\text{norm}}$ which fulfils $R_i =_4 R_m$.

To sum up, the recursive method of forming rules with uncertainty reminds us the method of forming formulae used in First Order Logic (FOL). However, let us notice that only four out of the five FOL connectives have their equivalents in the logic of rules with uncertainty, namely, the connective $\wedge$ has its equivalent in the form of the conjunction connective $\wedge_2$, $\vee$ in the form of the disjunction connective $\vee_3$, $\subseteq$ in the form of the subsumption operator $\leq_4$, and $\equiv$ in the form of the equality operator $=_4$. Furthermore, the proposed connectives $\wedge_2$ and $\vee_3$ are not global, but partial functions only.

Instead of the FOL negation connective $\neg$, in the logic of rules with uncertainty we propose a special, three-argument connective $\Rightarrow_4$. If, by means of the negation $\neg f$, one can express the truthfulness of the formula opposite to f, then by means of the rule $(f_1) \Rightarrow_4 (\bullet f_2, f_3)$, with the premise f_1 being true, one can express (except for the truthfulness of f_3) the lack of knowledge about the value of f_2. That is why, in the logic of rules with uncertainty, we can model “negation by absence” only. As a result, this logic is deductively incomplete and its expressive power is smaller than the expressive power of FOL. On the other hand, it offers us a possibility to make use of the notion of ignorance. The rules with uncertainty will be evaluated in the set {true, unknown}, rather than in the set {true, false}.

3. Knowledge Base of a Rule-Based System as a Model for the Logic of Rules with Uncertainty

Using the elements of $\mathbb{R}_{\text{norm}}$, we will represent both facts and ordinary rules, as well as meta-rules from the knowledge base of a rule-based system with uncertainty. To illustrate this claim, let us have a look at the rule-based system RiAD, which can aid prognosticating the effects of a bronchial asthma treatment (Jankowska, 2001; 2004). In the knowledge base of this system, one can find the following elements:

- facts, of temporary nature, informing about the intensity of disease symptoms in a patient with a bronchial asthma,
- ordinary rules, of persistent nature, defining dependencies between disease symptoms plus pharmacotherapy and effects expected after a one-year treatment (Deutsch *et al.*, 2001),
- meta-rules, of persistent nature, defining simple dependencies between ordinary rules.

Consequently, among the facts, the following uncertain fact might be found:

```
(assert (SMITH cough-before lev-c(3))
                              CF 0.8).
```
(5)

It informs us, with confidence `0.8`, that for a patient `SMITH`, before starting the treatment, the frequency of his or her night-time cough was at the level of several times a week (marked with the number `3`). The above fact is a model of the following rule with uncertainty:

$$(\top) \Rightarrow_4 (\bullet\bot, \langle \texttt{(SMITH cough-before lev-c(3))}, 0.8\rangle). \quad (6)$$

Now, as the example of an ordinary rule, we can use the following definition:

```
(defrule prognosis-1
    (X group 3)
    (X cough-before lev-c(3))
    (X wheezing-before lev-w(2))
    (X drugs 3-14)
⇒
    (assert (X cough-after lev-c(2 3))
                              CF 0.8)
    (assert (X pef-after lev-p(1 2))
                         CF 0.9)).      (7)
```

It tells us that in a patient initially classified into the group of chronic asthma of medium progress `(X group 3)`—with a night-time cough frequency of several times a week `(X cough-before lev-c(3))` and with an over wheezing frequency of several times a month `(X wheezing-before lev-w(2))`—after a prolonged (over one year) taking of the combination of drugs identified as 3-14 `(X drugs 3-14)`, the frequency of his or her night-time cough, with confidence 0.8, will decrease to several episodes a month or will stay at the same level `((X cough-after lev-c(2 3)) CF 0.8)`, and the patient's peak expiratory flow PEF will be placed, with confidence 0.9, within the range marked with the symbolic number 1 or within the range marked with the symbolic number 2 `((X pef-after lev-p(1 2)) CF 0.9)`.

The above definition is a model of the following rule with uncertainty:

$$\begin{aligned}&(\langle \texttt{(X group 3)}, 1.0\rangle \wedge_2\\ &\langle \texttt{(X cough-before lev-c(3))}, 1.0\rangle \wedge_2\\ &\langle \texttt{(X wheezing-before lev-w(2))}, 1.0\rangle \wedge_2\\ &\langle \texttt{(X drugs 3-14)}, 1.0\rangle)\\ \Rightarrow_4\\ &(\bullet\bot,\\ &\langle \texttt{(X cough-after lev-c(2 3))}, 0.8\rangle \wedge_2\\ &\langle \texttt{(X pef-after lev-p(1 2))}, 0.9\rangle). \quad (8)\end{aligned}$$

Examples of meta-rules will be considered later, after defining the sum and intersection operations on the $\mathbb{R}_{\text{norm}}$ set.

4. Algebra of Rules with Uncertainty

An essential component of most professional expert systems is a truth maintenance module. We consider a wider notion of the truth maintenance module. We assume that it not only keeps the track of dependencies among the elements of knowledge base, but it is also responsible for the knowledge base correctness and for the credibility of reasoning performed in the system. Such a truth maintenance module can detect the redundancy or inconsistency of information stored in the knowledge base. Optionally, it can also test the knowledge base regarding its completeness.

The contents of the knowledge base change in the course of expert system performance. Therefore, the truth maintenance module should strictly cooperate with the inference engine (Kahney *et al.*, 1989). It would be a smart solution to implement the knowledge base and the truth maintenance module together.

The logic of rules with uncertainty presented above allows modelling all the facts, all the ordinary rules and also some meta-rules from the knowledge base of a rule-based system with uncertainty. To obtain an additional possibility of modelling a high quality truth maintenance module, let us extend our formal system to the algebra. To this end, let us define the functions of the sum $\cup_4$ and the intersection $\cap_4$ of the rules in a normal form.

Accordingly, the sum $\cup_4 : \mathbb{R}_{\text{norm}} \times \mathbb{R}_{\text{norm}} \to \mathbb{R}_{\text{norm}}$ of the rules R_1 and $R_m \in \mathbb{R}_{\text{norm}}$ will be defined recurrently as follows:

- $R_i \cup_4 \top = \top \cup_4 R_i = R_i$,
- $R_i \cup_4 \bot = \bot \cup_4 R_i = \bot$,
- if $R_i = (FD_{i1}) \Rightarrow_4 (\bullet FD_{i2}, FD_{i3})$ and $R_m = (FD_{m1}) \Rightarrow_4 (\bullet FD_{m2}, FD_{m3})$, then $R_i \cup_4 R_m = \text{fnorm}_4(\text{fnorm}_3(FD_{i1} \cup_3 FD_{m1})) \Rightarrow_4 (\bullet\, \text{fnorm}_3(FD_{i2} \cup_3 FD_{m2}), \text{fnorm}_3(FD_{i3} \cap_3 FD_{m3})))$,
- if $R_i = (FD_{i1}) \Rightarrow_4 (\bullet FD_{i2}, FD_{i3})$ and $R_m = (R_{m1}, R_{m2}, \ldots, R_{mp}) \Rightarrow_4 (\bullet R_{m(p+1)}, R_{m(p+2)})$, then $R_i \cup_4 R_m = R_m \cup_4 R_i = \text{fnorm}_4((\top) \Rightarrow_4 (\bullet R_{m(p+1)}, ((FD_{i1}) \Rightarrow_4 (\bullet FD_{i2}, FD_{i3})) \cup_4 R_{m(p+2)}))$,
- if $R_i = (R_{i1}, R_{i2}, \ldots, R_{ik}) \Rightarrow_4 (\bullet R_{i(k+1)}, R_{i(k+2)})$ and $R_m = (R_{m1}, R_{m2}, \ldots, R_{mp}) \Rightarrow_4 (\bullet R_{m(p+1)}, R_{m(p+2)})$, then $R_i \cup_4 R_m = \text{fnorm}_4((R_{i1} \cap_4 R_{m1}, \ldots, R_{i1} \cap_4 R_{mp}, \ldots, R_{ik} \cap_4 R_{m1}, \ldots, R_{ik} \cap_4 R_{mp}) \Rightarrow_4 (\bullet(R_{i(k+1)} \cap_4 R_{m(p+1)}), R_{i(k+2)} \cup_4 R_{m(p+2)}))$.

Likewise, the intersection $\cap_4 : \mathbb{R}_{\text{norm}} \times \mathbb{R}_{\text{norm}} \to \mathbb{R}_{\text{norm}}$ of the rules R_i and $R_m \in \mathbb{R}_{\text{norm}}$ is defined as follows:

- $R_i \cap_4 \top = \top \cap_4 R_i = \top$,
- $R_i \cap_4 \bot = \bot \cap_4 R_i = R_i$,

- if $R_i = (FD_{i1}) \Rightarrow_4 (\bullet FD_{i2}, FD_{i3})$ and $R_m = (FD_{m1}) \Rightarrow_4 (\bullet FD_{m2}, FD_{m3})$, then $R_i \cap_4 R_m = \text{fnorm}_4((\text{fnorm}_3(FD_{i1} \cap_3 FD_{m1})) \Rightarrow_4 (\bullet \, \text{fnorm}_3(FD_{i2} \cap_3 FD_{m2}), \text{fnorm}_3(FD_{i3} \cup_3 FD_{m3})))$,
- if $R_i = (FD_{i1}) \Rightarrow_4 (\bullet FD_{i2}, FD_{i3})$ and $R_m = (R_{m1}, R_{m2}, \ldots, R_{mp}) \Rightarrow_4 (\bullet R_{m(p+1)}, R_{m(p+2)})$, then $R_i \cap_4 R_m = R_m \cap_4 R_i$ $= \text{fnorm}_4((R_{m1}, R_{m2}, \ldots, R_{mp}) \Rightarrow_4 (\bullet\bot, ((FD_{i1}) \Rightarrow_4 (\bullet FD_{i2}, FD_{i3})) \cap_4 R_{m(p+2)}))$,
- if $R_i = (R_{i1}, R_{i2}, \ldots, R_{ik}) \Rightarrow_4 (\bullet R_{i(k+1)}, R_{i(k+2)})$ and $R_m = (R_{m1}, R_{m2}, \ldots, R_{mp}) \Rightarrow_4 (\bullet R_{m(p+1)}, R_{m(p+2)})$, then $R_i \cap_4 R_m = \text{fnorm}_4((R_{i1} \cup_4 R_{m1}, \ldots, R_{i1} \cup_4 R_{mp}, \ldots, R_{ik} \cup_4 R_{m1}, \ldots, R_{ik} \cup_4 R_{mp}) \Rightarrow_4 (\bullet(R_{i(k+1)} \cup_4 R_{m(p+1)}), R_{i(k+2)} \cap_4 R_{m(p+2)}))$.

For the above functions of the sum $\cup_4$ and the intersection $\cap_4$ of the rules in a normal form, the following principles are satisfied:

$$\forall(R_i, R_m \in \mathbb{R}_{\text{norm}})\big((R_i \leq_4 R_i \cup_4 R_m) \wedge (R_m \leq_4 R_i \cup_4 R_m)\big),$$

$$\forall(R_i, R_m \in \mathbb{R}_{\text{norm}})\big((R_i \cap_4 R_m \leq_4 R_i) \wedge (R_i \cap_4 R_m \leq_4 R_m)\big).$$

Moreover, it can be proved that

$$\forall(R_i, R_m \in \mathbb{R}_{\text{norm}})\Big(\big(\exists(R_k \in \mathbb{R}_{\text{norm}})\big((R_i \leq_4 R_k \leq_4 R_i \cup_4 R_m) \wedge (R_m \leq_4 R_k \leq_4 R_i \cup_4 R_m)\big)\big) \Rightarrow \big(R_k =_4 (R_i \cup_4 R_m)\big)\Big)$$

and

$$\forall(R_i, R_m \in \mathbb{R}_{\text{norm}})\Big(\big(\exists(R_k \in \mathbb{R}_{\text{norm}})\big((R_i \cap_4 R_m \leq_4 R_k \leq_4 R_i) \wedge (R_i \cap_4 R_m \leq_4 R_k \leq_4 R_m)\big)\big) \Rightarrow \big(R_k =_4 (R_i \cap_4 R_m)\big)\Big).$$

From these principles we can draw some further auxiliary conclusions:

$$\forall(R_i, R_m \in \mathbb{R}_{\text{norm}})((R_i \cup_4 R_m) =_4 \sup\{R_i, R_m\})$$

and

$$\forall(R_i, R_m \in \mathbb{R}_{\text{norm}})((R_i \cap_4 R_m) =_4 \inf\{R_i, R_m\}),$$

and the final conclusion: the algebra $\mathbf{R} = (\mathbb{R}_{\text{norm}}, \cup_4, \cap_4)$ is a lattice.

The importance of the above conclusion is not diminished by the fact that the lattice $\mathbf{R}$ is determined on the set $\mathbb{R}_{\text{norm}}$, and not on the full set $\mathbb{R}$. Consider the rule $R = (r\text{-}p) \Rightarrow_4 (\bullet r\text{-}c_1, r\text{-}c_2)$. If the $r\text{-}c_2$ conclusion (being inserted) is "subsumed by" the $r\text{-}p$ premise and this premise is "subsumed by" the $r\text{-}c_2$ conclusion (being removed), then the rule R does not represent any value at all and it should be disregarded in the process of reasoning.

5. Truth Maintenance Module as a Model for the Algebra of Rules with Uncertainty

On the basis of the algebra $\mathbf{R} = (\mathbb{R}_{\text{norm}}, \cup_4, \cap_4)$, we can model a truth maintenance module of the rule-based system with uncertainty. Using the elements of the set $\mathbb{R}_{\text{norm}}$ and the functions of the sum $\cup_4$ and the intersection $\cap_4$, we will represent both facts and ordinary rules, as well as various meta-rules. The last ones, which specify the semantic constraints to be fulfilled by the facts and ordinary rules, form the truth maintenance module.

Therefore, any fact $f\text{-}kb$, after bringing it to the form of the disjunction of uncertain facts FD_{kb}, can be presented as the simple rule $(\top) \Rightarrow_4 (\bullet\bot, FD_{kb})$. Then the ordinary rule $r\text{-}kb = (p\text{-}kb) \Rightarrow (c1\text{-}kb, \neg c2\text{-}kb)$ will be given the form $(FD_{pkb}) \Rightarrow_4 (\bullet FD_{c1kb}, FD_{c2kb})$, in which FD_{pkb} is a formal representation of the $p\text{-}kb$ premises, whereas FD_{c1kb} and FD_{c2kb} are formal representations of the $c1\text{-}kb$ and the $c2\text{-}kb$ conclusions, respectively. The meta-rule $mr\text{-}kb = (mp\text{-}kb) \Rightarrow (mc1\text{-}kb, \neg mc2\text{-}kb)$ will be presented as the complex rule of the form $(R_{mp1kb}, R_{mp2kb}, \ldots, R_{mpkkb}) \Rightarrow_4 (\bullet R_{mc1kb}, R_{mc2kb})$ where $R_{mp1kb}, R_{mp2kb}, \ldots, R_{mpkkb}$ are formal representations of the rule's premises $mp\text{-}kb$, while R_{mc1kb} and R_{mc2kb} are formal representations of the rule conclusions $mc1\text{-}kb$ and $mc2\text{-}kb$, respectively.

Any proper subset $\mathbb{KB}$ of the set $\mathbb{R}_{\text{norm}}$ can represent a knowledge base of the expert system with uncertainty. Examining the $=_4$ and $\leq_4$ relations between elements of the set $\mathbb{KB}$, we can significantly improve the quality of this knowledge base.

For instance, finding any rules R_i and R_m with uncertainty which fulfil the relation $R_i \leq_4 R_m$ in the $\mathbb{KB}$ constitutes a pretext to delete the rule R_i from this set (and its equivalent from the knowledge base) because the knowledge represented by the rule R_i is also "included" in the rule R_m.

Next, the relation $=_4$ can be used to examine the consistency of the set of rules which create the $\mathbb{KB}$ knowledge base. If for any ordinary rules $R_{kb1}, R_{kb2} \in \mathbb{KB}$ there holds $R_{kb1} \cap_4 R_{kb2} =_4 (FD_{kb12}) \Rightarrow_4 (\bullet\bot, \top) =_4 \bot$, then these rules are contradictory to each other and they must not occur simultaneously in the $\mathbb{KB}$ set. Also, two any meta-rules $R_{mkb1}, R_{mkb2} \in \mathbb{KB}$, for which there holds $R_{mkb1} \cap_4 R_{mkb2} =_4 (R_{kb1}, R_{kb2}, \ldots, R_{kbk}) \Rightarrow_4 (\bullet\bot, \top) =_4 \bot$, are contradictory to each other. In case the $\mathbb{KB}$ does not have the feature of internal consistency, it should undergo a modification which would cause the recovery of this feature.

Using the algebra $\mathbf{R} = (\mathbb{R}_{\mathrm{norm}}, \cup_4, \cap_4)$, we can also examine other features of the $\mathbb{KB}$ knowledge base, for example, its completeness.

To illustrate the above deliberations, let us analyze the possibility of using the algebra $\mathbf{R}$ in designing and maintaining the truth maintenance module for the knowledge base of the system RiAD. Let us assume that in this knowledge base, apart from the fact (5) and the rule (7) defined above, there are the fact and the rule presented below:

```
(assert (SMITH cough-before lev-c(2 3))
                                  CF 0.7), (9)
```

```
(defrule prognosis-2
    (X group 3)
    (X cough-before lev-c(3))
    (X drugs 3-14)
⇒
    (assert (X cough-after lev-c(2))
                                  CF 0.8)
    (assert (X pef-after lev-p(1 2))
                             CF 0.95)).  (10)
```

Let us further assume that also the meta-fact (11) belongs to the RiAD knowledge base:

```
(assertmf if-then0
      ((X cough-before lev-c(Y Z)),
       (X cough-before lev-c(Y)))).  (11)
```

This meta-fact, which is a model of the following dependency from the logic of rules with uncertainty:

$$\big((\top) \Rightarrow_4 (\bullet\bot, \langle \texttt{(X cough-before (Y Z))}, 1.0\rangle)\big) \leq_4$$
$$\big((\top) \Rightarrow_4 (\bullet\bot, \langle \texttt{(X cough-before (Y))}, 1.0\rangle)\big), \quad (12)$$

forms an `if-then0` relation between the fact stating that a cough frequency of a person `X` before starting the treatment is set on the `Y` or `Z` level, and the fact stating that a cough frequency of this person is set right on the `Y` level.

As a result of the simultaneous occurrence of (5), (9) and (11) in the knowledge base, we can come to the conclusion that the fact (9) has a redundant character.

A similar situation will be observed in the case of the ordinary rules (7) and (10). If we assume that the meta-facts analogous to (11) are also in force for other measurable symptoms of the bronchial asthma disease (`wheezing-before`, `cough-after`, `pef-after`), we can easily deduce that the rule (7) is in the `if-then4` relation with the rule (10) (with weaker premises and a stronger conclusion).

What shall we do with the fact (9) and the rule (7), which are obviously redundant ones in the knowledge base? They must be removed from it!

In order to cope with their removal, it is sufficient to activate the meta-rule `subsumption-1`, defined by means of the extending construct `defrulem`:

```
(defrulem subsumption-1
    (rule R2)
    (if-then4(R1,R2))
    (not(if-and-only-if4(R1,R2)))
⇒
    (undefrule R1)),                    (13)
```

where `if-then4` and `if-and-only-if4` are predicate functions implementing the relations $\leq_4$ and $=_4$, respectively. The performance of this meta-rule will result in removing from the knowledge base each rule `R1` for which we can find any rule `R2` that is different from it and fulfils the dependency ‘`R1` is “subsumed by” `R2`’.

The meta-rule `subsumption-1` is a model for the following complex rule from the logic of rules with uncertainty:

$$((((R_2) \Rightarrow_4 (\bullet\bot, R_1)) \Rightarrow_4 (\bullet\bot, \top)), R_2,$$
$$(R_1) \Rightarrow_4 (\bullet\bot, R_2)) \Rightarrow_4 (\bullet R_1, \top). \quad (14)$$

Let us analyse its contents. Assuming that the relation $R_1 \leq_4 R_2$ holds, we obtain $((R_2) \Rightarrow_4 (\bullet\bot, R_1)) =_4 \bot$. In consequence, the first premise $(((R_2) \Rightarrow_4 (\bullet\bot, R_1)) \Rightarrow_4 (\bullet\bot, \top))$ is “the same as” $\top$. And, if the rule R_2 is in the knowledge base and the relation $R_1 =_4 R_2$ does not hold (which is guaranteed by fulfilling the third premise $(R_1) \Rightarrow_4 (\bullet\bot, R_2)$), then the rule R_1 (if it exists) will be removed from it.

In the opposite case, when the relation $R_1 \leq_4 R_2$ does not hold, the first premise $(((R_2) \Rightarrow_4 (\bullet\bot, R_1)) \Rightarrow_4 (\bullet\bot, \top))$ is “the same as” $\bot$. Similarly, fulfilling simultaneously both $R_1 \leq_4 R_2$ and $R_2 \leq_4 R_1$ (equivalent to $R_1 =_4 R_2$) makes the third premise $(R_1) \Rightarrow_4 (\bullet\bot, R_2)$ “the same as” $\bot$. If so, the complex rule considered will not become active.

Now let us investigate a new situation in which the meta-fact (15) co-occurs with the rules (16) and (17) in the RiAD knowledge base. The meta-fact asserts the presence of four separate levels of respiratory efficiency (marked with natural numbers `1`, `2`, `3`, `4`):

```
(assertmf if-and-only-if0 ((X pef-after
            lev-p(1 2 3 4)), true)).  (15)
```

Traditionally, the rules define the dependencies between the patient's states of health before and after the treatment:

```
(defrule prognosis-3
    (X group 3)
    (X cough-before lev-c(3 4))
    (X wheezing-before lev-w(3))
    (X drugs 3-14)
⇒
    (assert (X pef-after
              lev-p(2 3 4))),   (16)
```

```
(defrule prognosis-4
    (X group 3)
    (X cough-before lev-c(3))
    (X drugs 3-14)
⇒
    (assert (X pef-after
              lev-p(1 2 3))).   (17)
```

Let us assume that also the meta-fact (18) is in the RiAD knowledge base:

```
(assertmf if-and-only-if0
    ((X pef-after lev-p(Y Z)),
    or3((X pef-after lev-p(Y)),
    (X pef-after lev-p(Z))))).   (18)
```

This meta-fact is a model for the obvious dependency (19) from the logic of rules with uncertainty:

$$((\top) \Rightarrow_4 (\bullet\bot, \langle(\texttt{X pef-after lev-p(Y Z)}), 1.0\rangle)) =_4 ((\top) \Rightarrow_4 (\bullet\bot, \langle(\texttt{X pef-after lev-p(Y)}), 1.0\rangle \vee_3 \langle(\texttt{X pef-after lev-p(Z)}), 1.0\rangle)). \quad (19)$$

(Let us observe that the connective $\vee_3$—unlike $\wedge_2$—has in the RiAD its counterpart in the form of the function `or3`.)

As regards the rules (16) and (17) and the meta-fact (18), we obtain

```
intersect4(prognosis-3,prognosis-4) =
    (X group 3)
    (X cough-before lev-c(3))
    (X wheezing-before lev-w(3))
    (X drugs 3-14)
⇒
    (assert (X pef-after
              lev-p(1 2 3 4))),   (20)
```

where the `intersect4` function from RiAD is equivalent to the $\cap_4$ operator from the algebra of rules with uncertainty. In the light of the meta-fact (15), the intersection of the rules (16) and (17) equals `false`.

Apparently, this seems to be unlikely! However, after having a closer look at it, we can observe a contradiction between the rules (16) and (17). It results from imprecisely recognised levels of the respiratory efficiency `pef-after`. Most likely, the extreme levels `1` (the rule (17)) or `4` (the rule (16)) have been added to increase the "system safety". Therefore, both rules require some polishing. If not, the knowledge discovered in the reasoning process may turn out to be of no value!

Polishing the rules must be done with the assistance of a user. Obviously, the contradiction can be resolved by removing one of the two rules. However, if this is the case, the user must decide which rule is to be preserved in the knowledge base and which is to be removed from it. In the light of this comment, we think it is reasonable to design a meta-rule that removes, after the conflict has been detected, both rules from the knowledge base, and lets the user know about this modification. Such a meta-rule might have the following form:

```
(defrulem contradiction-1
    (rule R1)
    (rule R2)
    (if-and-only-if4(intersect4(R1,R2),
                                  false))
⇒
    (undefrule intersect4(R1,R2)).   (21)
```

Formally, this meta-rule performance results in removing from the knowledge base each rule `intersect4(R1, R2)`—obtained from a pair of the rules `R1` and `R2` from this base—which is "the same as" `false`. In fact, the knowledge base cannot contain any rule equal to `false`. Thus, the meta-rule performance consists in removing from the knowledge base all such rules `Ri` that fulfil the condition: `Ri` is "subsumed by" `intersection(R1, R2)`, first of all—the rules `R1` and `R2`. The recursive process of removing the rules will be performed by a special, independently designed meta-rule, which is initiated after entering the command `undefrule`.

The meta-rule (21) is a model for the following rule with uncertainty:

$$((R_1 \cap_4 R_2) \Rightarrow_4 (\bullet\bot, \top), R_1, R_2) \Rightarrow_4 (\bullet(R_1 \cap_4 R_2), \top). \quad (22)$$

We shall notice that the complex premise $(R_1 \cap_4 R_2) \Rightarrow_4 (\bullet\bot, \top)$ will be fulfilled if and only if $R_1 \cap_4 R_2 =_4 \bot$.

6. Final Conclusions

The knowledge base of a rule-based system, including a rule-based system with uncertainty, has a heterogeneous character from the point of view of a knowledge engineer. On the one hand, facts and ordinary rules with the contents defined by an expert in the field are placed in the base. The knowledge engineer is responsible for transforming the informal records into the form required by the inference engine.

On the other hand, a high quality knowledge base should be equipped with a truth maintenance module. It is a set of special meta-rules specifying the dependencies which ordinary rules must satisfy. This set is defined by the knowledge engineer alone on the basis of both the facts and ordinary rules, as well as reasoning algorithms.

Designing and maintaining the expert system knowledge base, especially the one with uncertainty, constitutes a difficult task (Jankowska, 2005; Santos and Santos, 1999). Taking this into account, we had an idea to build a tool which would aid the knowledge engineer to fulfil the task with a reference to rule-based systems.

The algebra of rules with uncertainty $\mathbf{R} = (\mathbb{R}_{\mathrm{norm}}, \cup_4, \cap_4)$ presented in this paper gives us some theoretical basis to build such a tool. The algebra, being a lattice, creates numerous possibilities to examine the facts, ordinary rules and meta-rules which are collected in the knowledge base of the system with uncertainty. Our paper shows these possibilities with reference to the knowledge base of the system prognosticating the progress in a bronchial asthma treatment.

The aim of the tool hereby presented is to aid designing a correct knowledge base. Base correctness must be verified both at the initial stage, while implementing the system, and later, while the system is already in action. That is why the tool should be used during system performance, too. To this end, all the simple and compound rules forming the tool must take part in the reasoning processes together with the system's ordinary rules. If necessary, the tool will influence the state of the knowledge base by itself.

Both truth maintenance module design and its implementation are a challenging task. Considering that the module is (in a sense) a supplement to the knowledge base, we should implement them jointly, in the programming language predestined to facilitate expert systems development (e.g. FuzzyCLIPS). The difficulty is that most of such languages cannot manage meta-rules. To ensure meta-rules management, it will be necessary to extend the set of the language constructs (e.g. `assertmf`, `defrulem`).

Moreover, we shall acknowledge the fact that implementing a full version of the presented truth maintenance module will obviously slow down the reasoning processes. The full test has time complexity $O(n^2)$, where n is the number of all rules (facts, ordinary rules and meta-rules) kept in the knowledge base. Taking this into account, instead of operating the module permanently, we should consider the possibility of turning it periodically on and off. For example, it can be turned on every time after a fixed, imposed number of reasoning steps, or its operating can be conditioned by changes in the knowledge base. In the designed system, in order to prognosticate the progress in a bronchial asthma treatment, we try to apply the first solution.

Summarizing, the tool presented here offers us a possibility to handle all the elements of the heterogeneous knowledge base of a rule-based system with uncertainty in a similar manner. Thanks to this possibility, we can comprehend well the inference engine performance. Moreover, this tool takes care of the knowledge base quality, not only while it is being designed, but also when it is already maintained. The consequence of implementing the tool is slowing down the whole system performance. However, in the case of expert systems which do not work in real time, this cost cannot be particularly important.

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