Int. J. Appl. Math. Comput. Sci., 2006, Vol. 16, No. 4, 441–462

ROBUST STABILIZATION OF DISCRETE LINEAR REPETITIVE PROCESSES WITH SWITCHED DYNAMICS

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Repetitive processes constitute a distinct class of 2D systems, i.e., systems characterized by information propagation in two independent directions, which are interesting in both theory and applications. They cannot be controlled by a direct extension of the existing techniques from either standard (termed 1D here) or 2D systems theories. Here we give new results on the design of physically based control laws. These results are for a sub-class of discrete linear repetitive processes with switched dynamics in both independent directions of information propagation.

Keywords: repetitive processes, $2D$ systems, switched dynamics, stabilization, uncertainty

1. Introduction

The essential unique characteristic of a repetitive process (also termed a multipass process in the early literature) can be illustrated by considering machining operations where the material or workpiece involved is processed by a series of sweeps, or passes, of the processing tool. Assuming the pass length $\alpha < +\infty$ to be constant, the output vector, or pass profile, $y_k(p)$, $p = 0, 1, \ldots, (\alpha - 1)$ (p being the independent spatial or temporal variable), generated on the pass k acts as a forcing function on, and hence contributes to, the dynamics of the new pass profile $y_{k+1}(p)$, $p = 0, 1, \ldots, (\alpha - 1)$, $k = 0, 1, \ldots$.

Industrial examples of repetitive processes include long-wall coal cutting operations and metal rolling operations (see the original papers cited in, e.g., (Rogers and Owens, 1992)). Also, a number of the so-called algorithmic examples exist where adopting a repetitive process setting for analysis has clear advantages over alternative approaches to systems related analysis. These include iterative learning control schemes (Amann *et al.*, 1998; Longman, 2003) and iterative solution algorithms for dynamic nonlinear optimal control problems based on the maximum principle (Roberts, 2002). In the former case, the sub-classes of the so-called differential and discrete linear repetitive processes form the basis for a rigorous analysis of a powerful class of such algorithms. In the latter, the repetitive process setting for analysis has led to the development of numerically reliable and computationally efficient solution algorithms.

Another possible area of application for repetitive processes regards the so-called spatially interconnected systems, which have already found numerous important physical applications, see, e.g., (D'Andrea and Dullerud, 2003) and the references therein. This arises from the fact that some of the state-space models in this area can be rewritten in the form of those for certain classes of discrete linear repetitive processes.

The unique control problem for these processes is that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass to pass (i.e., k) direction. Such behaviour is easily generated in simulation studies and in experiments on scaled models of industrial processes such as long-wall coal cutting. In this particular case, these oscillations are caused by the machine's weight as it comes to rest on the newly cut floor profile ready for the start of the next pass of the coal face.

The fact that the pass length is finite (and hence information in this direction only occurs over a finite duration) is a key difference with other classes of 2D discrete linear systems. Another is the fact that the pass initial conditions are reset before the start of each new pass and these can include explicit contributions from the previous pass. This feature has no 2D discrete linear systems counterpart, and overall large parts of the established systems theory for 2D discrete linear systems described by, in particular, the Roesser and Fornasini Marchesini state-space models either cannot be applied at all or only after appropriate modifications have been made. Hence there is a need to develop a systems theory for these processes for onward translation (where appropriate) into numerically reliable algorithms.

A rigorous stability theory for linear repetitive processes has been developed. This theory (Rogers and Owens, 1992) is based on an abstract model in a Banach space setting which includes a wide range of such processes as special cases. Also, the results of applying this theory to a broad range of cases have been reported. This has resulted in stability tests for some sub-classes of practical interest that can, if desired, be implemented by a direct application of well known 1D linear systems tests. This theory consists of two distinct concepts termed asymptotic stability and stability along the pass, where the former is a necessary condition for the latter.

Much of the work currently available on repetitive processes has focused on the definitions and characterizations of systems theoretic properties, but recently the design of control schemes has become an active research area. For example, it is physically meaningful to define the current pass error as the difference, at each point along the pass, between a specified reference trajectory for that pass, which in most cases will be the same on each pass, and the actual pass profile produced. Then it is possible to define the so-called current pass error actuated controller which uses the generated error vector to construct the current pass control input vector. In this context, preliminary work, see, e.g., (Benton, 2000), has shown that, except in a few very restrictive special cases, the controller used must be actuated by a combination of the current pass information and 'feedforward' information from the previous pass to guarantee even stability along the pass closed-loop. (Note also that in the iterative learning control application area the previous pass (or trial) output is an obvious signal to use as feedforward action in the overall control law.) Design algorithms for such control laws applied to discrete linear repetitive processes can be found in, e.g., (Gałkowski *et al.*, 2002).

Consider again the metal rolling operation. Here a number of passes may be completed under one regime and then the dynamics change to allow further processing to take place. One way of modelling such a case is by switching the dynamics from one state-space model to an alternative (or alternatives). More generally, there are (at least) two distinct possibilities for switching dynamics to occur in repetitive processes; either the switching occurs from pass to pass or along a pass. This paper continues the development of tools for the analysis of these two cases. Both of these are practically motivated, e.g., switching from pass to pass can occur when handling multiple operation robot arms or multiple metal rolling systems, and along the pass switching can arise in the analysis of iterative learning control applied to processes with periodic dynamics.

The previous work (Bochniak *et al.*, 2006) developed significant results in the areas of applicable stability tests and the design of control laws activated by information measured on the current and previous passes. This assumes that there is no uncertainty associated with the models used to model the dynamics but there will clearly be cases when this is not true, even for initial control related analysis. Hence, in this paper, the analysis of repetitive processes with switched dynamics in the presence of uncertainty in the models used is begun. As in other areas, the basic route is to assume that the uncertainty can be described by polytopic or norm bounded structures. The resulting design algorithms can be computed using Linear Matrix Inequalities (LMIs), and two examples are given to highlight such computations and, in particular, to show that they can be numerically reliable.

Throughout this paper, the null matrix and the identity matrix with the required dimensions are denoted by 0 and I, respectively. Moreover, $M > 0$ (< 0) denotes a real symmetric positive (negative) definite matrix.

2. Background

The basic form of the state-space model for discrete linear repetitive processes is given by

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + B_0 y_k(p), \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + D_0 y_k(p), \\ & p = 0, 1, \dots, (\alpha - 1), \quad k = 0, 1, \dots \end{aligned} \tag{1}$$

Here, on the pass k, $x_k(p)$ is the $n \times 1$ state vector, $y_k(p)$ is the $m \times 1$ pass profile vector, and $u_k(p)$ is the $r \times 1$ control input vector. To complete the process description, it is necessary to specify the boundary conditions, i.e., the initial state vector on each pass and the initial pass profile. Here no loss of generality arises from assuming $x_{k+1}(0) = d_{k+1},\ k \geq 0$, and $y_0(p) = f(p)$, where the

$n \times 1$ vector d_{k+1} has known constant entries and $f(p)$ is an $m \times 1$ vector whose entries are known functions of p.

The abstract model based stability theory (Rogers and Owens, 1992) for linear repetitive processes consists of two distinct concepts termed asymptotic stability and stability along the pass. Noting again the unique control problem for these processes, this theory demands that bounded sequences of inputs produce bounded sequences of pass profiles, where 'bounded' is defined in terms of the norm on the underlying function space. The essential difference between them is that asymptotic stability demands this property over the finite pass length whereas stability along the pass is stronger in that it demands this property uniformly, i.e., independent of the pass length.

In the case of processes described by (1), it can be shown (Rogers and Owens, 1992) that asymptotic stability holds if, and only if, $\mathbf{r}(D_0) \le 1$, where $\mathbf{r}(\cdot)$ denotes the spectral radius of its matrix argument. Also, if the example under consideration is asymptotically stable and the control input sequence applied $\{u_k\}_{k\ge1}$ converges strongly to u_∞ as $k \to \infty$, then the resulting output pass profile sequence $\{y_k\}_{k\ge1}$ converges strongly to y_∞—the so-called limit profile—defined (with $D = 0$ for ease of presentation) over $p = 0, 1, \ldots, (\alpha-1)$, by

$$\begin{aligned} x_\infty(p+1) &= (A + B_0(I-D_0)^{-1}C)x_\infty(p) \\ &\quad + Bu_\infty(p), \\ y_\infty(p) &= (I-D_0)^{-1}Cx_\infty(p). \end{aligned}$$

In effect, this result states that if a process is asymptotically stable then its repetitive dynamics can, after a 'sufficiently large' number of passes, be replaced by those of a 1D discrete linear system. Note, however, that this property does not guarantee that the limit profile is stable in the 1D linear systems sense, i.e., $\mathbf{r}\left(A + B_0(I_m - D_0)^{-1}C\right) \le 1$—a point which is easily illustrated by, e.g., the case when $A = -0.5, B = 0$, $B_0 = 0.5 + b_0, C = 1, D = D_0 = 0$, and the real scalar b_0 is chosen such that $|b_0| \ge 1$.

Stability along the pass prevents cases such as the simple example above from arising (by demanding that the bound is independent of the pass length), and the following characterization is known (Rogers and Owens, 1992).

Theorem 1. *A discrete linear repetitive process described by (1) is stable along the pass if and only if the so-called 2D characteristic polynomial*

$$\mathcal{C}(z_1, z_2) := \det \begin{bmatrix} I - z_1 A & -z_1 B_0 \\ -z_2 C & I - z_2 D_0 \end{bmatrix} \ne 0 \text{ in } \overline{U}^2, \tag{2}$$

where $\overline{U}^2 = \{(z_1, z_2) : |z_1| \le 1, |z_2| \le 1\}$.

In theory, a repetitive process evolves over a semi-infinite strip in the positive quadrant of the 2D domain, i.e., over $p = 0, 1, \ldots, (\alpha-1)$, $k \ge 0$. Stability along the pass, however, treats the process as evolving over the complete positive quadrant, i.e., both p and k are of unbounded duration. For this reason, stability along the pass can be too strong in some cases of practical interest—see, e.g., (Smyth, 1992) for a further discussion of this point and illustrative examples.

Previous works, see, e.g. (Gałkowski *et al.*, 2002), have used an LMI setting to design control laws of the following form for $p = 0, 1, \ldots, (\alpha-1)$, $k = 0, 1, \ldots$ for processes described by (1):

$$\begin{aligned} u_{k+1}(p) &= K_1 x_{k+1}(p) + K_2 y_k(p) \\ &= \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x_{k+1}(p) \\ y_k(p) \end{bmatrix}, \end{aligned} \tag{3}$$

where K_1 and K_2 are appropriately dimensioned matrices to be designed. In effect, this control law uses the feedback of the current state vector (which is assumed to be available for use) and the 'feedforward' of the previous pass profile vector. Note that in repetitive processes the term 'feedforward' is used to describe the case where state or pass profile information from the previous pass (or passes) is used as (part of) the input to a control law used on the current pass, i.e., to information which is propagated in the pass to pass (k) direction.

Next we give some well known results which will be extensively used in the analysis of this paper.

Lemma 1 (Schur's complement). (Gałkowski *et al.*, 2002; Boyd *et al.*, 1994) *Let $\mathbb{W}$, $\mathbb{L}$ and $\mathbb{V}$ be given matrices of appropriate dimensions with $\mathbb{W} = \mathbb{W}^T$, $\mathbb{V} > 0$. Then the matrix inequality*

$$\mathbb{W} + \mathbb{L}^T \mathbb{V} \mathbb{L} < 0 \tag{4}$$

holds if, and only if,

$$\begin{bmatrix} \mathbb{W} & \mathbb{L}^T \\ \mathbb{L} & -\mathbb{V}^{-1} \end{bmatrix} < 0. \tag{5}$$

Lemma 2. (Bachelier *et al.*, 1999) *Let $\mathbb{W}$, $\mathbb{L}$ and $\mathbb{V}$ be given matrices of appropriate dimensions with $\mathbb{W} = \mathbb{W}^T$, $\mathbb{V} > 0$. Then the matrix inequality (4) holds if, and only if,*

$$\begin{bmatrix} \mathbb{W} & \mathbb{L}^T G \\ G^T \mathbb{L} & \mathbb{V} - G - G^T \end{bmatrix} < 0, \tag{6}$$

where G is an arbitrary matrix of the same dimensions as $\mathbb{V}$.

Lemma 3. *(Du and Xie, 1999) Let Σ and F be known and unknown real matrices of appropriate dimensions, respectively, where F satisfies $\|F\| \leq 1$, i.e., $F^T F \leq I$. Then, for any scalar $\varepsilon > 0$, the matrix inequality*

$$\Sigma^T \begin{bmatrix} 0 & F \\ F^T & 0 \end{bmatrix} \Sigma \leq \Sigma^T \begin{bmatrix} \varepsilon I & 0 \\ 0 & \varepsilon^{-1} I \end{bmatrix} \Sigma \tag{7}$$

holds.

3. Processes with Dynamics Switched from Pass to Pass

All of the work which has been reported on the analysis and control of discrete repetitive processes either assumes that, once the state-space model is obtained, it remains fixed for the complete duration of passes, or (more recently) that it is subject to well defined uncertainty structures. In some cases, however, a more realistic scenario is that there are a number of regimes of operation, each of which has a state-space model description, and the process switches between them according to some given schedule. For example, in metal rolling it may be required to pass the workpiece through a series of passes which are described by different state-space models, i.e., complete a number of passes with one model in place and then switch to complete another number of passes described by a different model, and so on.

The dynamics in the scenario described above switch in the pass to pass direction and we now summarize some previously obtained (Bochniak *et al.*, 2006) relevant results on this case, where it is assumed that the process dynamics are described as follows for $p = 0, 1, \ldots, (\alpha - 1)$:

$$\begin{aligned} x_{l+1}(p+1) &= \begin{cases} A_1 x_{l+1}(p) + B_1 u_{l+1}(p) + B_{01} y_l(p), \\ \quad \text{for } l = 0, 2, \ldots, \text{ i.e., } l = 2k, \\ A_2 x_{l+1}(p) + B_2 u_{l+1}(p) + B_{02} y_l(p), \\ \quad \text{for } l = 1, 3, \ldots, \text{ i.e., } l = 2k+1, \end{cases} \\ y_{l+1}(p) &= \begin{cases} C_1 x_{l+1}(p) + D_1 u_{l+1}(p) + D_{01} y_l(p), \\ \quad \text{for } l = 0, 2, \ldots, \text{ i.e., } l = 2k, \\ C_2 x_{l+1}(p) + D_2 u_{l+1}(p) + D_{02} y_l(p), \\ \quad \text{for } l = 1, 3, \ldots, \text{ i.e., } l = 2k+1, \end{cases} \end{aligned} \tag{8}$$

where $x_l(p)$ is the $n \times 1$ state vector, $y_l(p)$ is the $m \times 1$ pass profile vector, and $u_l(p)$ is the $r \times 1$ control input vector. The boundary conditions are defined as for processes described by (1).

This model assumes that the dynamics switch on the completion of each pass profile. This is clearly not the most general case but, given the absence of any previous results in this area, it will act as a starting point with the possibility that the experience gained will lead to straightforward generalizations to other cases.

One obvious approach to the analysis of the process model given above is to attempt to transform it into an equivalent model of the form (1) and then directly apply the existing results. Introduce, therefore, the following new state, pass profile and input vectors:

$$X_{l+1}(p) = \begin{bmatrix} x_{2k+1}(p) \\ x_{2k+2}(p) \end{bmatrix}, \quad U_{l+1}(p) = \begin{bmatrix} u_{2k+1}(p) \\ u_{2k+2}(p) \end{bmatrix},$$

$$Y_l(p) = y_{2k}(p), \qquad Y_{l+1}(p) = y_{2k+2}(p).$$

Then the equivalent model of the form (1) for processes described by (8) is

$$\begin{aligned} X_{l+1}(p+1) &= \widehat{A} X_{l+1}(p) + \widehat{B} U_{l+1}(p) + \widehat{B}_0 Y_l(p), \\ Y_{l+1}(p) &= \widehat{C} X_{l+1}(p) + \widehat{D} U_{l+1}(p) + \widehat{D}_0 Y_l(p), \end{aligned} \tag{9}$$

where

$$\widehat{A} = \begin{bmatrix} A_1 & 0 \\ B_{02} C_1 & A_2 \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} B_1 & 0 \\ B_{02} D_1 & B_2 \end{bmatrix},$$

$$\widehat{B}_0 = \begin{bmatrix} B_{01} \\ B_{02} D_{01} \end{bmatrix}, \quad \widehat{C} = \begin{bmatrix} D_{02} C_1 & C_2 \end{bmatrix},$$

$$\widehat{D} = \begin{bmatrix} D_{02} D_1 & D_2 \end{bmatrix}, \quad \widehat{D}_0 = D_{02} D_{01}.$$

Now it is possible to give the conditions for stability along the pass of processes described by (8). Of the numerous sets of conditions which have been developed, the most relevant here is the following one (see also (Gałkowski *et al.*, 2002)) based on the use of LMIs. This condition is a sufficient one but, unlike necessary and sufficient alternatives, it leads easily to control law design algorithms (as shown below). As a preliminary step, it is convenient to introduce the following matrices relating to (9):

$$\widehat{\Phi} = \begin{bmatrix} \widehat{A} & \widehat{B}_0 \\ \widehat{C} & \widehat{D}_0 \end{bmatrix}, \quad \widehat{\Pi} = \begin{bmatrix} \widehat{B} \\ \widehat{D} \end{bmatrix}. \tag{10}$$

The matrices $\widehat{\Phi}$ and $\widehat{\Pi}$ are termed the augmented process matrix and the augmented input matrix, respectively.

The following sufficient condition for stability along the pass of processes described by (9) can now be stated.

Theorem 2. (Rogers and Owens, 1992; Gałkowski *et al.*, 2002) *A discrete linear repetitive process whose state-space model can be written in the form (9) is stable along the pass if there exist matrices $W_1 > 0$ and $W_2 > 0$ such that*

$$\widehat{\Phi}^T W \widehat{\Phi} - W < 0 \tag{11}$$

holds, where $W = W_1 \oplus W_2$, with $\oplus$ denoting the direct sum, i.e., $W = \mathbf{diag}\,(W_1, W_2)$.

Note also that (11) is (one formulation of) the so-called 2D Lyapunov equation for these processes (Gałkowski *et al.*, 2002).

To apply a control action to these processes of the above form, consider a switched control law of the form

$$u_{l+1}(p) = \begin{cases} K_1^1 x_{l+1}(p) + K_2^1 y_l(p), \\ \qquad \text{for } l = 0, 2, \dots, \ \text{ i.e., } l = 2k, \\ \\ K_1^2 x_{l+1}(p) + K_2^2 y_l(p), \\ \qquad \text{for } l = 1, 3, \dots, \ \text{ i.e., } l = 2k+1, \end{cases} \tag{12}$$

or

$$U_{l+1}(p) = \widehat{K}_1 X_{l+1}(p) + \widehat{K}_2 Y_l(p), \tag{13}$$

where

$$\widehat{K}_1 = \begin{bmatrix} K_1^1 & 0 \\ K_2^2(C_1 + D_1 K_1^1) & K_1^2 \end{bmatrix},$$

$$\widehat{K}_2 = \begin{bmatrix} K_2^1 \\ K_2^2(D_{01} + D_1 K_2^1) \end{bmatrix}.$$

On applying this control law to (9), the resulting controlled process state-space model can be written as

$$\begin{aligned} X_{l+1}(p+1) &= \widehat{A}_{\mathrm{new}} X_{l+1}(p) + \widehat{B}_{0\mathrm{new}} Y_l(p), \\ Y_{l+1}(p) &= \widehat{C}_{\mathrm{new}} X_{l+1}(p) + \widehat{D}_{0\mathrm{new}} Y_l(p), \end{aligned} \tag{14}$$

where

$$\begin{aligned} \widehat{A}_{\mathrm{new}} &= \widehat{A} + \widehat{B}\widehat{K}_1 \\ &= \begin{bmatrix} A_1 + B_1 K_1^1 & 0 \\ (B_{02} + B_2 K_2^2)(C_1 + D_1 K_1^1) & A_2 + B_2 K_1^2 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \widehat{B}_{0\mathrm{new}} &= \widehat{B}_0 + \widehat{B}\widehat{K}_2 \\ &= \begin{bmatrix} B_{01} + B_1 K_2^1 \\ (B_{02} + B_2 K_2^2)(D_{01} + D_1 K_2^1) \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \widehat{C}_{\mathrm{new}} &= \widehat{C} + \widehat{D}\widehat{K}_1 \\ &= \begin{bmatrix} (D_{02} + D_2 K_2^2)(C_1 + D_1 K_1^1) & C_2 + D_2 K_1^2 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \widehat{D}_{0\mathrm{new}} &= \widehat{D}_0 + \widehat{D}\widehat{K}_2 \\ &= (D_{02} + D_2 K_2^2)(D_{01} + D_1 K_2^1). \end{aligned}$$

The augmented process matrix for this last state-space model can be written in the following form:

$$\begin{aligned} \widehat{\Phi}_{\mathrm{new}} &= \begin{bmatrix} \widehat{A}_{\mathrm{new}} & \widehat{B}_{0\mathrm{new}} \\ \widehat{C}_{\mathrm{new}} & \widehat{D}_{0\mathrm{new}} \end{bmatrix} \\ &= \begin{bmatrix} \widehat{A} & \widehat{B}_0 \\ \widehat{C} & \widehat{D}_0 \end{bmatrix} + \begin{bmatrix} \widehat{B} \\ \widehat{D} \end{bmatrix} \begin{bmatrix} \widehat{K}_1 & \widehat{K}_2 \end{bmatrix} \\ &= \widehat{\Phi} + \widehat{\Pi}\widehat{K}, \end{aligned} \tag{15}$$

and we can now rewrite (15) as

$$\widehat{\Phi}_{\mathrm{new}} = \Phi_1 + \Phi_2^1 \Phi_2^2, \tag{16}$$

where

$$\begin{aligned} \Phi_1 &= \begin{bmatrix} A_1 + B_1 K_1^1 & 0 & B_{01} + B_1 K_2^1 \\ 0 & A_2 + B_2 K_1^2 & 0 \\ 0 & C_2 + D_2 K_1^2 & 0 \end{bmatrix} \\ &= \bar{A}_1 + \bar{B}_1 \bar{K}_1, \end{aligned}$$

$$\begin{aligned} \Phi_2^1 &= \begin{bmatrix} 0 & 0 & 0 \\ B_{02} + B_2 K_2^2 & 0 & B_{02} + B_2 K_2^2 \\ D_{02} + D_2 K_2^2 & 0 & D_{02} + D_2 K_2^2 \end{bmatrix} \\ &= \bar{A}_2^1 + \bar{B}_2^1 \bar{K}_2, \end{aligned}$$

$$\begin{aligned} \Phi_2^2 &= \begin{bmatrix} C_1 + D_1 K_1^1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_{01} + D_1 K_2^1 \end{bmatrix} \\ &= \bar{A}_2^2 + \bar{B}_2^2 \bar{K}_1, \end{aligned}$$

and

$$\bar{A}_1 = \begin{bmatrix} A_1 & 0 & B_{01} \\ 0 & A_2 & 0 \\ 0 & C_2 & 0 \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} B_1 & 0 & B_1 \\ 0 & B_2 & 0 \\ 0 & D_2 & 0 \end{bmatrix},$$

$$\bar{A}_2^1 = \begin{bmatrix} 0 & 0 & 0 \\ B_{02} & 0 & B_{02} \\ D_{02} & 0 & D_{02} \end{bmatrix}, \quad \bar{B}_2^1 = \begin{bmatrix} 0 & 0 & 0 \\ B_2 & 0 & B_2 \\ D_2 & 0 & D_2 \end{bmatrix},$$

$$\bar{A}_2^2 = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_{01} \end{bmatrix}, \quad \bar{B}_2^2 = \begin{bmatrix} D_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_1 \end{bmatrix},$$

$$\bar{K}_1 = \begin{bmatrix} K_1^1 & 0 & 0 \\ 0 & K_1^2 & 0 \\ 0 & 0 & K_2^1 \end{bmatrix}, \quad \bar{K}_2 = \begin{bmatrix} K_2^2 & 0 & 0 \\ 0 & K_2^2 & 0 \\ 0 & 0 & K_2^2 \end{bmatrix}.$$

Now we are in a position to give the following result, which is a less conservative form of that in (Bochniak *et al.*, 2006).

Theorem 3. *Suppose that a a control law of the form (13) is applied to a discrete linear repetitive process whose state-space model can be written in the form (9). Then the resulting closed-loop process is stable along the pass if there exist a matrix* $X > 0$*, non-singular matrices* $\bar{V}$*,* $\bar{Z}$*, and rectangular matrices* $\bar{L}$*,* $\bar{N}$ *such that*

$$\begin{bmatrix} -X & \bar{A}_2^1\bar{Z}+\bar{B}_2^1\bar{N} & \bar{A}\bar{V}+\bar{B}\bar{L} \\ \bar{Z}^T\bar{A}_2^{1T}+\bar{N}^T\bar{B}_2^{1T} & -\bar{Z}-\bar{Z}^T & \bar{A}_2^2\bar{V}+\bar{B}_2^2\bar{L} \\ \bar{V}^T\bar{A}^T+\bar{L}^T\bar{B}^T & \bar{V}^T\bar{A}_2^{2T}+\bar{L}^T\bar{B}_2^{2T} & X-\bar{V}-\bar{V}^T \end{bmatrix} < 0, \quad (17)$$

where

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}, \qquad \bar{V} = \begin{bmatrix} V_1 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & V_3 \end{bmatrix},$$

$$\bar{Z} = \begin{bmatrix} Z & 0 & 0 \\ 0 & Z & 0 \\ 0 & 0 & Z \end{bmatrix}, \qquad \bar{L} = \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{bmatrix},$$

$$\bar{N} = \begin{bmatrix} N & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & N \end{bmatrix}. \quad (18)$$

If this condition holds, the control law matrices are given by

$$\begin{aligned} \bar{K}_1 &= \bar{L}\bar{V}^{-1} = \begin{bmatrix} K_1^1 & 0 & 0 \\ 0 & K_1^2 & 0 \\ 0 & 0 & K_2^1 \end{bmatrix}, \\ \bar{K}_2 &= \bar{N}\bar{Z}^{-1} = \begin{bmatrix} K_2^2 & 0 & 0 \\ 0 & K_2^2 & 0 \\ 0 & 0 & K_2^2 \end{bmatrix}. \end{aligned} \quad (19)$$

Proof. A sufficient condition for the stability along the pass of the controlled process is, cf. Theorem 2, the existence of positive definite matrices $X_1 > 0$, $X_2 > 0$ such that

$$\widehat{\Phi}_{\text{new}} X \widehat{\Phi}_{\text{new}}^T - X < 0,$$

where $\widehat{\Phi}_{\text{new}}$ given by (16). Also

$$\widehat{\Phi}_{\text{new}} X \widehat{\Phi}_{\text{new}}^T - X = \left(\Phi_1+\Phi_2^1\Phi_2^2\right) X \left(\Phi_1+\Phi_2^1\Phi_2^2\right)^T - X$$

$$= \begin{bmatrix} I & \Phi_2^1 \end{bmatrix} \begin{bmatrix} -X+\Phi_1 X\Phi_1^T & \Phi_2^1\bar{Z}+\Phi_1 X\Phi_2^{2T} \\ \bar{Z}^T\Phi_2^{1T}+\Phi_2^2 X\Phi_1^T & \Phi_2^2 X\Phi_2^{2T}-\bar{Z}-\bar{Z}^T \end{bmatrix} \times \begin{bmatrix} I \\ \Phi_2^{1T} \end{bmatrix} < 0,$$

where $\bar{Z}$ is a compatibly dimensioned nonsymmetric and nonsingular matrix. Hence

$$\begin{bmatrix} -X+\Phi_1 X\Phi_1^T & \Phi_2^1\bar{Z}+\Phi_1 X\Phi_2^{2T} \\ \bar{Z}^T\Phi_2^{1T}+\Phi_2^2 X\Phi_1^T & \Phi_2^2 X\Phi_2^{2T}-\bar{Z}-\bar{Z}^T \end{bmatrix} < 0,$$

i.e.,

$$\begin{bmatrix} -X & \Phi_2^1\bar{Z} \\ \bar{Z}^T\Phi_2^{1T} & -\bar{Z}-\bar{Z}^T \end{bmatrix} + \begin{bmatrix} \Phi_1 \\ \Phi_2^2 \end{bmatrix} X \begin{bmatrix} \Phi_1^T & \Phi_2^{2T} \end{bmatrix} < 0.$$

Applying Lemma 2 to this last expression, we obtain

$$\begin{bmatrix} -X & \Phi_2^1\bar{Z} & \Phi_1\bar{V} \\ \bar{Z}^T\Phi_2^{1T} & -\bar{Z}-\bar{Z}^T & \Phi_2^2\bar{V} \\ \bar{V}^T\Phi_1^T & \bar{V}^T\Phi_2^{2T} & X-\bar{V}-\bar{V}^T \end{bmatrix} < 0,$$

where $\bar{V}$ is a compatibly dimensioned nonsymmetric and nonsingular matrix.

Finally, the use of

$$\begin{aligned} \Phi_1\bar{V} &= \bar{A}_1\bar{V}+\bar{B}_1\bar{K}_1\bar{V} = \bar{A}_1\bar{V}+\bar{B}_1\bar{L}, \\ \Phi_2^1\bar{Z} &= \bar{A}_2^1\bar{Z}+\bar{B}_2^1\bar{K}_2\bar{Z} = \bar{A}_2^1\bar{Z}+\bar{B}_2^1\bar{N}, \\ \Phi_2^2\bar{V} &= \bar{A}_2^2\bar{V}+\bar{B}_2^2\bar{K}_1\bar{V} = \bar{A}_2^2\bar{V}+\bar{B}_2^2\bar{L} \end{aligned} \quad (20)$$

leads to (17) with the control law matrices given by (19), and the proof is complete. ■

3.1. Robustness Analysis.

In what follows, we extend the analysis of the control law design given above to cases where the uncertainty associated with the process state-space model is of the polytopic or norm-bounded type.

Consider first the polytopic form. Here we assume that the matrices which define the sub-processes in the state-space model of (9) belong to a polytope of matrices (the convex hull of a finite set of matrices), i.e.,

$$\begin{bmatrix} A_1 & B_1 & B_{01} \\ A_2 & B_2 & B_{02} \\ C_1 & D_1 & D_{01} \\ C_2 & D_2 & D_{02} \end{bmatrix}$$

$$\in \mathbf{Co}\left\{ \begin{bmatrix} A_1^i & B_1^i & B_{01}^i \\ A_2^i & B_2^i & B_{02}^i \\ C_1^i & D_1^i & D_{01}^i \\ C_2^i & D_2^i & D_{02}^i \end{bmatrix}, \ i=1,\dots,P \right\}$$

$$= \left\{ \sum_{i=1}^{P} \ell_i \begin{bmatrix} A_1 & B_1 & B_{01} \\ A_2 & B_2 & B_{02} \\ C_1 & D_1 & D_{01} \\ C_2 & D_2 & D_{02} \end{bmatrix} : \ \ell_i \geq 0, \ \sum_{i=1}^{P} \ell_i = 1 \right\}. \quad (21)$$

The matrices A_j^i, B_j^i, B_{0j}^i for $j=1,2$ and $i=1,\dots,P$ are assumed known and are termed the vertices of the polytope, the non-negative numbers ℓ_i for $i=1,\dots,P$ are termed the polytopic coordinates, where P denotes the number of the polytope vertices. Note that here the uncertainties for odd k, i.e., in the matrices $\left[\begin{smallmatrix} A_1 & B_1 & B_{01} \\ C_1 & D_1 & D_{01}\end{smallmatrix}\right]$ and even k, i.e., in the matrices $\left[\begin{smallmatrix} A_2 & B_2 & B_{02} \\ C_2 & D_2 & D_{02}\end{smallmatrix}\right]$ are embedded into the joint polytope.

Now, we have the following result.

Theorem 4. *An uncertain discrete linear repetitive process described by (9) in the presence of uncertainty which can be modelled by (21) is stable along the pass if there exist a matrix $X>0$, nonsingular matrices $\bar{V}$ and $\bar{Z}$ such that for all $i=1,\dots,P$ (i.e., for each vertex of the polytope),*

$$\begin{bmatrix} -X & \bar{A}_2^{1i}\bar{Z} & \bar{A}_1^i\bar{V} \\ \bar{Z}^T\bar{A}_2^{1iT} & -\bar{Z}-\bar{Z}^T & \bar{A}_2^{2i}\bar{V} \\ \bar{V}^T\bar{A}_1^{iT} & \bar{V}^T\bar{A}_2^{2iT} & X-\bar{V}-\bar{V}^T \end{bmatrix} < 0, \quad (22)$$

where

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$$

and

$$\bar{A}_1^i = \begin{bmatrix} A_1^i & 0 & B_{01}^i \\ 0 & A_2^i & 0 \\ 0 & C_2^i & 0 \end{bmatrix}, \quad \bar{A}_2^{1i} = \begin{bmatrix} 0 & 0 & 0 \\ B_{02}^i & 0 & B_{02}^i \\ D_{02}^i & 0 & D_{02}^i \end{bmatrix},$$

$$\bar{A}_2^{2i} = \begin{bmatrix} C_1^i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_{01}^i \end{bmatrix}. \quad (23)$$

Proof. Interpreting Theorem 2 in this case gives stability along the pass if there exist $X_1>0$, $X_2>0$ such that

$$\widehat{\Phi}X\widehat{\Phi}^T - X < 0,$$

with $X = \mathbf{diag}(X_1, X_2)$ and $\widehat{\Phi} \in \mathbf{Co}\big\{\widehat{\Phi}^i,\ i=1,\dots,P\big\}$. This last matrix inequality is equivalent to

$$\widehat{\Phi}^iX\widehat{\Phi}^{iT} - X < 0, \quad i=1,2,\dots,P.$$

Using steps analogous to Theorem 3, we prove stability along the pass of each vertex of the polytope and the proof is complete. ∎

Suppose now that a control law of the form (12) is applied. Then the uncertainty in the model matrices C_1, D_{01} and D_1 requires us to use the form of (13) with

$$\begin{aligned} \widehat{K}_1 &\in \mathbf{Co}\Big\{\widehat{K}_1^i,\ i=1,\dots,P\Big\}, \\ \widehat{K}_1^i &= \begin{bmatrix} K_1^1 & 0 \\ K_2^2(C_1^i + D_1^iK_1^1) & K_1^2 \end{bmatrix}, \\ \widehat{K}_2 &\in \mathbf{Co}\Big\{\widehat{K}_2^i,\ i=1,\dots,P\Big\}, \\ \widehat{K}_2^i &= \begin{bmatrix} K_2^1 \\ K_2^2(D_{01}^i + D_1^iK_2^1) \end{bmatrix}. \end{aligned} \quad (24)$$

The resulting controlled process state-space model can be written in the form

$$\begin{aligned} X_{l+1}(p+1) &= \widehat{A}_{\rm new}X_{l+1}(p) + \widehat{B}_{0\rm new}Y_l(p), \\ Y_{l+1}(p) &= \widehat{C}_{\rm new}X_{l+1}(p) + \widehat{D}_{0\rm new}Y_l(p), \end{aligned} \quad (25)$$

where

$$\widehat{A}_{\rm new} \in \mathbf{Co}\Big\{\widehat{A}_{\rm new}^i,\ i=1,\dots,P\Big\},$$

$$\widehat{B}_{0\rm new} \in \mathbf{Co}\Big\{\widehat{B}_{0\rm new}^i,\ i=1,\dots,P\Big\},$$

$$\widehat{C}_{\rm new} \in \mathbf{Co}\Big\{\widehat{C}_{\rm new}^i,\ i=1,\dots,P\Big\},$$

$$\widehat{D}_{0\rm new} \in \mathbf{Co}\Big\{\widehat{D}_{0\rm new}^i,\ i=1,\dots,P\Big\},$$

$$\begin{aligned} \widehat{A}_{\rm new}^i &= \widehat{A}^i + \widehat{B}^i\widehat{K}_1^i \\ &= \begin{bmatrix} A_1^i + B_1^iK_1^1 & 0 \\ (B_{02}^i + B_2^iK_2^2)(C_1^i + D_1^iK_1^1) & A_2^i + B_2^iK_1^2 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \widehat{B}_{0\rm new}^i &= \widehat{B}_0^i + \widehat{B}^i\widehat{K}_2^i \\ &= \begin{bmatrix} B_{01}^i + B_1^iK_2^1 \\ (B_{02}^i + B_2^iK_2^2)(D_{01}^i + D_1^iK_2^1) \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \widehat{C}_{\rm new}^i &= \widehat{C}^i + \widehat{D}^i\widehat{K}_1^i \\ &= \begin{bmatrix} (D_{02}^i + D_2^iK_2^2)(C_1^i + D_1^iK_1^1) & C_2^i + D_2^iK_1^2 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \widehat{D}_{0\rm new}^i &= \widehat{D}_0^i + \widehat{D}^i\widehat{K}_2^i \\ &= (D_{02}^i + D_2^iK_2^2)(D_{01}^i + D_1^iK_2^1). \end{aligned}$$

Also, the augmented process matrix in this case can be written as

$$\begin{aligned}\widehat{\Phi}^i_{\mathrm{new}} &= \begin{bmatrix} \widehat{A}^i_{\mathrm{new}} & \widehat{B}^i_{0\mathrm{new}} \\ \widehat{C}^i_{\mathrm{new}} & \widehat{D}^i_{0\mathrm{new}} \end{bmatrix} \\ &= \begin{bmatrix} \widehat{A}^i & \widehat{B}^i_0 \\ \widehat{C}^i & \widehat{D}^i_0 \end{bmatrix} + \begin{bmatrix} \widehat{B}^i \\ \widehat{D}^i \end{bmatrix} \begin{bmatrix} \widehat{K}^i_1 & \widehat{K}^i_2 \end{bmatrix} \\ &= \widehat{\Phi}^i + \widehat{\Pi}^i \widehat{K}^i, \end{aligned} \tag{26}$$

or

$$\widehat{\Phi}^i_{\mathrm{new}} = \Phi^i_1 + \Phi^{1i}_2 \Phi^{2i}_2 \tag{27}$$

for $i = 1, 2, \dots, P$, where

$$\begin{aligned}\Phi^i_1 &= \begin{bmatrix} A^i_1 + B^i_1 K^1_1 & 0 & B^i_{01} + B^i_1 K^1_2 \\ 0 & A^i_2 + B^i_2 K^2_1 & 0 \\ 0 & C^i_2 + D^i_2 K^2_1 & 0 \end{bmatrix} \\ &= \bar{A}^i_1 + \bar{B}^i_1 \bar{K}_1,\end{aligned}$$

$$\begin{aligned}\Phi^{1i}_2 &= \begin{bmatrix} 0 & 0 & 0 \\ B^i_{02} + B^i_2 K^2_2 & 0 & B^i_{02} + B^i_2 K^2_2 \\ D^i_{02} + D^i_2 K^2_2 & 0 & D^i_{02} + D^i_2 K^2_2 \end{bmatrix} \\ &= \bar{A}^{1i}_2 + \bar{B}^{1i}_2 \bar{K}_2,\end{aligned}$$

$$\begin{aligned}\Phi^{2i}_2 &= \begin{bmatrix} C^i_1 + D^i_1 K^1_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D^i_{01} + D^i_1 K^1_2 \end{bmatrix} \\ &= \bar{A}^{2i}_2 + \bar{B}^{2i}_2 \bar{K}_1,\end{aligned}$$

where $\bar{A}^i_1$, $\bar{A}^{1i}_2$, $\bar{A}^{2i}_2$ are as in (23), and

$$\bar{B}^i_1 = \begin{bmatrix} B^i_1 & 0 & B^i_1 \\ 0 & B^i_2 & 0 \\ 0 & D^i_2 & 0 \end{bmatrix}, \qquad \bar{B}^{1i}_2 = \begin{bmatrix} 0 & 0 & 0 \\ B^i_2 & 0 & B^i_2 \\ D^i_2 & 0 & D^i_2 \end{bmatrix},$$

$$\bar{B}^{2i}_2 = \begin{bmatrix} D^i_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D^i_1 \end{bmatrix},$$

$$\bar{K}^i_1 = \begin{bmatrix} K^1_1 & 0 & 0 \\ 0 & K^2_1 & 0 \\ 0 & 0 & K^1_2 \end{bmatrix}, \qquad \bar{K}_2 = \begin{bmatrix} K^2_2 & 0 & 0 \\ 0 & K^2_2 & 0 \\ 0 & 0 & K^2_2 \end{bmatrix}.$$

Now we have the following result, which enables the control law considered here to be designed for stability along the pass.

Theorem 5. *Suppose that a a control law of the form (12) is applied to a discrete linear repetitive process whose state-space model can be written in the form (9) in the presence of uncertainty modelled by (21). Then the resulting controlled process is stable along the pass if there exist a matrix $X > 0$, nonsingular matrices $\bar{V}$ and $\bar{Z}$, and rectangular matrices $\bar{L}$, $\bar{N}$ such that for all $i = 1, \dots, P$ (i.e., for each vertex of the polytope),*

$$\begin{bmatrix} -X & \bar{A}^{1i}_2 \bar{Z} + \bar{B}^{1i}_2 \bar{N} & \bar{A}^i_1 \bar{V} + \bar{B}^i_1 \bar{L} \\ \bar{Z}^T \bar{A}^{1iT}_2 + \bar{N}^T \bar{B}^{1iT}_2 & -\bar{Z} - \bar{Z}^T & \bar{A}^{2i}_2 \bar{V} + \bar{B}^{2i}_2 \bar{L} \\ \bar{V}^T \bar{A}^{iT}_1 + \bar{L}^T \bar{B}^{iT}_1 & \bar{V}^T \bar{A}^{2iT}_2 + \bar{L}^T \bar{B}^{2iT}_2 & X - \bar{V} - \bar{V}^T \end{bmatrix} < 0, \tag{28}$$

where the matrices X, $\bar{V}$, $\bar{Z}$, $\bar{L}$ and $\bar{N}$ are given in (18). If these conditions hold, the control law matrices are given by (19).

Proof. Interpreting Theorem 2 in terms of the controlled process (25) shows that it is stable along the pass if there exist matrices $X_1 > 0$ and $X_2 > 0$ such that

$$\widehat{\Phi}_{\mathrm{new}} X \widehat{\Phi}^T_{\mathrm{new}} - X < 0,$$

where $X = \mathbf{diag}\,(X_1, X_2)$ and $\widehat{\Phi}_{\mathrm{new}} \in \mathbf{Co}\{\widehat{\Phi}^i_{\mathrm{new}},\ i = 1, \dots, P\}$. This last matrix inequality is equivalent to

$$\widehat{\Phi}^i_{\mathrm{new}} X \widehat{\Phi}^{iT}_{\mathrm{new}} - X < 0, \quad i = 1, \dots, P.$$

Following the same steps as in the proof of Theorem 3 shows that all vertices of the polytope, and hence all convex combinations of them, are stable along the pass and the proof is complete. ■

3.1.1. Norm-Bounded Uncertainty. In what follows, we consider an uncertain discrete linear repetitive process with dynamics switched from pass to pass described by the following state-space model over $p = 0, 1, \dots, (\alpha - 1)$ and $l = 0, 1, \dots$:

$$\begin{aligned} X_{l+1}(p+1) &= (\widehat{A} + \Delta\widehat{A}) X_{l+1}(p) + (\widehat{B} + \Delta\widehat{B}) U_{l+1}(p) \\ &\quad + (\widehat{B}_0 + \Delta\widehat{B}_0) Y_l(p), \\ Y_{l+1}(p) &= (\widehat{C} + \Delta\widehat{C}) X_{l+1}(p) + (\widehat{D} + \Delta\widehat{D}) U_{l+1}(p) \\ &\quad + (\widehat{D}_0 + \Delta\widehat{D}_0) Y_l(p), \end{aligned} \tag{29}$$

where

$$\widehat{A} + \Delta\widehat{A} = \begin{bmatrix} A_1 + \Delta A_1 & 0 \\ (B_{02} + \Delta B_{02})(C_1 + \Delta C_1) & A_2 + \Delta A_2 \end{bmatrix},$$

$$\widehat{B} + \Delta\widehat{B} = \begin{bmatrix} B_1 + \Delta B_1 & 0 \\ (B_{02} + \Delta B_{02})(D_1 + \Delta D_1) & B_2 + \Delta B_2 \end{bmatrix},$$

$$\widehat{B}_0 + \Delta\widehat{B}_0 = \begin{bmatrix} B_{01} + \Delta B_{01} \\ (B_{02} + \Delta B_{02})(D_{01} + \Delta D_{01}) \end{bmatrix},$$

$$\widehat{C}+\Delta\widehat{C}=\Big[(D_{02}+\Delta D_{02})(C_1+\Delta C_1)\ C_2+\Delta C_2\Big],$$

$$\widehat{D}+\Delta\widehat{D}=\Big[(D_{02}+\Delta D_{02})(D_1+\Delta D_1)\ D_2+\Delta D_2\Big],$$

$$\widehat{D}_0+\Delta\widehat{D}_0=(D_{02}+\Delta D_{02})(D_{01}+\Delta D_{01}).$$

The perturbations in each sub-process state-space model, which may vary with t, are assumed to satisfy the norm bounded structure

$$\left[\begin{array}{cc|c} \Delta A_1 & \Delta B_{01} & \Delta B_1 \\ \Delta A_2 & \Delta B_{02} & \Delta B_2 \\ \hline \Delta C_1 & \Delta D_{01} & \Delta D_1 \\ \Delta C_2 & \Delta D_{02} & \Delta D_2 \end{array}\right] = \left[\begin{array}{c} H_1 \\ H_2 \\ \hline H_3 \\ H_4 \end{array}\right] F \left[\begin{array}{cc|c} E_{11} & E_{12} & E_2 \end{array}\right], \quad (30)$$

where F is some unknown real matrix which satisfies

$$\|F\| \le 1, \quad \text{i.e.,} \quad F^T F \le I, \quad (31)$$

and H_1, H_2, H_3, H_4 and E_{11}, E_{12}, E_2 are known real constant matrices of compatible dimensions.

The augmented process matrix $(\widehat{\Phi}+\Delta\widehat{\Phi})$ for the uncontrolled process here is

$$\begin{aligned}\widehat{\Phi}+\Delta\widehat{\Phi} &= \begin{bmatrix} \widehat{A}+\Delta\widehat{A} & \widehat{B}_0+\Delta\widehat{B}_0 \\ \widehat{C}+\Delta\widehat{C} & \widehat{D}_0+\Delta\widehat{D}_0 \end{bmatrix} \\ &= (\bar{A}_1+\Delta\bar{A}_1) \\ &\quad + (\bar{A}_2^1+\Delta\bar{A}_2^1)(\bar{A}_2^2+\Delta\bar{A}_2^2), \end{aligned} \quad (32)$$

where

$$\Delta\bar{A}_1 = \begin{bmatrix} \Delta A_1 & 0 & \Delta B_{01} \\ 0 & \Delta A_2 & 0 \\ 0 & \Delta C_2 & 0 \end{bmatrix} = \bar{H}_1 \bar{F} \bar{E}_1^1,$$

$$\Delta\bar{A}_2^1 = \begin{bmatrix} 0 & 0 & 0 \\ \Delta B_{02} & 0 & \Delta B_{02} \\ \Delta D_{02} & 0 & \Delta D_{02} \end{bmatrix} = \bar{H}_3 \bar{F} \bar{E}_1^2, \quad (33)$$

$$\Delta\bar{A}_2^2 = \begin{bmatrix} \Delta C_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta D_{01} \end{bmatrix} = \bar{H}_2 \bar{F} \bar{E}_1^1,$$

with

$$\bar{H}_1 = \begin{bmatrix} H_1 & 0 & H_1 \\ 0 & H_2 & 0 \\ 0 & H_4 & 0 \end{bmatrix}, \quad \bar{H}_2 = \begin{bmatrix} H_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & H_3 \end{bmatrix},$$

$$\bar{H}_3 = \begin{bmatrix} 0 & 0 & 0 \\ H_2 & 0 & H_2 \\ H_3 & 0 & H_3 \end{bmatrix}, \qquad \bar{F} = \begin{bmatrix} F & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix},$$

$$\bar{E}_1^1 = \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & E_{11} & 0 \\ 0 & 0 & E_{12} \end{bmatrix}, \quad \bar{E}_1^2 = \begin{bmatrix} E_{12} & 0 & 0 \\ 0 & E_{12} & 0 \\ 0 & 0 & E_{12} \end{bmatrix}.$$

Now, we can state the following sufficient condition for stability along the pass of uncertain processes described by (29).

Theorem 6. *An uncertain discrete linear repetitive process whose state-space model (29), with the uncertainty structure modelled by (30), is stable along the pass if there exist a matrix $X > 0$, non-singular matrices $\bar{V}$ and $\bar{Z}$ and scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that*

$$\begin{bmatrix} -X & \bar{A}_2^1\bar{Z} & \varepsilon_1\bar{H}_3 & 0 & \bar{A}_1\bar{V} & \varepsilon_2\bar{H}_1 & 0 \\ \bar{Z}^T\bar{A}_2^{1T} & -\bar{Z}-\bar{Z}^T & 0 & \bar{Z}^T\bar{E}_1^{2T} & \bar{A}_2^2\bar{V} & \varepsilon_2\bar{H}_2 & 0 \\ \varepsilon_1\bar{H}_3^T & 0 & -\varepsilon_1 I & 0 & 0 & 0 & 0 \\ 0 & \bar{E}_1^2\bar{Z} & 0 & -\varepsilon_1 I & 0 & 0 & 0 \\ \bar{V}^T\bar{A}_1^T & \bar{V}^T\bar{A}_2^{2T} & 0 & 0 & X-\bar{V}-\bar{V}^T & 0 & \bar{V}^T\bar{E}_1^{1T} \\ \varepsilon_2\bar{H}_1^T & \varepsilon_2\bar{H}_2^T & 0 & 0 & 0 & -\varepsilon_2 I & 0 \\ 0 & 0 & 0 & 0 & \bar{E}_1^1\bar{V} & 0 & -\varepsilon_2 I \end{bmatrix} < 0, \quad (34)$$

where $X = \mathbf{diag}\,(X_1, X_2)$.

Proof. Theorem 2 applied to this case gives stability along the pass if there exist matrices $X_1 > 0$ and $X_2 > 0$ such that

$$(\widehat{\Phi}+\Delta\widehat{\Phi})X(\widehat{\Phi}+\Delta\widehat{\Phi})^T - X < 0,$$

with the matrix $(\widehat{\Phi}+\Delta\widehat{\Phi})$ given by (32).

Using Theorem 3, we have

$$\begin{bmatrix} -X+(\bar{A}_1+\Delta\bar{A}_1)X(\bar{A}_1+\Delta\bar{A}_1)^T & (\bar{A}_2^1+\Delta\bar{A}_2^1)\bar{Z}+(\bar{A}_1+\Delta\bar{A}_1)X(\bar{A}_2^2+\Delta\bar{A}_2^2)^T \\ \bar{Z}^T(\bar{A}_2^1+\Delta\bar{A}_2^1)^T+(\bar{A}_2^2+\Delta\bar{A}_2^2)X(\bar{A}_1+\Delta\bar{A}_1)^T & (\bar{A}_2^2+\Delta\bar{A}_2^2)X(\bar{A}_2^2+\Delta\bar{A}_2^2)^T-\bar{Z}-\bar{Z}^T \end{bmatrix} < 0,$$

where $\bar{Z}$ is given by (18) and Z is a nonsingular, possibly nonsymmetric, matrix of compatible dimensions. Also, this last condition can be rewritten as

$$\begin{bmatrix} -X + (\bar{A}_1 + \Delta\bar{A}_1)X(\bar{A}_1 + \Delta\bar{A}_1)^T & \bar{A}_2^1\bar{Z} + (\bar{A}_1 + \Delta\bar{A}_1)X(\bar{A}_2^2 + \Delta\bar{A}_2^2)^T \\ \bar{Z}^T\bar{A}_2^{1T} + (\bar{A}_2^2 + \Delta\bar{A}_2^2)X(\bar{A}_1 + \Delta\bar{A}_1)^T & (\bar{A}_2^2 + \Delta\bar{A}_2^2)X(\bar{A}_2^2 + \Delta\bar{A}_2^2)^T - \bar{Z} - \bar{Z}^T \end{bmatrix} + \begin{bmatrix} 0 & (\Delta\bar{A}_2^1)\bar{Z} \\ \bar{Z}^T(\Delta\bar{A}_2^1)^T & 0 \end{bmatrix} < 0,$$

or

$$\begin{bmatrix} -X + (\bar{A}_1 + \Delta\bar{A}_1)X(\bar{A}_1 + \Delta\bar{A}_1)^T & \bar{A}_2^1\bar{Z} + (\bar{A}_1 + \Delta\bar{A}_1)X(\bar{A}_2^2 + \Delta\bar{A}_2^2)^T \\ \bar{Z}^T\bar{A}_2^{1T} + (\bar{A}_2^2 + \Delta\bar{A}_2^2)X(\bar{A}_1 + \Delta\bar{A}_1)^T & (\bar{A}_2^2 + \Delta\bar{A}_2^2)X(\bar{A}_2^2 + \Delta\bar{A}_2^2)^T - \bar{Z} - \bar{Z}^T \end{bmatrix} + \begin{bmatrix} \bar{H}_3 & 0 \\ 0 & \bar{Z}^T\bar{E}_1^{2T} \end{bmatrix} \begin{bmatrix} 0 & \bar{F} \\ \bar{F}^T & 0 \end{bmatrix} \begin{bmatrix} \bar{H}_3^T & 0 \\ 0 & \bar{E}_1^2\bar{Z} \end{bmatrix} < 0.$$

Applying Lemma 3 to this last expression and then Lemma 1 to the result gives

$$\begin{bmatrix} -X & \bar{A}_2^1\bar{Z} & \bar{H}_3 & 0 \\ \bar{Z}^T\bar{A}_2^{1T} & -\bar{Z} - \bar{Z}^T & 0 & \bar{Z}^T\bar{E}_1^{2T} \\ \bar{H}_3^T & 0 & -\varepsilon_1^{-1}I & 0 \\ 0 & \bar{E}_1^2\bar{Z} & 0 & -\varepsilon_1 I \end{bmatrix} + \begin{bmatrix} \bar{A}_1+\Delta\bar{A}_1 \\ \bar{A}_2^2+\Delta\bar{A}_2^2 \\ 0 \\ 0 \end{bmatrix} \times X \begin{bmatrix} (\bar{A}_1 + \Delta\bar{A}_1)^T & (\bar{A}_2^2 + \Delta\bar{A}_2^2)^T & 0 & 0 \end{bmatrix} < 0$$

or, using Lemma 2,

$$\begin{bmatrix} -X & \bar{A}_2^1\bar{Z} & \bar{H}_3 & 0 & \bar{A}_1\bar{V} \\ \bar{Z}^T\bar{A}_2^{1T} & -\bar{Z}-\bar{Z}^T & 0 & \bar{Z}^T\bar{E}_1^{2T} & \bar{A}_2^2\bar{V} \\ \bar{H}_3^T & 0 & -\varepsilon_1^{-1}I & 0 & 0 \\ 0 & \bar{E}_1^2\bar{Z} & 0 & -\varepsilon_1 I & 0 \\ \bar{V}^T\bar{A}_1^T & \bar{V}^T\bar{A}_2^{2T} & 0 & 0 & X-\bar{V}-\bar{V}^T \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & \Delta\bar{A}_1\bar{V} \\ 0 & 0 & 0 & 0 & \Delta\bar{A}_2^2\bar{V} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \bar{V}^T\Delta\bar{A}_1^T & \bar{V}^T\Delta\bar{A}_2^{2T} & 0 & 0 & 0 \end{bmatrix} < 0,$$

where $\bar{V}$ is a nonsymmetric and nonsingular matrix of compatible dimensions.

This last inequality can be rewritten in the form

$$\begin{bmatrix} -X & \bar{A}_2^1\bar{Z} & \bar{H}_3 & 0 & \bar{A}_1\bar{V} \\ \bar{Z}^T\bar{A}_2^{1T} & -\bar{Z}-\bar{Z}^T & 0 & \bar{Z}^T\bar{E}_1^{2T} & \bar{A}_2^2\bar{V} \\ \bar{H}_3^T & 0 & -\varepsilon_1^{-1}I & 0 & 0 \\ 0 & \bar{E}_1^2\bar{Z} & 0 & -\varepsilon_1 I & 0 \\ \bar{V}^T\bar{A}_1^T & \bar{V}^T\bar{A}_2^{2T} & 0 & 0 & X-\bar{V}-\bar{V}^T \end{bmatrix} + \begin{bmatrix} \bar{H}_1 & 0 \\ \bar{H}_2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \bar{V}^T\bar{E}_1^{1T} \end{bmatrix} \times \begin{bmatrix} 0 & \bar{F} \\ \bar{F}^T & 0 \end{bmatrix} \begin{bmatrix} \bar{H}_1^T & \bar{H}_2^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{E}_1^1\bar{V} \end{bmatrix} < 0.$$

The application of Lemma 3 and then Lemma 1 to the result now yields

$$\begin{bmatrix} -X & \bar{A}_2^1\bar{Z} & \bar{H}_3 & 0 & \bar{A}_1\bar{V} & \bar{H}_1 & 0 \\ \bar{Z}^T\bar{A}_2^{1T} & -\bar{Z}-\bar{Z}^T & 0 & \bar{Z}^T\bar{E}_1^{2T} & \bar{A}_2^2\bar{V} & \bar{H}_2 & 0 \\ \bar{H}_3^T & 0 & -\varepsilon_1^{-1}I & 0 & 0 & 0 & 0 \\ 0 & \bar{E}_1^2\bar{Z} & 0 & -\varepsilon_1 I & 0 & 0 & 0 \\ \bar{V}^T\bar{A}_1^T & \bar{V}^T\bar{A}_2^{2T} & 0 & 0 & X-\bar{V}-\bar{V}^T & 0 & \bar{V}^T\bar{E}_1^{1T} \\ \bar{H}_1^T & \bar{H}_2^T & 0 & 0 & 0 & -\varepsilon_2^{-1}I & 0 \\ 0 & 0 & 0 & 0 & \bar{E}_1^1\bar{V} & 0 & -\varepsilon_2 I \end{bmatrix} < 0.$$

Finally, premultiplying and postmultiplying this last inequality by $\mathbf{diag}\,(I, I, \varepsilon_1 I, I, I, \varepsilon_2 I, I)$ yields (34), and the proof is complete. ■

Consider now the application of a control law of the form (12) but, because of the uncertainty in the process state-space model, modified (via (13)) to the form

$$U_{l+1}(p) = (\widehat{K}_1 + \Delta\widehat{K}_1)X_{l+1}(p) + (\widehat{K}_2 + \Delta\widehat{K}_2)Y_l(p), \quad (35)$$

where

$$\widehat{K}_1+\Delta\widehat{K}_1 = \begin{bmatrix} K_1^1 & 0 \\ K_2^2(C_1+\Delta C_1+(D_1+\Delta D_1)K_1^1) & K_1^2 \end{bmatrix}, \quad (36)$$

$$\widehat{K}_2+\Delta\widehat{K}_2 = \begin{bmatrix} K_2^1 \\ K_2^2(D_{01}+\Delta D_{01}+(D_1+\Delta D_1)K_2^1) \end{bmatrix}.$$

On the application of this control law, the resulting controlled process state-space model is

$$\begin{aligned} X_{l+1}(p+1) &= (\widehat{A}_{\text{new}} + \Delta\widehat{A}_{\text{new}})X_{l+1}(p) \\ &\quad + (\widehat{B}_{0\text{new}} + \Delta\widehat{B}_{0\text{new}})Y_l(p), \\ Y_{l+1}(p) &= (\widehat{C}_{\text{new}} + \Delta\widehat{C}_{\text{new}})X_{l+1}(p) \\ &\quad + (\widehat{D}_{0\text{new}} + \Delta\widehat{D}_{0\text{new}})Y_l(p), \end{aligned} \tag{37}$$

where

$$\widehat{A}_{\text{new}} + \Delta\widehat{A}_{\text{new}} = \begin{bmatrix} A_1+\Delta A_1+(B_1+\Delta B_1)K_1^1 & 0 \\ \big(B_{02}+\Delta B_{02}+(B_2+\Delta B_2)K_2^2\big) \times\big(C_1+\Delta C_1+(D_1+\Delta D_1)K_1^1\big) & A_2+\Delta A_2+(B_2+\Delta B_2)K_1^2 \end{bmatrix},$$

$$\widehat{B}_{0\text{new}} + \Delta\widehat{B}_{0\text{new}} = \begin{bmatrix} B_{01}+\Delta B_{01}+(B_1+\Delta B_1)K_2^1 \\ \big(B_{02}+\Delta B_{02}+(B_2+\Delta B_2)K_2^2\big) \times\big(D_{01}+\Delta D_{01}+(D_1+\Delta D_1)K_2^1\big) \end{bmatrix},$$

$$\widehat{C}_{\text{new}} + \Delta\widehat{C}_{\text{new}} = \begin{bmatrix} \big(D_{02}+\Delta D_{02}+(D_2+\Delta D_2)K_2^2\big) \times\big(C_1+\Delta C_1+(D_1+\Delta D_1)K_1^1\big) & C_2+\Delta C_2+(D_2+\Delta D_2)K_1^2 \end{bmatrix},$$

$$\widehat{D}_{0\text{new}} + \Delta\widehat{D}_{0\text{new}} = \big(D_{02}+\Delta D_{02}+(D_2+\Delta D_2)K_2^2\big) \times\big(D_{01}+\Delta D_{01}+(D_1+\Delta D_1)K_2^1\big).$$

The augmented process matrix corresponding to this last model is given by

$$\widehat{\Phi}_{\text{new}}+\Delta\widehat{\Phi}_{\text{new}} = \begin{bmatrix} \widehat{A}_{\text{new}}+\Delta\widehat{A}_{\text{new}} & \widehat{B}_{0\text{new}}+\Delta\widehat{B}_{0\text{new}} \\ \widehat{C}_{\text{new}}+\Delta\widehat{C}_{\text{new}} & \widehat{D}_{0\text{new}}+\Delta\widehat{D}_{0\text{new}} \end{bmatrix} \tag{38}$$

or

$$\widehat{\Phi}_{\text{new}} + \Delta\widehat{\Phi}_{\text{new}} = \Phi_1 + \Delta\Phi_1 + (\Phi_2^1 + \Delta\Phi_2^1)(\Phi_2^2 + \Delta\Phi_2^2), \tag{39}$$

where

$$\begin{aligned} \Delta\Phi_1 &= \begin{bmatrix} \Delta A_1+\Delta B_1K_1^1 & 0 & \Delta B_{01}+\Delta B_1K_2^1 \\ 0 & \Delta A_2+\Delta B_2K_1^2 & 0 \\ 0 & \Delta C_2+\Delta D_2K_1^2 & 0 \end{bmatrix} \\ &= \Delta\bar{A}_1+\Delta\bar{B}_1\bar{K}_1, \end{aligned}$$

$$\begin{aligned} \Delta\Phi_2^1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta B_{02}+\Delta B_2K_2^2 & 0 \\ 0 & 0 & \Delta D_{02}+\Delta D_2K_2^2 \end{bmatrix} \\ &= \Delta\bar{A}_2^1 + \Delta\bar{B}_2^1\bar{K}_2, \end{aligned}$$

$$\begin{aligned} \Delta\Phi_2^2 &= \begin{bmatrix} 0 & 0 & 0 \\ \Delta C_1+\Delta D_1K_1^1 & 0 & \Delta D_{01}+\Delta D_1K_2^1 \\ \Delta C_1+\Delta D_1K_1^1 & 0 & \Delta D_{01}+\Delta D_1K_2^1 \end{bmatrix} \\ &= \Delta\bar{A}_2^2 + \Delta\bar{B}_2^2\bar{K}_1, \end{aligned}$$

where $\Delta\bar{A}_1$, $\Delta\bar{A}_2^1$, $\Delta\bar{A}_2^2$ are as in (33), and

$$\begin{aligned} \Delta\bar{B}_1 &= \begin{bmatrix} \Delta B_1 & 0 & \Delta B_1 \\ 0 & \Delta B_2 & 0 \\ 0 & \Delta D_2 & 0 \end{bmatrix} = \bar{H}_1\bar{F}\bar{E}_2, \\ \Delta\bar{B}_2^1 &= \begin{bmatrix} 0 & 0 & 0 \\ \Delta B_2 & 0 & \Delta B_2 \\ \Delta D_2 & 0 & \Delta D_2 \end{bmatrix} = \bar{H}_3\bar{F}\bar{E}_2, \\ \Delta\bar{B}_2^2 &= \begin{bmatrix} \Delta D_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta D_1 \end{bmatrix} = \bar{H}_2\bar{F}\bar{E}_2, \end{aligned} \tag{40}$$

with

$$\bar{E}_2 = \begin{bmatrix} E_2 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_2 \end{bmatrix}.$$

Now we have the following result.

Theorem 7. *Suppose that a a control law of the form (12) is applied to a discrete linear repetitive process described by (29) with an uncertainty structure satisfying (30). Then the resulting controlled process is stable along the pass if there exist a matrix $X > 0$, nonsingular matrices $\bar{V}$ and $\bar{Z}$, rectangular matrices $\bar{L}$ and $\bar{N}$, and scalars $\varepsilon_1 > 0$,*

$\varepsilon_2 > 0$ *such that*

$$\begin{bmatrix} -X & \bar{A}_2^1\bar{Z}+\bar{B}_2^1\bar{N} & \varepsilon_1\bar{H}_3 & 0 & \bar{A}_1\bar{V}+\bar{B}_1\bar{L} & \varepsilon_2\bar{H}_1 & 0 \\ \bar{Z}^T\bar{A}_2^{1T}+\bar{N}^T\bar{B}_2^{1T} & -\bar{Z}-\bar{Z}^T & 0 & \bar{Z}^T\bar{E}_1^{2T}+\bar{N}^T\bar{E}_2^T & \bar{A}_2^2\bar{V}+\bar{B}_2^2\bar{L} & \varepsilon_2\bar{H}_2 & 0 \\ \varepsilon_1\bar{H}_3^T & 0 & -\varepsilon_1 I & 0 & 0 & 0 & 0 \\ 0 & \bar{E}_1^2\bar{Z}+\bar{E}_2\bar{N} & 0 & -\varepsilon_1 I & 0 & 0 & 0 \\ \bar{V}^T\bar{A}_1^T+\bar{L}^T\bar{B}_1^T & \bar{V}^T\bar{A}_2^{2T}+\bar{L}^T\bar{B}_2^{2T} & 0 & 0 & X-\bar{V}-\bar{V}^T & 0 & \bar{V}^T\bar{E}_1^{1T}+\bar{L}^T\bar{E}_2^T \\ \varepsilon_2\bar{H}_1^T & \varepsilon_2\bar{H}_2^T & 0 & 0 & 0 & -\varepsilon_2 I & 0 \\ 0 & 0 & 0 & 0 & \bar{E}_1^1\bar{V}+\bar{E}_2\bar{L} & 0 & -\varepsilon_2 I \end{bmatrix} < 0, \quad (41)$$

where the matrices X, $\bar{V}$, $\bar{Z}$, $\bar{L}$ and $\bar{N}$ are given in (18). If this condition holds, the control law matrices are given by (19).

Proof. Theorem 2 applied to this case gives that stability along the pass holds if there exist matrices $X_1 > 0$ and $X_2 > 0$ such that

$$(\widehat{\Phi}_{\text{new}} + \Delta\widehat{\Phi}_{\text{new}})X(\widehat{\Phi}_{\text{new}} + \Delta\widehat{\Phi}_{\text{new}})^T - X < 0$$

with the matrix $(\widehat{\Phi}_{\text{new}} + \Delta\widehat{\Phi}_{\text{new}})$ given by (39).

We can rewrite the above inequality (see Theorem 3) as

$$\begin{bmatrix} -X+(\Phi_1+\Delta\Phi_1)X(\Phi_1+\Delta\Phi_1)^T & (\Phi_2^1+\Delta\Phi_2^1)\bar{Z}+(\Phi_1+\Delta\Phi_1)X(\Phi_2^2+\Delta\Phi_2^2)^T \\ \bar{Z}^T(\Phi_2^1+\Delta\Phi_2^1)^T+(\Phi_2^2+\Delta\Phi_2^2)X(\Phi_1+\Delta\Phi_1)^T & (\Phi_2^2+\Delta\Phi_2^2)X(\Phi_2^2+\Delta\Phi_2^2)^T-\bar{Z}-\bar{Z}^T \end{bmatrix} < 0,$$

where $\bar{Z}$ is given by (18) and Z is a nonsingular, possibly nonsymmetric, matrix of compatible dimensions. Equivalently, we have that

$$\begin{bmatrix} -X + (\Phi_1 + \Delta\Phi_1)X(\Phi_1 + \Delta\Phi_1)^T & \Phi_2^1\bar{Z} + (\Phi_1 + \Delta\Phi_1)X(\Phi_2^2 + \Delta\Phi_2^2)^T \\ \bar{Z}^T\Phi_2^{1T} + (\Phi_2^2 + \Delta\Phi_2^2)X(\Phi_1 + \Delta\Phi_1)^T & (\Phi_2^2 + \Delta\Phi_2^2)X(\Phi_2^2 + \Delta\Phi_2^2)^T - \bar{Z} - \bar{Z}^T \end{bmatrix} + \begin{bmatrix} 0 & (\Delta\Phi_2^1)\bar{Z} \\ \bar{Z}^T(\Delta\Phi_2^1)^T & 0 \end{bmatrix} < 0,$$

or

$$\begin{bmatrix} -X + (\Phi_1 + \Delta\Phi_1)X(\Phi_1 + \Delta\Phi_1)^T & \Phi_2^1\bar{Z} + (\Phi_1 + \Delta\Phi_1)X(\Phi_2^2 + \Delta\Phi_2^2)^T \\ \bar{Z}^T\Phi_2^{1T} + (\Phi_2^2 + \Delta\Phi_2^2)X(\Phi_1 + \Delta\Phi_1)^T & (\Phi_2^2 + \Delta\Phi_2^2)X(\Phi_2^2 + \Delta\Phi_2^2)^T - \bar{Z} - \bar{Z}^T \end{bmatrix}$$
$$+ \begin{bmatrix} \bar{H}_3 & 0 \\ 0 & \bar{Z}^T(\bar{E}_1^{2T} + \bar{K}_2^T\bar{E}_2^T) \end{bmatrix} \begin{bmatrix} 0 & \bar{F} \\ \bar{F}^T & 0 \end{bmatrix} \times \begin{bmatrix} \bar{H}_3^T & 0 \\ 0 & (\bar{E}_1^2 + \bar{E}_2\bar{K}_2)\bar{Z} \end{bmatrix} < 0.$$

Applying Lemma 3 to this last expression and then Lemma 1 to the result yields

$$\begin{bmatrix} -X & \Phi_2^1\bar{Z} & \bar{H}_3 & 0 \\ \bar{Z}^T\Phi_2^{1T} & -\bar{Z}-\bar{Z}^T & 0 & \bar{Z}^T(\bar{E}_1^{2T}+\bar{K}_2^T\bar{E}_2^T) \\ \bar{H}_3^T & 0 & -\varepsilon_1^{-1}I & 0 \\ 0 & (\bar{E}_1^2+\bar{E}_2\bar{K}_2)\bar{Z} & 0 & -\varepsilon_1 I \end{bmatrix}$$
$$+ \begin{bmatrix} \Phi_1+\Delta\Phi_1 \\ \Phi_2^2+\Delta\Phi_2^2 \\ 0 \\ 0 \end{bmatrix} X \begin{bmatrix} (\Phi_1+\Delta\Phi_1)^T & (\Phi_2^2+\Delta\Phi_2^2)^T & 0 & 0 \end{bmatrix} < 0.$$

Using Lemma 2, we now have that

$$\begin{bmatrix} -X & \Phi_2^1\bar{Z} & \bar{H}_3 & 0 & \Phi_1\bar{V} \\ \bar{Z}^T\Phi_2^{1T} & -\bar{Z}-\bar{Z}^T & 0 & \bar{Z}^T(\bar{E}_1^{2T}+\bar{K}_2^T\bar{E}_2^T) & \Phi_2^2\bar{V} \\ \bar{H}_3^T & 0 & -\varepsilon_1^{-1}I & 0 & 0 \\ 0 & (\bar{E}_1^2+\bar{E}_2\bar{K}_2)\bar{Z} & 0 & -\varepsilon_1 I & 0 \\ \bar{V}^T\Phi_1^T & \bar{V}^T\Phi_2^{2T} & 0 & 0 & X-\bar{V}-\bar{V}^T \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & \Delta\Phi_1\bar{V} \\ 0 & 0 & 0 & 0 & \Delta\Phi_2^2\bar{V} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \bar{V}^T\Delta\Phi_1^T & \bar{V}^T\Delta\Phi_2^{2T} & 0 & 0 & 0 \end{bmatrix} < 0,$$

where $\bar{V}$ is given by (18) and V_1, V_2 and V_3 are nonsingular, possibly nonsymmetric, matrices of compatible dimensions.

Now, rewrite the last inequality as

$$
\begin{bmatrix}
-X & \Phi_2^1\bar{Z} & \bar{H}_3 & 0 & \Phi_1\bar{V} \\
\bar{Z}^T\Phi_2^{1T} & -\bar{Z}-\bar{Z}^T & 0 & \bar{Z}^T(\bar{E}_1^{2T}+\bar{K}_2^T\bar{E}_2^T) & \Phi_2^2\bar{V} \\
\bar{H}_3^T & 0 & -\varepsilon_1^{-1}I & 0 & 0 \\
0 & (\bar{E}_1^2+\bar{E}_2\bar{K}_2)\bar{Z} & 0 & -\varepsilon_1 I & 0 \\
\bar{V}^T\Phi_1^T & \bar{V}^T\Phi_2^{2T} & 0 & 0 & X-\bar{V}-\bar{V}^T
\end{bmatrix}
+
\begin{bmatrix}
\bar{H}_1 & 0 \\
\bar{H}_2 & 0 \\
0 & 0 \\
0 & 0 \\
0 & \bar{V}^T(\bar{E}_1^{1T}+\bar{K}_1^T\bar{E}_2^T)
\end{bmatrix}
\begin{bmatrix}
0 & \bar{F} \\
\bar{F}^T & 0
\end{bmatrix}
\times
\begin{bmatrix}
\bar{H}_1^T & \bar{H}_2^T & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & (\bar{E}_1^1+\bar{E}_2\bar{K}_1)\bar{V}
\end{bmatrix}
< 0.
$$

The application of Lemma 3 to this last expression and Lemma 1 to the result yields

$$
\begin{bmatrix}
-X & \Phi_2^1\bar{Z} & \bar{H}_3 & 0 & \Phi_1\bar{V} & \bar{H}_1 & 0 \\
\bar{Z}^T\Phi_2^{1T} & -\bar{Z}-\bar{Z}^T & 0 & \bar{Z}^T(\bar{E}_1^{2T}+\bar{K}_2^T\bar{E}_2^T) & \Phi_2^2\bar{V} & \bar{H}_2 & 0 \\
\bar{H}_3^T & 0 & -\varepsilon_1^{-1}I & 0 & 0 & 0 & 0 \\
0 & (\bar{E}_1^2+\bar{E}_2\bar{K}_2)\bar{Z} & 0 & -\varepsilon_1 I & 0 & 0 & 0 \\
\bar{V}^T\Phi_1^T & \bar{V}^T\Phi_2^{2T} & 0 & 0 & X-\bar{V}-\bar{V}^T & 0 & \bar{V}^T(\bar{E}_1^{1T}+\bar{K}_1^T\bar{E}_2^T) \\
\bar{H}_1^T & \bar{H}_2^T & 0 & 0 & 0 & -\varepsilon_2^{-1}I & 0 \\
0 & 0 & 0 & 0 & (\bar{E}_1^1+\bar{E}_2\bar{K}_1)\bar{V} & 0 & -\varepsilon_2 I
\end{bmatrix}
< 0.
$$

Next, premultiplying and postmultiplying this inequality by $\mathbf{diag}\,(I, I, \varepsilon_1 I, I, I, \varepsilon_2 I, I)$ gives

$$
\begin{bmatrix}
-X & \Phi_2^1\bar{Z} & \varepsilon_1\bar{H}_3 & 0 & \Phi_1\bar{V} & \varepsilon_2\bar{H}_1 & 0 \\
\bar{Z}^T\Phi_2^{1T} & -\bar{Z}-\bar{Z}^T & 0 & \bar{Z}^T(\bar{E}_1^{2T}+\bar{K}_2^T\bar{E}_2^T) & \Phi_2^2\bar{V} & \varepsilon_2\bar{H}_2 & 0 \\
\varepsilon_1\bar{H}_3^T & 0 & -\varepsilon_1 I & 0 & 0 & 0 & 0 \\
0 & (\bar{E}_1^2+\bar{E}_2\bar{K}_2)\bar{Z} & 0 & -\varepsilon_1 I & 0 & 0 & 0 \\
\bar{V}^T\Phi_1^T & \bar{V}^T\Phi_2^{2T} & 0 & 0 & X-\bar{V}-\bar{V}^T & 0 & \bar{V}^T(\bar{E}_1^{1T}+\bar{K}_1^T\bar{E}_2^T) \\
\varepsilon_2\bar{H}_1^T & \varepsilon_2\bar{H}_2^T & 0 & 0 & 0 & -\varepsilon_2 I & 0 \\
0 & 0 & 0 & 0 & (\bar{E}_1^1+\bar{E}_2\bar{K}_1)\bar{V} & 0 & -\varepsilon_2 I
\end{bmatrix}
< 0.
$$

Finally, using

$$
\begin{aligned}
\Phi_1\bar{V} &= \bar{A}_1\bar{V} + \bar{B}_1\bar{K}_1\bar{V} = \bar{A}_1\bar{V} + \bar{B}_1\bar{L}, \\
\Phi_2^1\bar{Z} &= \bar{A}_2^1\bar{Z} + \bar{B}_2^1\bar{K}_2\bar{Z} = \bar{A}_2^1\bar{Z} + \bar{B}_2^1\bar{N}, \\
\Phi_2^2\bar{V} &= \bar{A}_2^2\bar{V} + \bar{B}_2^2\bar{K}_1\bar{V} = \bar{A}_2^2\bar{V} + \bar{B}_2^2\bar{L}, \\
(\bar{E}_1^1 + \bar{E}_2\bar{K}_1)\bar{V} &= \bar{E}_1^1\bar{V} + \bar{E}_2\bar{K}_1\bar{V} = \bar{E}_1^1\bar{V} + \bar{E}_2\bar{L}, \\
(\bar{E}_1^2 + \bar{E}_2\bar{K}_2)\bar{Z} &= \bar{E}_1^2\bar{Z} + \bar{E}_2\bar{K}_2\bar{Z} = \bar{E}_1^2\bar{Z} + \bar{E}_2\bar{N}
\end{aligned}
\tag{42}
$$

leads to (41) with the control law matrices given by (19), and the proof is complete. ■

The following example provides an application of Theorem 7.

Example 1. Consider the following discrete linear repetitive process state-space model where all but the matrices B_{01}, B_{02}, D_{01} and D_{02} have been obtained from a 1D differential linear system of one axis of a gantry robot used in iterative learning control experiments (Ratcliffe *et al.*, 2005) with a sample time $T_s = 0.01$ sec:

$$
A_1 = \begin{bmatrix}
-0.5707\cdot 10^{-1} & 1 & 0 & 0 & 0 & 0 & 0 \\
-0.2446 & -0.5707\cdot 10^{-1} & -0.6013 & 0.1625 & 0.3896 & 0.3688 & 0.7939 \\
0 & 0 & 0.1384 & 1 & 0 & 0 & 0 \\
0 & 0 & -0.7884 & 0.1384 & 0.3850 & 0.3645 & 0.7845 \\
0 & 0 & 0 & 0 & 0.6236 & 1 & 0 \\
0 & 0 & 0 & 0 & -0.3977 & 0.6236 & 0.6758 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
$$

$$B_1 = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\4 \end{bmatrix}, \quad B_{01} = \begin{bmatrix} 0.10\\-0.30\\-0.20\\0.10\\-0.10\\0.40\\0.30 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 3.1505\\-0.2620\\-3.0157\\0.8150\\1.9542\\1.8499\\3.9819 \end{bmatrix}^T,$$

$$D_1 = 0, \qquad D_{01} = 1.20,$$

$$A_2 = \left[\begin{matrix} 0.2653 & 1 & 0\\ -0.1375 & 0.2653 & 0.3123\\ 0 & 0 & 0.6635\cdot 10^{-2}\\ 0 & 0 & -0.1396\cdot 10^{-1}\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{matrix}\right.$$

$$\left.\begin{matrix} 0 & 0 & 0 & 0\\ -0.8535 & 1.4479 & 1.0859 & 1.2343\\ 1 & 0 & 0 & 0\\ 0.6635\cdot 10^{-2} & 0.8637 & 0.6478 & 0.7363\\ 0 & 1 & 1.0321 & 1.1731\\ 0 & 0 & 1 & 0.8798\\ 0 & 0 & 0 & 1 \end{matrix}\right],$$

$$B_2 = 10^{-2}\begin{bmatrix} 0\\0\\0\\0\\0\\0\\0.3906 \end{bmatrix}, \quad B_{02} = \begin{bmatrix} -0.40\\0.30\\-0.10\\0.20\\0.30\\-0.20\\0.10 \end{bmatrix},$$

$$C_2 = 10^{-2}\begin{bmatrix} 0.1623\\-0.5927\\0.1260\\-0.3443\\0.5840\\0.4380\\0.4979 \end{bmatrix}^T, \quad D_2 = 0, \qquad D_{02} = 0.60,$$

and the uncertainty structure modelled by

$$H = \begin{bmatrix} H_1\\H_2\\H_3\\H_4 \end{bmatrix} = \begin{bmatrix} 0.1 \end{bmatrix}_{(2n+2m)\times 1},$$

$$E = \begin{bmatrix} E_{11} \mid E_{12} \mid E_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0.01 \end{bmatrix}_{1\times n} \mid 0.01 \mid 0.01 \end{bmatrix}.$$

Using the MATLAB LMI Control Toolbox, we can check that the stabilization condition of Theorem 7 holds with (43) and

$$L_1 = \begin{bmatrix} 52.9058 & -25.0560 & -19.4657 & 54.6730 \\ & 39.8828 & 28.1598 & -101.7701 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} -936.9583 & -893.6913 & -513.6079 & -633.7263 \\ & -559.0882 & 911.8489 & -824.8886 \end{bmatrix},$$

$$L_3 = -8.1018,$$
$$Z = 57.6722, \qquad N = -745.3623,$$
$$\varepsilon_1 = 17.7970, \qquad \varepsilon_2 = 19.6019,$$

and

$$\mathbf{eig}\,(X) = \{ \begin{matrix} 0.7158, & 1.1292, & 37.5499,\\ 44.3078, & 72.0117, & 79.3489,\\ 150.2692, & 170.3241, & 314.2136,\\ 327.6514, & 351.9829, & 558.9057,\\ 667.8880, & 928.4967, & 1255.5043 \end{matrix} \},$$

where $\mathbf{eig}\,(\cdot)$ denotes the eigenvalues of its matrix argument, which confirms that matrix X is positive definite.

The only remaining potential numerical difficulty is associated with forming the inverses of the square matrices $\bar{V}$ and $\bar{Z}$ of (18) to compute the control law gain matrices. The condition numbers of these matrices are given by

$$\mathbf{cond}\left(\bar{V}\right) = 1184.3982, \quad \mathbf{cond}\left(\bar{Z}\right) = 1.0000$$

and hence no difficulties of this type arise. Finally, the control law matrices are given by

$$K_1^1 = \begin{bmatrix} 0.0050 & -0.1983 & 0.0612 & 0.1804 \\ & -0.0438 & -0.2085 & -0.3628 \end{bmatrix},$$

$$K_1^2 = \begin{bmatrix} -0.0547 & 1.8972 & 0.2814 & -0.0144 \\ & -17.8868 & -120.3157 & -357.2174 \end{bmatrix},$$

$$K_2^1 = -0.1826, \qquad K_2^2 = -12.9241,$$

which produces the final controlled switched process stable along the pass. The respective simulation results are shown in Fig. 1.

$$X_1 = \begin{bmatrix} 265.0370 & 3.2621 & 124.5642 & 19.9530 & -7.9962 & 22.9728 & -127.6870 & -17.4785 & -0.5425 & -12.7266 & -2.7633 & -1.2564 & 1.7970 & -0.7897 \\ 3.2621 & 242.0984 & -11.9722 & 144.3611 & -17.1366 & 23.5575 & -29.0969 & 34.4636 & -20.9129 & 11.8933 & -15.5503 & -21.8787 & 12.8103 & -3.5208 \\ 124.5642 & -11.9722 & 368.4182 & -51.3197 & 177.8854 & -116.7439 & 158.2551 & 73.3648 & -20.4005 & 33.0491 & -13.1771 & -28.1982 & 14.7954 & -3.5859 \\ 19.9530 & 144.3611 & -51.3197 & 246.5962 & 8.7653 & 97.9338 & -146.7649 & -42.7514 & 27.4117 & -13.0525 & 19.5248 & 29.1214 & -17.5577 & 4.9810 \\ -7.9962 & -17.1366 & 177.8854 & 8.7653 & 475.9322 & -110.0790 & -37.4967 & 40.9310 & 6.3315 & 26.5892 & 7.2460 & 0.2742 & -3.1597 & 1.6899 \\ 22.9728 & 23.5575 & -116.7439 & 97.9338 & -110.0790 & 341.3245 & -233.4411 & -68.4696 & 31.5829 & -26.5117 & 21.5341 & 36.2554 & -20.4317 & 5.3847 \\ -127.6870 & -29.0969 & 158.2551 & -146.7649 & -37.4967 & -233.4411 & 378.8219 & 95.7716 & -40.5744 & 39.1009 & -26.9603 & -47.8465 & 27.0030 & -7.1344 \\ -17.4785 & 34.4636 & 73.3648 & -42.7514 & 40.9310 & -68.4696 & 95.7716 & 986.3738 & 54.9630 & 336.3364 & 75.6992 & 38.5814 & 12.0750 & -1.8076 \\ -0.5425 & -20.9129 & -20.4005 & 27.4117 & 6.3315 & 31.5829 & -40.5744 & 54.9630 & 465.3885 & 57.1786 & 217.5740 & 225.1623 & -92.2416 & 25.0249 \\ -12.7266 & 11.8933 & 33.0491 & -13.0525 & 26.5892 & -26.5117 & 39.1009 & 336.3364 & 57.1786 & 501.0665 & 128.1086 & 80.4877 & -32.9451 & 11.8289 \\ -2.7633 & -15.5503 & -13.1771 & 19.5248 & 7.2460 & 21.5341 & -26.9603 & 75.6992 & 217.5740 & 128.1086 & 269.6901 & 202.8584 & -80.1919 & 20.8023 \\ -1.2564 & -21.8787 & -28.1982 & 29.1214 & 0.2742 & 36.2554 & -47.8465 & 38.5814 & 225.1623 & 80.4877 & 202.8584 & 281.1896 & -109.5404 & 26.1347 \\ 1.7970 & 12.8103 & 14.7954 & -17.5577 & -3.1597 & -20.4317 & 27.0030 & 12.0750 & -92.2416 & -32.9451 & -80.1919 & -109.5404 & 84.0842 & -26.1693 \\ -0.7897 & -3.5208 & -3.5859 & 4.9810 & 1.6899 & 5.3847 & -7.1344 & -1.8076 & 25.0249 & 11.8289 & 20.8023 & 26.1347 & -26.1693 & 9.9700 \end{bmatrix}, \tag{43a}$$

$$X_2 = 44.3078, \tag{43b}$$

$$V_1 = \begin{bmatrix} 261.7018 & 4.7001 & 110.5979 & 2.2956 & -27.8408 & 17.3028 & -119.8872 \\ 3.0288 & 253.2669 & 16.6811 & 94.2706 & -21.7773 & -9.8883 & 20.4174 \\ 120.4596 & 1.8425 & 306.3218 & -73.5666 & 122.7114 & -108.3803 & 141.7747 \\ 14.7726 & 126.4219 & 19.3522 & 251.0069 & 65.0976 & 74.3826 & -106.1499 \\ -6.3846 & -12.3672 & 96.4814 & -18.0123 & 399.1164 & -83.6090 & -69.5487 \\ 16.7381 & 7.3868 & 1.2933 & 91.3309 & -27.0557 & 297.9124 & -157.4820 \\ -125.1051 & -8.8567 & 94.9489 & -140.1001 & -77.9619 & -214.4360 & 337.7773 \end{bmatrix}, \tag{43c}$$

$$V_2 = \begin{bmatrix} 690.1699 & 33.7436 & 107.5294 & 37.1964 & 60.2176 & 5.2953 & -2.5731 \\ -7.7851 & 444.6480 & 6.5624 & 155.6922 & 203.7368 & -83.4781 & 23.3039 \\ 292.8675 & 41.3942 & 382.8377 & 99.8370 & 91.3793 & -35.8418 & 11.5799 \\ 13.1630 & 208.9687 & 39.8069 & 258.0101 & 198.3998 & -77.1550 & 19.9475 \\ -66.6670 & 230.9958 & -41.5460 & 160.2523 & 269.9527 & -101.4096 & 23.6145 \\ 36.5857 & -101.7159 & 12.8268 & -65.2533 & -102.3273 & 80.5270 & -25.2972 \\ -6.2784 & 27.5752 & -0.4838 & 16.6175 & 23.6500 & -25.0667 & 9.7797 \end{bmatrix}, \tag{43d}$$

$$V_3 = 44.3791, \tag{43e}$$

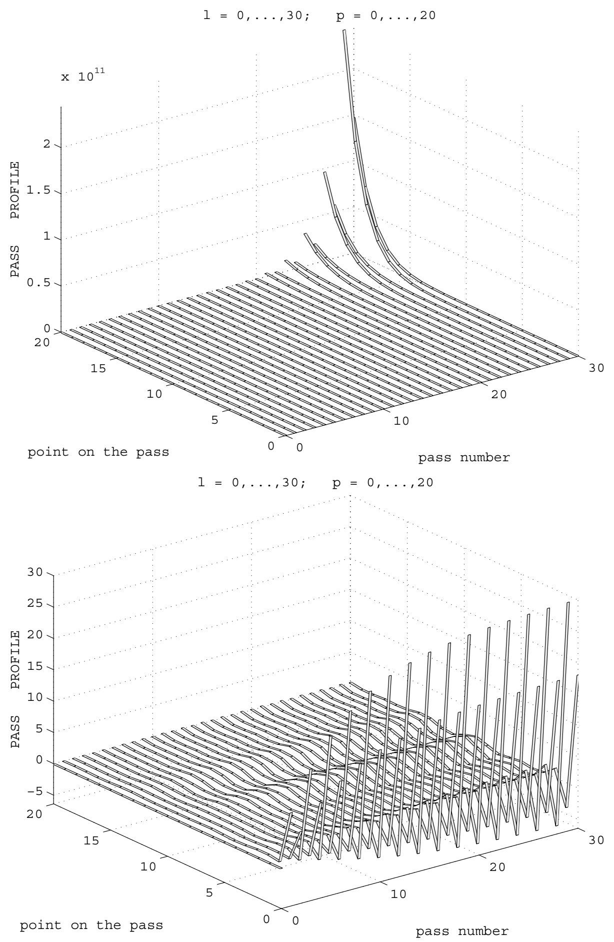

Fig. 1. Pass profile sequence of the model of Example 1 before (left plot) and after (right plot) stabilization. The boundary conditions are $y_0(p) = 0$, and each entry in $x_{l+1}(0)$ is equal to unity.

4. Processes with Dynamics Switched Along the Pass

The analysis of the previous section can be adopted for application to discrete linear repetitive processes with switching in the along the pass direction as described by

$$x_{k+1}(s+1) = \begin{cases} A_1 x_{k+1}(s) + B_1 u_{k+1}(s) + B_{01} y_k(s), \\ \qquad \text{for } s = 0, 2, \dots, \text{ i.e., } s = 2p, \\ A_2 x_{k+1}(s) + B_2 u_{k+1}(s) + B_{02} y_k(s), \\ \qquad \text{for } s = 1, 3, \dots, \text{ i.e., } s = 2p+1, \end{cases} \tag{44}$$

$$y_{k+1}(s) = \begin{cases} C_1 x_{k+1}(s) + D_1 u_{k+1}(s) + D_{01} y_k(s), \\ \qquad \text{for } s = 0, 2, \dots, \text{ i.e., } s = 2p, \\ C_2 x_{k+1}(s) + D_2 u_{k+1}(s) + D_{02} y_k(s), \\ \qquad \text{for } s = 1, 3, \dots, \text{ i.e., } s = 2p+1, \end{cases}$$

where $s = 0, 1, \dots, (\alpha - 1)$ denotes points along the pass, $p = 0, 1, \dots, (\alpha - 2)/2$, and all other symbols have the meaning assigned to them in the previous section. Alternatively, we can use the state-space model

$$\begin{aligned} X_{k+1}(s+1) &= \check{A} X_{k+1}(s) + \check{B} U_{k+1}(s) + \check{B}_0 Y_k(s), \\ Y_{k+1}(s) &= \check{C} X_{k+1}(s) + \check{D} U_{k+1}(s) + \check{D}_0 Y_k(s), \end{aligned} \tag{45}$$

where

$$X_{k+1}(s) = x_{k+1}(2p), \qquad X_{k+1}(s+1) = x_{k+1}(2p+2),$$

$$U_{k+1}(s) = \begin{bmatrix} u_{k+1}(2p) \\ u_{k+1}(2p+1) \end{bmatrix}, \quad Y_{k+1}(s) = \begin{bmatrix} y_{k+1}(2p) \\ y_{k+1}(2p+1) \end{bmatrix}$$

and

$$\check{A} = A_2 A_1, \ \check{B} = \begin{bmatrix} A_2 B_1 & B_2 \end{bmatrix}, \ \check{B}_0 = \begin{bmatrix} A_2 B_{01} & B_{02} \end{bmatrix},$$

$$\check{C} = \begin{bmatrix} C_1 \\ C_2 A_1 \end{bmatrix}, \ \check{D} = \begin{bmatrix} D_1 & 0 \\ C_2 B_1 & D_2 \end{bmatrix}, \ \check{D}_0 = \begin{bmatrix} D_{01} & 0 \\ C_2 B_{01} & D_{02} \end{bmatrix}.$$

For stability along the pass we have the following result, which follows immediately if we interpret Theorem 2 in terms of this last state-space model.

Theorem 8. (Gałkowski *et al.*, 2002; Rogers and Owens, 1992) *A discrete linear repetitive process which can be written in the form (44) is stable along the pass if there exist matrices $W_1 > 0$ and $W_2 > 0$ such that*

$$\check{\Phi}^T W \check{\Phi} - W < 0, \tag{46}$$

where the augmented process matrix $\check{\Phi}$ is given by

$$\check{\Phi} = \begin{bmatrix} \check{A} & \check{B}_0 \\ \check{C} & \check{D}_0 \end{bmatrix} \tag{47}$$

and $W = \mathbf{diag}\,(W_1, W_2)$.

Consider also the use of a switched control law of the form

$$u_{k+1}(s) = \begin{cases} K_1^1 x_{k+1}(s) + K_2^1 y_k(s), \\ \qquad \text{for } s = 0, 2, \dots, \text{ i.e., } s = 2p, \\ K_1^2 x_{k+1}(s) + K_2^2 y_k(s), \\ \qquad \text{for } s = 1, 3, \dots, \text{ i.e., } s = 2p+1, \end{cases} \tag{48}$$

or

$$U_{k+1}(s) = \check{K}_1 X_{k+1}(s) + \check{K}_2 Y_k(s), \tag{49}$$

where

$$\check{K}_1 = \begin{bmatrix} K_1^1 \\ K_1^2 (A_1 + B_1 K_1^1) \end{bmatrix},$$

$$\check{K}_2 = \begin{bmatrix} K_2^1 & 0 \\ K_1^2 (B_{01} + B_1 K_2^1) & K_2^2 \end{bmatrix}.$$

The resulting controlled process state-space model can be written as

$$\begin{aligned} X_{k+1}(s+1) &= \check{A}_{\text{new}} X_{k+1}(s) + \check{B}_{0\text{new}} Y_k(s), \\ Y_{k+1}(s) &= \check{C}_{\text{new}} X_{k+1}(s) + \check{D}_{0\text{new}} Y_k(s), \end{aligned} \tag{50}$$

where

$$\check{A}_{\text{new}} = \check{A} + \check{B}\check{K}_1 = \left[\ (A_2 + B_2K_1^2)(A_1 + B_1K_1^1)\ \right],$$

$$\begin{aligned} \check{B}_{0\text{new}} &= \check{B}_0 + \check{B}\check{K}_2 \\ &= \left[(A_2+B_2K_1^2)(B_{01}+B_1K_2^1) \quad B_{02}+B_2K_2^2 \right], \end{aligned}$$

$$\check{C}_{\text{new}} = \check{C} + \check{D}\check{K}_1 = \begin{bmatrix} C_1 + D_1K_1^1 \\ (C_2 + D_2K_1^2)(A_1 + B_1K_1^1) \end{bmatrix},$$

$$\begin{aligned} \check{D}_{0\text{new}} &= \check{D}_0 + \check{D}\check{K}_2 \\ &= \begin{bmatrix} D_{01}+D_1K_2^1 & 0 \\ (C_2+D_2K_1^2)(B_{01}+B_1K_2^1) & D_{02}+D_2K_2^2 \end{bmatrix}. \end{aligned}$$

Also, introduce

$$\begin{aligned} \check{\Phi}_{\text{new}} &= \begin{bmatrix} \check{A}_{\text{new}} & \check{B}_{0\text{new}} \\ \check{C}_{\text{new}} & \check{D}_{0\text{new}} \end{bmatrix} \\ &= \begin{bmatrix} \check{A} & \check{B}_0 \\ \check{C} & \check{D}_0 \end{bmatrix} + \begin{bmatrix} \check{B} \\ \check{D} \end{bmatrix} \left[\ \check{K}_1 \quad \check{K}_2\ \right]. \end{aligned} \tag{51}$$

Then we can rewrite (51) in the following form:

$$\check{\Phi}_{\text{new}} = \Phi_1 + \Phi_2^1 \Phi_2^2, \tag{52}$$

where

$$\begin{aligned} \Phi_1 &= \begin{bmatrix} 0 & 0 & B_{02} + B_2K_2^2 \\ C_1 + D_1K_1^1 & D_{01} + D_1K_2^1 & 0 \\ 0 & 0 & D_{02} + D_2K_2^2 \end{bmatrix} \\ &= \widetilde{A}_1 + \widetilde{B}_1\widetilde{K}_1, \\ \Phi_2^1 &= \begin{bmatrix} A_2 + B_2K_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_2 + D_2K_1^2 \end{bmatrix} \\ &= \widetilde{A}_2^1 + \widetilde{B}_2^1\widetilde{K}_2, \\ \Phi_2^2 &= \begin{bmatrix} A_1 + B_1K_1^1 & B_{01} + B_1K_2^1 & 0 \\ 0 & 0 & 0 \\ A_1 + B_1K_1^1 & B_{01} + B_1K_2^1 & 0 \end{bmatrix} \\ &= \widetilde{A}_2^2 + \widetilde{B}_2^2\widetilde{K}_1, \end{aligned}$$

and

$$\widetilde{A} = \begin{bmatrix} 0 & 0 & B_{02} \\ C_1 & D_{01} & 0 \\ 0 & 0 & D_{02} \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} 0 & 0 & B_2 \\ D_1 & D_1 & 0 \\ 0 & 0 & D_2 \end{bmatrix},$$

$$\widetilde{A}_2^1 = \begin{bmatrix} A_2 & A_2 & 0 \\ 0 & 0 & 0 \\ C_2 & C_2 & 0 \end{bmatrix}, \quad \widetilde{B}_2^1 = \begin{bmatrix} B_2 & B_2 & 0 \\ 0 & 0 & 0 \\ D_2 & D_2 & 0 \end{bmatrix},$$

$$\widetilde{A}_2^2 = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & B_{01} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \widetilde{B}_2^2 = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{K}_1 = \begin{bmatrix} K_1^1 & 0 & 0 \\ 0 & K_2^1 & 0 \\ 0 & 0 & K_2^2 \end{bmatrix}, \quad \widetilde{K}_2 = \begin{bmatrix} K_1^2 & 0 & 0 \\ 0 & K_1^2 & 0 \\ 0 & 0 & K_1^2 \end{bmatrix}.$$

Now we are in a position to establish the following result:

Theorem 9. *Suppose that a control law of the form (49) is applied to a discrete linear repetitive process which can be written in the form (45). Then the resulting controlled process is stable along the pass if there exist a matrix* $X > 0$*, nonsingular matrices* $\widetilde{V}$ *and* $\widetilde{Z}$*, and rectangular matrices* $\widetilde{L}$ *and* $\widetilde{N}$ *such that*

$$\begin{bmatrix} -X & \widetilde{A}_2^1\widetilde{Z}+\widetilde{B}_2^1\widetilde{N} & \widetilde{A}\widetilde{V}+\widetilde{B}\widetilde{L} \\ \widetilde{Z}^T\widetilde{A}_2^{1T}+\widetilde{N}^T\widetilde{B}_2^{1T} & -\widetilde{Z}-\widetilde{Z}^T & \widetilde{A}_2^2\widetilde{V}+\widetilde{B}_2^2\widetilde{L} \\ \widetilde{V}^T\widetilde{A}^T+\widetilde{L}^T\widetilde{B}^T & \widetilde{V}^T\widetilde{A}_2^{2T}+\widetilde{L}^T\widetilde{B}_2^{2T} & X-\widetilde{V}-\widetilde{V}^T \end{bmatrix} < 0, \tag{53}$$

where

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}, \quad \widetilde{V} = \begin{bmatrix} V_1 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & V_3 \end{bmatrix},$$

$$\widetilde{L} = \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{bmatrix}, \quad \widetilde{N} = \begin{bmatrix} N & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & N \end{bmatrix},$$

$$\widetilde{Z} = \begin{bmatrix} Z & 0 & 0 \\ 0 & Z & 0 \\ 0 & 0 & Z \end{bmatrix}. \tag{54}$$

If this condition holds, the control law matrices are given by

$$\widetilde{K}_1 = \widetilde{L}\widetilde{V}^{-1} = \begin{bmatrix} K_1^1 & 0 & 0 \\ 0 & K_2^1 & 0 \\ 0 & 0 & K_2^2 \end{bmatrix}, \qquad (55)$$
$$\widetilde{K}_2 = \widetilde{N}\widetilde{Z}^{-1} = \begin{bmatrix} K_1^2 & 0 & 0 \\ 0 & K_1^2 & 0 \\ 0 & 0 & K_1^2 \end{bmatrix}.$$

Proof. The proof proceeds using steps identical to those of Theorem 3, with

$$\begin{aligned} \Phi_1\widetilde{V} &= \widetilde{A}\widetilde{V} + \widetilde{B}\widetilde{K}_1\widetilde{V} = \widetilde{A}\widetilde{V} + \widetilde{B}\widetilde{L}, \\ \Phi_2^1\widetilde{Z} &= \widetilde{A}_2^1\widetilde{Z} + \widetilde{B}_2^1\widetilde{K}_2\widetilde{Z} = \widetilde{A}_2^1\widetilde{Z} + \widetilde{B}_2^1\widetilde{N}, \qquad (56) \\ \Phi_2^2\widetilde{V} &= \widetilde{A}_2^2\widetilde{V} + \widetilde{B}_2^2\widetilde{K}_1\widetilde{V} = \widetilde{A}_2^2\widetilde{V} + \widetilde{B}_2^2\widetilde{L}, \end{aligned}$$

and hence the details are omitted here. ∎

Theorem 9 is a modification of a result in (Bochniak *et al.*, 2006), where the degree of conservativeness is reduced to a chance of numerical difficulties.

4.1. Robustness Analysis. Here we expand the robustness analysis developed so far in this paper to the case of dynamics switching along the pass. In the case of polytopic uncertainty, we first have the following result.

Theorem 10. *Consider a discrete linear repetitive process described by (45), with the uncertainty structure modelled by (21). Then the stability along the pass holds if there exist a matrix $X > 0$ and nonsingular matrices $\widetilde{V}$ and $\widetilde{Z}$ such that for all $i = 1, \ldots, P$, i.e., for each vertex of the polytope, we have*

$$\begin{bmatrix} -X & \widetilde{A}_2^{1i}\widetilde{Z} & \widetilde{A}_1^i\widetilde{V} \\ \widetilde{Z}^T\widetilde{A}_2^{1iT} & -\widetilde{Z} - \widetilde{Z}^T & \widetilde{A}_2^{2i}\widetilde{V} \\ \widetilde{V}^T\widetilde{A}_1^{iT} & \widetilde{V}^T\widetilde{A}_2^{2iT} & X - \widetilde{V} - \widetilde{V}^T \end{bmatrix} < 0, \qquad (57)$$

where

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$$

and

$$\widetilde{A}_1^i = \begin{bmatrix} 0 & 0 & B_{02}^i \\ C_1^i & D_{01}^i & 0 \\ 0 & 0 & D_{02}^i \end{bmatrix}, \quad \widetilde{A}_2^{1i} = \begin{bmatrix} A_2^i & A_2^i & 0 \\ 0 & 0 & 0 \\ C_2^i & C_2^i & 0 \end{bmatrix},$$
$$\widetilde{A}_2^{2i} = \begin{bmatrix} A_1^i & 0 & 0 \\ 0 & B_{01}^i & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (58)$$

Proof. Up to routine changes, the proof proceeds in the same way as that of Theorem 4, and hence the details are omitted here. ∎

In what follows, we consider the application of a control law of the form (49) but, because of the the uncertainty in the process state-space model, modified (via (49)) to the form

$$\begin{aligned} \check{K}_1 &\in \mathbf{Co}\left\{\check{K}_1^i,\ i = 1, \ldots, P\right\}, \\ \check{K}_1^i &= \begin{bmatrix} K_1^1 \\ K_1^2(A_1^i + B_1^iK_1^1) \end{bmatrix}, \\ \check{K}_2 &\in \mathbf{Co}\left\{\check{K}_2^i,\ i = 1, \ldots, P\right\}, \qquad (59) \\ \check{K}_2^i &= \begin{bmatrix} K_2^1 & 0 \\ K_1^2(B_{01}^i + B_1^iK_2^1) & K_2^2 \end{bmatrix}. \end{aligned}$$

The resulting controlled discrete linear repetitive process is given by (50) with

$$\begin{aligned} \check{A}_{\text{new}} &\in \mathbf{Co}\left\{\check{A}_{\text{new}}^i,\ i = 1, \ldots, P\right\}, \\ \check{B}_{0\text{new}} &\in \mathbf{Co}\left\{\check{B}_{0\text{new}}^i,\ i = 1, \ldots, P\right\}, \\ \check{C}_{\text{new}} &\in \mathbf{Co}\left\{\check{C}_{\text{new}}^i,\ i = 1, \ldots, P\right\}, \\ \check{D}_{0\text{new}} &\in \mathbf{Co}\left\{\check{D}_{0\text{new}}^i,\ i = 1, \ldots, P\right\}, \\ \check{A}_{\text{new}}^i &= \check{A}^i + \check{B}^i\check{K}_1^i = (A_2^i + B_2^iK_1^2)(A_1^i + B_1^iK_1^1), \\ \check{B}_{0\text{new}}^i &= \check{B}_0^i + \check{B}^i\check{K}_2^i \\ &= \left[(A_2^i + B_2^iK_1^2)(B_{01}^i + B_1^iK_2^1) \quad B_{02}^i + B_2^iK_2^2\right], \\ \check{C}_{\text{new}}^i &= \check{C}^i + \check{D}^i\check{K}_1^i \\ &= \begin{bmatrix} C_1^i + D_1^iK_1^1 \\ (C_2^i + D_2^iK_1^2)(A_1^i + B_1^iK_1^1) \end{bmatrix}, \\ \check{D}_{0\text{new}}^i &= \check{D}_0^i + \check{D}^i\check{K}_2^i \\ &= \begin{bmatrix} D_{01}^i + D_1^iK_2^1 & 0 \\ (C_2^i + D_2^iK_1^2)(B_{01}^i + B_1^iK_2^1) & D_{02}^i + D_2^iK_2^2 \end{bmatrix}. \end{aligned}$$

The associated augmented process matrix is given by

$$\begin{aligned} \check{\Phi}_{\text{new}}^i &= \begin{bmatrix} \check{A}_{\text{new}}^i & \check{B}_{0\text{new}}^i \\ \check{C}_{\text{new}}^i & \check{D}_{0\text{new}}^i \end{bmatrix} \\ &= \begin{bmatrix} \check{A}^i & \check{B}_0^i \\ \check{C}^i & \check{D}_0^i \end{bmatrix} + \begin{bmatrix} \check{B}^i \\ \check{D}^i \end{bmatrix}\begin{bmatrix} \check{K}_1^i & \check{K}_2^i \end{bmatrix} \\ &= \check{\Phi}^i + \check{\Pi}^i\check{K}^i, \qquad (60) \end{aligned}$$

which can be written in the form

$$\check{\Phi}^i_{\text{new}} = \Phi^i_1 + \Phi^{1i}_2 \Phi^{2i}_2 \tag{61}$$

for $i = 1, 2, \dots, P$, where

$$\begin{aligned}
\Phi^i_1 &= \begin{bmatrix} 0 & 0 & B^i_{02}+B^i_2 K^2_2 \\ C^i_1+D^i_1 K^1_1 & D^i_{01}+D^i_1 K^1_2 & 0 \\ 0 & 0 & D^i_{02}+D^i_2 K^2_2 \end{bmatrix} \\
&= \widetilde{A}^i_1 + \widetilde{B}^i_1 \widetilde{K}_1, \\
\Phi^{1i}_2 &= \begin{bmatrix} A^i_2 + B^i_2 K^2_1 & A^i_2 + B^i_2 K^2_1 & 0 \\ 0 & 0 & 0 \\ C^i_2 + D^i_2 K^2_1 & C^i_2 + D^i_2 K^2_1 & 0 \end{bmatrix}, \\
&= \widetilde{A}^{1i}_2 + \widetilde{B}^{1i}_2 \widetilde{K}_2, \\
\Phi^{2i}_2 &= \begin{bmatrix} A^i_1 + B^i_1 K^1_1 & 0 & 0 \\ 0 & B^i_{01} + B^i_1 K^1_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \widetilde{A}^{2i}_2 + \widetilde{B}^{2i}_2 \widetilde{K}_1,
\end{aligned}$$

where $\widetilde{A}^i_1$, $\widetilde{A}^{1i}_2$, $\widetilde{A}^{2i}_2$ are given by (58), and

$$\widetilde{B}^i_1 = \begin{bmatrix} 0 & 0 & B^i_2 \\ D^i_1 & D^i_1 & 0 \\ 0 & 0 & D^i_2 \end{bmatrix}, \quad \widetilde{B}^{1i}_2 = \begin{bmatrix} B^i_2 & B^i_2 & 0 \\ 0 & 0 & 0 \\ D^i_2 & D^i_2 & 0 \end{bmatrix},$$

$$\widetilde{B}^{2i}_2 = \begin{bmatrix} B^i_1 & 0 & 0 \\ 0 & B^i_1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{K}_1 = \begin{bmatrix} K^1_1 & 0 & 0 \\ 0 & K^1_2 & 0 \\ 0 & 0 & K^2_2 \end{bmatrix}, \quad \widetilde{K}_2 = \begin{bmatrix} K^2_1 & 0 & 0 \\ 0 & K^2_1 & 0 \\ 0 & 0 & K^2_1 \end{bmatrix}.$$

Now we have the following result.

Theorem 11. *Suppose that a control law of the form (48) is applied to a discrete linear repetitive process written in the form (45), with the uncertainty structure modelled by (21). Then the resulting uncertain controlled process is stable along the pass if there exist a matrix $X > 0$, nonsingular matrices $\widetilde{V}$ and $\widetilde{Z}$, and rectangular matrices $\widetilde{L}$, $\widetilde{N}$ such that for all $i = 1, \dots, P$ (i.e., for each vertex of the polytope) we have*

$$\begin{bmatrix} -X & \widetilde{A}^{1i}_2 \widetilde{Z} + \widetilde{B}^{1i}_2 \widetilde{N} & \widetilde{A}^i_1 \widetilde{V} + \widetilde{B}^i_1 \widetilde{L} \\ \widetilde{Z}^T \widetilde{A}^{1iT}_2 + \widetilde{N}^T \widetilde{B}^{1iT}_2 & -\widetilde{Z} - \widetilde{Z}^T & \widetilde{A}^{2i}_2 \widetilde{V} + \widetilde{B}^{2i}_2 \widetilde{L} \\ \widetilde{V}^T \widetilde{A}^{iT}_1 + \widetilde{L}^T \widetilde{B}^{iT}_1 & \widetilde{V}^T \widetilde{A}^{2iT}_2 + \widetilde{L}^T \widetilde{B}^{2iT}_2 & X - \widetilde{V} - \widetilde{V}^T \end{bmatrix} < 0, \tag{62}$$

where the matrices X, $\widetilde{V}$, $\widetilde{Z}$, $\widetilde{L}$ and $\widetilde{N}$ are given in (54). If this condition holds, the control law matrices are again given by (55).

Proof. The proof follows the steps of that for Theorem 5, and hence is omitted here. ■

For the norm-bounded uncertainty case, we start with the state-space model over $k = 0, 1, \dots,$ $s = 0, 1, \dots, (\alpha - 1)$:

$$\begin{aligned}
X_{k+1}(s+1) &= (\check{A}+\Delta\check{A})X_{k+1}(s) + (\check{B}+\Delta\check{B})U_{k+1}(s) \\
&\quad + (\check{B}_0 + \Delta\check{B}_0)Y_k(s), \\
Y_{k+1}(s) &= (\check{C}+\Delta\check{C})X_{k+1}(s) + (\check{D}+\Delta\check{D})U_{k+1}(s) \\
&\quad + (\check{D}_0 + \Delta\check{D}_0)Y_k(s),
\end{aligned} \tag{63}$$

where

$$\begin{aligned}
\check{A}+\Delta\check{A} &= (A_2 + \Delta A_2)(A_1 + \Delta A_1), \\
\check{B}+\Delta\check{B} &= \begin{bmatrix} (A_2+\Delta A_2)(B_1+\Delta B_1)(B_2+\Delta B_2) \end{bmatrix}, \\
\check{B}_0+\Delta\check{B}_0 &= \begin{bmatrix} (A_2+\Delta A_2)(B_{01}+\Delta B_{01}) & (B_{02}+\Delta B_{02}) \end{bmatrix}, \\
\check{C}+\Delta\check{C} &= \begin{bmatrix} (C_1 + \Delta C_1) \\ (C_2 + \Delta C_2)(A_1 + \Delta A_1) \end{bmatrix}, \\
\check{D}+\Delta\check{D} &= \begin{bmatrix} (D_1 + \Delta D_1) & 0 \\ (C_2+\Delta C_2)(B_1+\Delta B_1) & (D_2+\Delta D_2) \end{bmatrix}, \\
\check{D}_0+\Delta\check{D}_0 &= \begin{bmatrix} (D_{01} + \Delta D_{01}) & 0 \\ (C_2+\Delta C_2)(B_{01}+\Delta B_{01}) & (D_{02}+\Delta D_{02}) \end{bmatrix},
\end{aligned}$$

and (30) and (31) also apply. Hence the augmented process matrix $(\check{\Phi} + \Delta\check{\Phi})$ in this case can be written as

$$\begin{aligned}
\check{\Phi} + \Delta\check{\Phi} &= \begin{bmatrix} \check{A} + \Delta\check{A} & \check{B}_0 + \Delta\check{B}_0 \\ \check{C} + \Delta\check{C} & \check{D}_0 + \Delta\check{D}_0 \end{bmatrix} \\
&= (\widetilde{A}_1 + \Delta\widetilde{A}_1) + (\widetilde{A}^1_2 + \Delta\widetilde{A}^1_2)(\widetilde{A}^2_2 + \Delta\widetilde{A}^2_2),
\end{aligned} \tag{64}$$

where

$$\Delta\widetilde{A}_1 = \begin{bmatrix} 0 & 0 & \Delta B_{02} \\ \Delta C_1 & \Delta D_{01} & 0 \\ 0 & 0 & \Delta D_{02} \end{bmatrix} = \widetilde{H}_1 \widetilde{F} \widetilde{E}^1_1,$$

$$\Delta\widetilde{A}^1_2 = \begin{bmatrix} \Delta A_2 & \Delta A_2 & 0 \\ 0 & 0 & 0 \\ \Delta C_2 & \Delta C_2 & 0 \end{bmatrix} = \widetilde{H}_3 \widetilde{F} \widetilde{E}^2_1, \tag{65}$$

$$\Delta\widetilde{A}^2_2 = \begin{bmatrix} \Delta A_1 & 0 & 0 \\ 0 & \Delta B_{01} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \widetilde{H}_2 \widetilde{F} \widetilde{E}^1_1,$$

with

$$\widetilde{H}_1 = \begin{bmatrix} 0 & 0 & H_2 \\ H_3 & H_3 & 0 \\ 0 & 0 & H_4 \end{bmatrix}, \quad \widetilde{H}_2 = \begin{bmatrix} H_1 & 0 & 0 \\ 0 & H_1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{H}_3 = \begin{bmatrix} H_2 & H_2 & 0 \\ 0 & 0 & 0 \\ H_4 & H_4 & 0 \end{bmatrix}, \quad \widetilde{F} = \begin{bmatrix} F & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix},$$

$$\widetilde{E}_1^1 = \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & E_{12} & 0 \\ 0 & 0 & E_{12} \end{bmatrix}, \quad \widetilde{E}_1^2 = \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & E_{11} & 0 \\ 0 & 0 & E_{11} \end{bmatrix}.$$

Now we have the following result.

Theorem 12. *A discrete linear repetitive process described by (63) with the uncertainty structure modelled by (30) is stable along the pass if there exist a matrix* $X > 0$*, nonsingular matrices* $\widetilde{V}$ *and* $\widetilde{Z}$*, and scalars* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$ *such that*

$$\begin{bmatrix} -X & \widetilde{A}_2^1\widetilde{Z} & \varepsilon_1\widetilde{H}_3 & 0 & \widetilde{A}_1\widetilde{V} & \varepsilon_2\widetilde{H}_1 & 0 \\ \widetilde{Z}^T\widetilde{A}_2^{1T} & -\widetilde{Z}-\widetilde{Z}^T & 0 & \widetilde{Z}^T\widetilde{E}_1^{2T} & \widetilde{A}_2^2\widetilde{V} & \varepsilon_2\widetilde{H}_2 & 0 \\ \varepsilon_1\widetilde{H}_3^T & 0 & -\varepsilon_1 I & 0 & 0 & 0 & 0 \\ 0 & \widetilde{E}_1^2\widetilde{Z} & 0 & -\varepsilon_1 I & 0 & 0 & 0 \\ \widetilde{V}^T\widetilde{A}_1^T & \widetilde{V}^T\widetilde{A}_2^{2T} & 0 & 0 & X-\widetilde{V}-\widetilde{V}^T & 0 & \widetilde{V}^T\widetilde{E}_1^{1T} \\ \varepsilon_2\widetilde{H}_1^T & \varepsilon_2\widetilde{H}_2^T & 0 & 0 & 0 & -\varepsilon_2 I & 0 \\ 0 & 0 & 0 & 0 & \widetilde{E}_1^1\widetilde{V} & 0 & -\varepsilon_2 I \end{bmatrix} < 0, \qquad (66)$$

where $X = \mathbf{diag}\,(X_1, X_2)$.

Proof. The proof proceeds as that of Theorem 6, and hence the details are omitted here. ■

Consider now the control law

$$\begin{aligned} U_{k+1}(s) = {} & (\check{K}_1 + \Delta\check{K}_1)X_{k+1}(s) \\ & + (\check{K}_2 + \Delta\check{K}_2)Y_k(s), \end{aligned} \qquad (67)$$

where

$$\begin{aligned} & \check{K}_1 + \Delta\check{K}_1 \\ & \quad = \begin{bmatrix} K_1^1 \\ K_1^2(A_1 + \Delta A_1 + (B_1 + \Delta B_1)K_1^1) \end{bmatrix}, \\ & \check{K}_2 + \Delta\check{K}_2 \\ & \quad = \begin{bmatrix} K_2^1 & 0 \\ K_1^2(B_{01} + \Delta B_{01} + (B_1 + \Delta B_1)K_2^1) & K_2^2 \end{bmatrix}. \end{aligned} \qquad (68)$$

The resulting controlled process state-space model can be written as

$$\begin{aligned} X_{k+1}(s+1) = {} & (\check{A}_{\text{new}} + \Delta\check{A}_{\text{new}})X_{k+1}(s) \\ & + (\check{B}_{0\text{new}} + \Delta\check{B}_{0\text{new}})Y_k(s), \\ Y_{k+1}(s) = {} & (\check{C}_{\text{new}} + \Delta\check{C}_{\text{new}})X_{k+1}(s) \\ & + (\check{D}_{0\text{new}} + \Delta\check{D}_{0\text{new}})Y_k(s), \end{aligned} \qquad (69)$$

where

$$\begin{aligned} \check{A}_{\text{new}} + \Delta\check{A}_{\text{new}} = {} & \big(A_2 + \Delta A_2 + (B_2 + \Delta B_2)K_1^2\big) \\ & \times \big(A_1 + \Delta A_1 + (B_1 + \Delta B_1)K_1^1\big), \end{aligned}$$

$$\begin{aligned} & \check{B}_{0\text{new}} + \Delta\check{B}_{0\text{new}} \\ & = \Big[\begin{matrix} \big(A_2 + \Delta A_2 + (B_2 + \Delta B_2)K_1^2\big) \\ \times\big(B_{01} + \Delta B_{01} + (B_1 + \Delta B_1)K_2^1\big) \end{matrix} \\ & \qquad B_{02} + \Delta B_{02} + (B_2 + \Delta B_2)K_2^2 \Big], \end{aligned}$$

$$\check{C}_{\text{new}} + \Delta\check{C}_{\text{new}} = \begin{bmatrix} C_1 + \Delta C_1 + (D_1 + \Delta D_1)K_1^1 \\ \big(C_2 + \Delta C_2 + (D_2 + \Delta D_2)K_1^2\big) \\ \times\big(A_1 + \Delta A_1 + (B_1 + \Delta B_1)K_1^1\big) \end{bmatrix},$$

$$\begin{aligned} & \check{D}_{0\text{new}} + \Delta\check{D}_{0\text{new}} \\ & = \begin{bmatrix} D_{01} + \Delta D_{01} + (D_1 + \Delta D_1)K_2^1 & 0 \\ \big(C_2 + \Delta C_2 + (D_2 + \Delta D_2)K_1^2\big) \\ \times\big(B_{01} + \Delta B_{01} + (B_1 + \Delta B_1)K_2^1\big) & D_{02} + \Delta D_{02} + (D_2 + \Delta D_2)K_2^2 \end{bmatrix}, \end{aligned}$$

and the associated augmented process matrix as

$$\begin{aligned} & \check{\Phi}_{\text{new}} + \Delta\check{\Phi}_{\text{new}} \\ & \quad = \begin{bmatrix} \check{A}_{\text{new}} + \Delta\check{A}_{\text{new}} & \check{B}_{0\text{new}} + \Delta\check{B}_{0\text{new}} \\ \check{C}_{\text{new}} + \Delta\check{C}_{\text{new}} & \check{D}_{0\text{new}} + \Delta\check{D}_{0\text{new}} \end{bmatrix}, \end{aligned} \qquad (70)$$

or

$$\begin{aligned} & \check{\Phi}_{\text{new}} + \Delta\check{\Phi}_{\text{new}} \\ & \quad = \Phi_1 + \Delta\Phi_1 + (\Phi_2^1 + \Delta\Phi_2^1)(\Phi_2^2 + \Delta\Phi_2^2), \end{aligned} \qquad (71)$$

where

$$
\begin{aligned}
&\Delta\Phi_1\\
&=\begin{bmatrix} 0 & 0 & \Delta B_{02}+\Delta B_2 K_2^2 \\ \Delta C_1+\Delta D_1 K_1^1 & \Delta D_{01}+\Delta D_1 K_2^1 & 0 \\ 0 & 0 & \Delta D_{02}+\Delta D_2 K_2^2 \end{bmatrix}\\
&\qquad\qquad = \Delta\widetilde{A} + \Delta\widetilde{B}\widetilde{K}_1,
\end{aligned}
$$

$$
\begin{aligned}
&\Delta\Phi_2^1\\
&=\begin{bmatrix} \Delta A_2+\Delta B_2 K_1^2 & \Delta A_2+\Delta B_2 K_1^2 & 0 \\ 0 & 0 & 0 \\ \Delta C_2+\Delta D_2 K_1^2 & \Delta C_2+\Delta D_2 K_1^2 & 0 \end{bmatrix}\\
&\qquad\qquad = \Delta\widetilde{A}_2^1 + \Delta\widetilde{B}_2^1\widetilde{K}_2,
\end{aligned}
$$

$$
\begin{aligned}
&\Delta\Phi_2^2\\
&=\begin{bmatrix} \Delta A_1+\Delta B_1 K_1^1 & 0 & 0 \\ 0 & \Delta B_{01}+\Delta B_1 K_2^1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\\
&\qquad\qquad = \Delta\widetilde{A}_2^2 + \Delta\widetilde{B}_2^2\widetilde{K}_1,
\end{aligned}
$$

and $\Delta\widetilde{A}_1$, $\Delta\widetilde{A}_2^1$, $\Delta\widetilde{A}_2^2$ are given by (65), and

$$
\Delta\widetilde{B}_1 = \begin{bmatrix} 0 & 0 & \Delta B_2 \\ \Delta D_1 & \Delta D_1 & 0 \\ 0 & 0 & \Delta D_2 \end{bmatrix} = \widetilde{H}_1\widetilde{F}\widetilde{E}_2,
$$

$$
\Delta\widetilde{B}_2^1 = \begin{bmatrix} \Delta B_2 & \Delta B_2 & 0 \\ 0 & 0 & 0 \\ \Delta D_2 & \Delta D_2 & 0 \end{bmatrix} = \widetilde{H}_3\widetilde{F}\widetilde{E}_2, \tag{72}
$$

$$
\Delta\widetilde{B}_2^2 = \begin{bmatrix} \Delta B_1 & 0 & 0 \\ 0 & \Delta B_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \widetilde{H}_2\widetilde{F}\widetilde{E}_2,
$$

with

$$
\widetilde{E}_2 = \begin{bmatrix} E_2 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_2 \end{bmatrix}.
$$

Now we have the following result.

Theorem 13. *Suppose that a control law of the form (48) is applied to a discrete linear repetitive process described by (63), with the uncertainty structure modelled by (30). Then the resulting closed-loop process is stable along the pass if there exist a matrix $X > 0$, nonsingular matrices $\widetilde{V}$ and $\widetilde{Z}$, rectangular matrices $\widetilde{L}$, $\widetilde{N}$, and scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that*

$$
\left[\begin{array}{ccccccc}
-X & \widetilde{A}_2^1\widetilde{Z}+\widetilde{B}_2^1\widetilde{N} & \varepsilon_1\widetilde{H}_3 & 0 & \widetilde{A}_1\widetilde{V}+\widetilde{B}_1\widetilde{L} & \varepsilon_2\widetilde{H}_1 & 0 \\
\widetilde{Z}^T\widetilde{A}_2^{1T}+\widetilde{N}^T\widetilde{B}_2^{1T} & -\widetilde{Z}-\widetilde{Z}^T & 0 & \widetilde{Z}^T\widetilde{E}_1^{2T}+\widetilde{N}^T\widetilde{E}_2^T & \widetilde{A}_2^2\widetilde{V}+\widetilde{B}_2^2\widetilde{L} & \varepsilon_2\widetilde{H}_2 & 0 \\
\varepsilon_1\widetilde{H}_3^T & 0 & -\varepsilon_1 I & 0 & 0 & 0 & 0 \\
0 & \widetilde{E}_1^2\widetilde{Z}+\widetilde{E}_2\widetilde{N} & 0 & -\varepsilon_1 I & 0 & 0 & 0 \\
\widetilde{V}^T\widetilde{A}_1^T+\widetilde{L}^T\widetilde{B}_1^T & \widetilde{V}^T\widetilde{A}_2^{2T}+\widetilde{L}^T\widetilde{B}_2^{2T} & 0 & 0 & X-\widetilde{V}-\widetilde{V}^T & 0 & \widetilde{V}^T\widetilde{E}_1^{1T}+\widetilde{L}^T\widetilde{E}_2^T \\
\varepsilon_2\widetilde{H}_1^T & \varepsilon_2\widetilde{H}_2^T & 0 & 0 & 0 & -\varepsilon_2 I & 0 \\
0 & 0 & 0 & 0 & \widetilde{E}_1^1\widetilde{V}+\widetilde{E}_2\widetilde{L} & 0 & -\varepsilon_2 I
\end{array}\right] < 0, \tag{73}
$$

where the matrices X, $\widetilde{V}$, $\widetilde{Z}$, $\widetilde{L}$ and $\widetilde{N}$ are given by (54). If this condition holds, the control law matrices are again given by (55).

Proof. The proof is similar to that of Theorem 7 and hence the details are omitted here. ■

5. Conclusions and Future Work

We have developed LMI based control law design algorithms for uncertain discrete linear repetitive processes with switching in the dynamics either from pass to pass or along the pass. This general problem area is of significant theoretical and practical importance and is much more complicated than the corresponding problems in the absence of switching. The main source of difficulty here is the fact that even if the uncertainty present in the switched dynamics can be represented by convex uncertainty regions, these regions for the overall process are nonconvex. The methods developed in this paper provide appropriate approximations and allow a process within this general class to be stabilized.

The results here have been obtained under the assumption that a constant uncertainty independent Lyapunov function (the basis of LMI stability conditions which have not been covered explicitly in this paper) and static control laws can be used, but at the cost of some conservativeness. One possible means of reducing the effects of this is to use a parameter variable Lyapunov function and control law. This is the subject of ongoing work and will be reported in due course. Also, there is a clear need to extend the design for stability along the pass to this fundamental property plus performance objectives, disturbance rejection or attenuation and tracking given reference signals. It is also possible to use the pass to pass switching model to describe the dynamics of the so-called bi-directional processes, where, instead of resetting before the start of each new pass, the process produces the next pass in the reverse direction, and so on. Such processes occur in industrial examples, e.g., in distillation columns (Edwards, 1974), and cannot be studied by uni-directional models, which have been the focus of attention in the currently available literature.

References

Amann N., Owens D.H. and Rogers E. (1998): *Predictive optimal iterative learnig control*. — Int. J. Contr., Vol. 69, No. 2, pp. 203–226.

Bachelier O., Bernussou J., de Oliveira M.C. and Geromel J.C. (1999): *Parameter dependent Lyapunov control design: Numerical evaluation*. — Proc. 38-th Conf. *Decision and Control*, Phoenix, USA, pp. 293–297.

Benton S.E. (2000): *Analysis and Control of Linear Repetitive Processes*. — Ph.D. thesis, University of Southampton, UK.

Bochniak J., Gałkowski K., Rogers E., Mehdi D., Bachelier O. and Kummert A. (2006): *Stabilization of discrete linear repetitive processes with switched dynamics*. — Multidim. Syst. Signal Process., Vol. 17, No. 2–3, pp. 271–293.

Boyd S., Feron E., El Ghaoui L. and Balakrishnan V. (1994): *Linear Matrix Inequalities in System and Control Theory*. — Philadelphia: SIAM.

D'Andrea R. and Dullerud G.E. (2003): *Distributed control design for spatially interconnected systems*. — IEEE Trans. Automat. Contr., Vol. 48, No. 9, pp. 1478–1495.

Du C. and Xie L. (1999): *Stability analysis and stabilisation of uncertain two-dimensional discrete systems: An LMI approach*. — IEEE Trans. Circ. Syst. I: Fundam. Theory Applic., Vol. 46, No. 11, pp. 1371–1374.

Edwards J.B. (1974): *Stability problems in the control of multipass processes*. — Proc. IEE, Vol. 121, No. 11, pp. 1425–1432.

Gałkowski K., Rogers E., Xu S., Lam J. and Owens D.H. (2002): *LMIs—A fundamental tool in analysis and controller design for discrete linear repetitive processes*. — IEEE Trans. Circ. Syst. I: Fundam. Theory Applic., Vol. 49, No. 6, pp. 768–778.

Longman R. (2003): *On the interaction between theory, experiments and simulation in developing practical learning control algorithms*. — Int. J. Appl. Math. Comput. Sci., Vol. 13, No. 1, pp. 101–112.

Ratcliffe J.D., Hatonen J.J., Lewin P.L., Rogers E., Harte T.J. and Owens D.H. (2005): *P-type iterative learning control for systems that contain resonance*. — Int. J. Adapt. Contr. Sig. Process., Vol. 19, No. 10, pp. 769–796.

Roberts P.D. (2002): *Two-dimensional analysis of an iterative nonlinear optimal control algorithm*. — IEEE Trans. Circ. Syst. I: Fundam. Theory Applic., Vol. 49, No. 6, pp. 872–878.

Rogers E. and Owens D.H. (1992): *Stability Analysis for Linear Repetitive Processes*. — Lect. Notes Contr. Inf. Sci., Vol. 175, Berlin, Germany: Springer-Verlag.

Smyth K.J. (1992): *Computer Aided Analysis for Linear Repetitive Processes*. — Ph.D. thesis, University of Strathclyde, Glasgow, UK.

Received: 18 November 2005
Revised: 26 August 2006