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OBSERVER–BASED CONTROLLER DESIGN OF TIME–DELAY SYSTEMS WITH AN INTERVAL TIME–VARYING DELAY

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This paper considers the problem of designing an observer-based output feedback controller to exponentially stabilize a class of linear systems with an interval time-varying delay in the state vector. The delay is assumed to vary within an interval with known lower and upper bounds. The time-varying delay is not required to be differentiable, nor should its lower bound be zero. By constructing a set of Lyapunov–Krasovskii functionals and utilizing the Newton–Leibniz formula, a delay-dependent stabilizability condition which is expressed in terms of Linear Matrix Inequalities (LMIs) is derived to ensure the closed-loop system is exponentially stable with a prescribed α-convergence rate. The design of an observer-based output feedback controller can be carried out in a systematic and computationally efficient manner via the use of an LMI-based algorithm. A numerical example is given to illustrate the design procedure.

Keywords: observer-based feedback control, interval time-varying delay, linear matrix inequalities, Lyapunov–Krasovskii functionals, exponential stability.

1. Introduction

Time-delay systems have attracted widespread attention, and many fundamental problems including stability, stabilization and estimation have been reported and studied in the literature (see, for example, the works of Blizorukova *et al.* (2001), Busłowicz (2010), Fridman and Shaked (2002), Gu *et al.* (2003), Kaczorek and Busłowicz (2004), Kowalewski (2009), Park (1999), Raja *et al.* (2011), Richard (2003), Tokarzewski (2009), Trinh (1994; 1997; 1999; 2010), Xiang *et al.* (2010) and the references therein). In particular, a problem of theoretical and practical importance is the design of observer-based controllers or dynamic output feedback controllers for time-delay systems. This problem stems from a practical constraint that not all of the state variables are available for feedback control, and therefore either an observer-based or a dynamical output feedback controller, which uses

only the available output information, is employed to achieve the desired closed-loop system performance. In this regard, the Lyapunov–Krasovskii functional approach, which resulted in Riccati equations, LMIs or matrix inequalities, has been among some popular and effective tools used in the design of observer-based and dynamic output feedback controllers for time-delay systems (Baser and Kizilsac, 2007; Chen, 2007; Ivanescu *et al.*, 2000; Kwon *et al.*, 2006; Park, 2004; Tong *et al.*, 2011). So far, the existing works have treated the time delay as either constant (Chen, 2007; Ivanescu *et al.*, 2000; Kwon *et al.*, 2006; Park, 2004) or time-varying (Baser and Kizilsac, 2007), but with the assumption that the time-varying delay is a continuously differentiable function satisfying some boundedness conditions on its derivative.

There has been growing research interest in stability analysis and stabilization of systems subject to an interval

time-varying delay (see, for example, Botmart *et al.*, 2011; Phat *et al.*, 2012; Shao, 2009; Shao and Han, 2012). An interval time-varying delay is a time delay that varies within an interval whose lower bound is not restricted to be zero and its time derivative can even be undefined or unknown. Recently, exponential stability of linear systems with an interval time-varying non-differentiable delay was considered by Phat *et al.* (2012). Also, the problem of designing memoryless state feedback controllers to exponentially stabilize a class of linear uncertain systems with an interval time-varying delay was studied by Botmart *et al.* (2011). When not all of the state variables are available for feedback control, a state observer may be used to provide an estimate of the state vector and thus an observer-based output feedback controller can be realized. Nevertheless, compared with the state feedback design problem (Botmart *et al.*, 2011), the observer-based output feedback design problem involves more control parameters and it is more complicated and difficult to solve using a convex programming approach.

This paper specifically focuses on interval time-varying delays. We study a class of linear systems subject to an interval time-varying delay in the state vector. The time-varying delay varies within an interval with known lower and upper bounds. Here, the delay is not required to be differentiable nor should its lower bound be zero. Under the practical constraint that not all of the state variables of the system are available for feedback control, we consider the problem of designing an observer-based output feedback controller to exponentially stabilize the closed-loop system with a prescribed α-convergence rate. Since the real-time knowledge of the delay is not available and completely unknown, it is clear that no proposed structure for the state observer can contain any internal delay. This is in contrast to the existing observer-based design methods (Chen, 2007; Ivanescu *et al.*, 2000; Kwon *et al.*, 2006; Park, 2004), which treated a known constant time delay and thus their proposed state observers containing an internal time delay.

In this paper, we propose to use a completely *memoryless* full-order Luenberger-type state observer to reconstruct the state vector for the feedback control purpose. Our objective is to systematically derive both the observer and controller gain matrices to ensure that the closed-loop system is exponentially stable. By constructing a set of Lyapunov–Krasovskii functionals and utilizing the Newton–Leibniz formula, a delay-dependent LMI stabilizability condition is derived. As a result, both the controller and observer parameters can be derived via the use of an efficient LMI-based algorithm.

Notation. The following notation will be used in this paper: $\mathbb{R}^+$ denotes the set of all real non-negative numbers; $\mathbb{R}^n$ denotes the n-dimensional space with the vector norm $\|\cdot\|$; $M^{n\times r}$ denotes the space of all $(n \times r)$-dimensional matrices; A^T denotes the transpose of matrix A; A is symmetric if $A = A^T$; I denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A) = \max\{\text{Re}\lambda; \lambda \in \lambda(A)\}$;

$$x_t := \{x(t+s) : s \in [-h, 0]\},$$
$$\|x_t\| = \sup_{s\in[-h,0]}\|x(t+s)\|$$

$\mathcal{C}^1([0,t], \mathbb{R}^n)$ denotes the set of all $\mathbb{R}^n$-valued continuously differentiable functions on $[0,t]$. A matrix A is called semi-positive definite $(A \geq 0)$ if $\langle Ax, x \rangle \geq 0$, for all $x \in \mathbb{R}^n$; A is positive definite $(A > 0)$ if $\langle Ax, x \rangle > 0$ for all $x \neq 0$; $A > B$ means $A - B > 0$. The symmetric term in a matrix is denoted by $*$.

2. Problem statement and preliminaries

Consider a linear system with an interval time-varying delay in the state vector, where

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - h(t)) + Bu(t), \ t \geq 0,$$
$$y(t) = Cx(t), \tag{1}$$

with the initial condition function

$$x(t) = \phi(t), \quad t \in [-h_2, 0], \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is a control vector, $y(t) \in \mathbb{R}^q$ is the output vector, $A_0, A_1 \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{q\times n}$ are constant matrices, $\phi(t) \in \mathcal{C}^1([-h_2, 0], \mathbb{R}^n)$ is the initial function with the norm

$$\|\phi\|_{C^1} = \sup_{-h_2 \leq t \leq 0}\{\|\phi(t)\|, \|\dot{\phi}(t)\|\}.$$

The time-varying delay function $h(t)$ satisfies

$$0 < h_1 \leq h(t) \leq h_2, \quad h_1 < h_2, \quad t \geq 0.$$

It is worth noting that the time delay is assumed to be a continuous function belonging to a given interval, which means that the lower and upper bounds to the time-varying delay are available, but the lower bound is not restricted to being zero. Furthermore, the time-varying delay can be non-differentiable.

Since not all of the state variables are available for feedback control and the real-time knowledge of the delay, $h(t)$, is not available, we consider a completely memoryless full-order Luenberger state observer for the system (1),

$$\dot{\hat{x}}(t) = A_0\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)), \quad t \geq 0,$$
$$\hat{x}(t) = 0, \quad \forall t \in [-h_2, 0], \tag{3}$$

and the following control law for the system (1):

$$u(t) = -K\hat{x}(t), \quad t \geq 0, \tag{4}$$

in which $\hat{x}(t) \in \mathbb{R}^n$ is the observer state vector, $K \in \mathbb{R}^{m \times n}$ and $L \in \mathbb{R}^{n \times q}$ are the constant controller and observer gain matrices, respectively.

Define an error vector $\Theta(t) = x(t) - \hat{x}(t)$, which denotes the difference between the actual state and estimated states. Then we have the following closed-loop system:

$$\dot{\Theta}(t) = (A_0 - LC)\Theta(t) + A_1 x(t - h(t)), \tag{5}$$

$$\dot{x}(t) = (A_0 - BK)x(t) + A_1 x(t - h(t)) + BK\Theta(t). \tag{6}$$

The objective is to derive K and L so that the above closed-loop system is exponentially stable with a prescribed α-convergence rate.

Definition 1. (*Phat et al., 2012*) Given $\alpha > 0$, the system (1), where $u(t) = 0$, is α-*stable* if there exists a positive number $\beta > 0$ such that every solution $x(t, \phi)$ satisfies the condition

$$\|x(t, \phi)\| \leq \beta e^{-\alpha t} \|\phi\|_{\mathcal{C}^1}, \quad \forall t \geq 0.$$

We also introduce the following technical well-known propositions, which will be used in the proof of our results.

Proposition 1. (Jensen's inequality (Gu *et al.*, 2003)) *For any symmetric positive definite matrix $M > 0$, a scalar $\gamma > 0$ and a vector function $\omega : [0, \gamma] \rightarrow \mathbb{R}^n$ such that the integrations concerned are well defined, the following inequality holds:*

$$\left(\int_0^\gamma \omega(s) \, ds \right)^T M \left(\int_0^\gamma \omega(s) \, ds \right)$$
$$\leq \gamma \left(\int_0^\gamma \omega^T(s) M \omega(s) \, ds \right).$$

Proposition 2. (Schur complement lemma) *Given constant matrices X, Y, Z with appropriate dimensions satisfying $X = X^T, Y = Y^T$ and $Y > 0$, we have $X + Z^T Y^{-1} Z < 0$ if and only if*

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0 \quad or \quad \begin{pmatrix} -Y & Z \\ Z^T & X \end{pmatrix} < 0.$$

3. Main results

In this section, using the Lyapunov–Krasovskii method, we establish a delay-dependent criterion in terms of linear matrix inequalities for the exponential stabilization of the linear system (1) via the observer-based controller (3)–(4).

Now, we synthesize the gain matrices K and L of controller and observer as follows:

$$K = YP^{-1}, \quad L = -\frac{1}{2}PC^T, \tag{7}$$

where P is a positive-definite matrix and Y are to be designed.

Write

$$h_{12} = h_2 - h_1,$$

$$\mathcal{H} = \begin{bmatrix} M_1^T & M_2^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\mathcal{L} = \begin{bmatrix} N_1^T & N_2^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & -N_1 & \Omega_{15} & BY & 0 \\ * & \Omega_{22} & M_2 & -N_2 & PA_1^T & PA_1^T & 0 \\ * & * & \Omega_{33} & 0 & 0 & 0 & 0 \\ * & * & * & \Omega_{44} & 0 & 0 & 0 \\ * & * & * & * & \Omega_{55} & BY & 0 \\ * & * & * & * & * & \Omega_{66} & PC^T \\ * & * & * & * & * & * & -I \end{bmatrix},$$

with

$$\Omega_{11} = A_0 P + PA_0^T + 2\alpha P - BY - Y^T B^T$$
$$+ Q_1 + Q_2 - e^{-2\alpha h_1} S_1,$$

$$\Omega_{12} = N_1 - M_1 + A_1 P, \quad \Omega_{13} = M_1 + e^{-2\alpha h_1} S_1,$$

$$\Omega_{15} = PA_0^T - Y^T B^T,$$

$$\Omega_{22} = N_2 + N_2^T - M_2 - M_2^T,$$

$$\Omega_{33} = -e^{-2\alpha h_1} Q_1 - e^{-2\alpha h_1} S_1,$$

$$\Omega_{44} = -e^{-2\alpha h_2} Q_2, \quad \Omega_{55} = h_1^2 S_1 + h_{12} S_2 - 2P,$$

$$\Omega_{66} = A_0 P + PA_0^T + 2\alpha P,$$

$$\lambda = \lambda_{\min}(P^{-1}),$$

$$\Lambda = 2\lambda_{\max}(P^{-1}) + h_1 \lambda_{\max}(P^{-1} Q_1 P^{-1})$$
$$+ h_2 \lambda_{\max}(P^{-1} Q_2 P^{-1}) + \frac{1}{2} h_1^3 \lambda_{\max}(P^{-1} S_1 P^{-1})$$
$$+ \frac{1}{2}(h_2^2 - h_1^2) \lambda_{\max}(P^{-1} S_2 P^{-1}).$$

Theorem 1. *Given $\alpha > 0$, $u(t) = -YP^{-1}\hat{x}(t)$ is an observer-based controller for the system (1) if there exist symmetric positive-definite matrices P, S_1, S_2, Q_1, Q_2, and matrices M_1, M_2, N_1, N_2, Y satisfying the following LMIs:*

$$\Omega_1 = \begin{bmatrix} \Omega & h_{12}\mathcal{H} \\ h_{12}\mathcal{H}^T & -h_{12}e^{-2\alpha h_2} S_2 \end{bmatrix} < 0,$$

$$\Omega_2 = \begin{bmatrix} \Omega & h_{12}\mathcal{L} \\ h_{12}\mathcal{L}^T & -h_{12}e^{-2\alpha h_2} S_2 \end{bmatrix} < 0.$$

Moreover, the solution $x(t, \phi)$ satisfies

$$\|x(t, \phi)\| \leq \sqrt{\frac{\Lambda}{\lambda}} e^{-\alpha t} \|\phi\|_{\mathcal{C}^1}, \quad t \in \mathbb{R}^+.$$

Proof. For $i = 1, 2$ set

$$\overline{Q_i} = P^{-1} Q_i P^{-1}, \qquad \overline{S_i} = P^{-1} S_i P^{-1},$$
$$\overline{M_i} = P^{-1} M_i P^{-1}, \qquad \overline{N_i} = P^{-1} N_i P^{-1}.$$

Consider the following Lyapunov–Krasovskii functional:

$$V = \sum_{i=1}^{5} V_i,$$

where

$$V_1 = x^T(t) P^{-1} x(t) + \Theta^T(t) P^{-1} \Theta(t),$$
$$V_2 = \int_{t-h_1}^{t} e^{2\alpha(s-t)} x^T(s) \overline{Q_1} x(s) \, \mathrm{d}s,$$
$$V_3 = \int_{t-h_2}^{t} e^{2\alpha(s-t)} x^T(s) \overline{Q_2} x(s) \, \mathrm{d}s,$$
$$V_4 = \int_{-h_1}^{0} \int_{t+s}^{t} h_1 e^{2\alpha(\tau-t)} \dot{x}^T(\tau) \overline{S_1} \dot{x}(\tau) \, \mathrm{d}\tau \, \mathrm{d}s,$$
$$V_5 = \int_{-h_2}^{-h_1} \int_{t+s}^{t} e^{2\alpha(\tau-t)} \dot{x}^T(\tau) \overline{S_2} \dot{x}(\tau) \, \mathrm{d}\tau \, \mathrm{d}s.$$

Taking the derivative of $V_i, i = 1, \ldots, 5$, we have

$$\begin{aligned}
\dot{V}_1 = {} & x^T(t) \Big[P^{-1}[A_0 - BK] + [A_0 - BK]^T P^{-1} \Big] x(t) \\
& + 2x^T(t) P^{-1} A_1 x(t - h(t)) + 2x^T(t) P^{-1} BK\Theta(t) \\
& + \Theta^T(t) \Big[P^{-1}[A_0 - LC] + [A_0 - LC]^T P^{-1} \Big] \Theta(t) \\
& + 2\Theta^T(t) P^{-1} A_1 x(t - h(t)),
\end{aligned} \tag{8}$$

$$\begin{aligned}
\dot{V}_2 = {} & -2\alpha V_2 + x^T(t) \overline{Q_1} x(t) \\
& - e^{-2\alpha h_1} x^T(t - h_1) \overline{Q_1} x(t - h_1),
\end{aligned} \tag{9}$$

$$\begin{aligned}
\dot{V}_3 = {} & -2\alpha V_3 + x^T(t) \overline{Q_2} x(t) \\
& - e^{-2\alpha h_2} x^T(t - h_2) \overline{Q_2} x(t - h_2),
\end{aligned} \tag{10}$$

$$\begin{aligned}
\dot{V}_4 = {} & -2\alpha V_4 + h_1^2 \dot{x}^T(t) \overline{S_1} \dot{x}(t) \\
& - e^{-2\alpha h_1} \int_{t-h_1}^{t} h_1 \dot{x}^T(s) \overline{S_1} \dot{x}(s) \, \mathrm{d}s,
\end{aligned} \tag{11}$$

$$\begin{aligned}
\dot{V}_5 \leq {} & -2\alpha V_5 + h_{12} \dot{x}^T(t) \overline{S_2} \dot{x}(t) \\
& - e^{-2\alpha h_2} \int_{t-h_2}^{t-h_1} \dot{x}^T(s) \overline{S_2} \dot{x}(s) \, \mathrm{d}s.
\end{aligned} \tag{12}$$

Using Proposition 1, we get

$$\begin{aligned}
& -\int_{t-h_1}^{t} h_1 \dot{x}^T(s) \overline{S_1} \dot{x}(s) \, \mathrm{d}s \\
& \leq -\big(x(t) - x(t - h_1)\big)^T \overline{S_1} \big(x(t) - x(t - h_1)\big).
\end{aligned}$$

By the Leibniz–Newton formula, we have

$$\begin{aligned}
& -e^{-2\alpha h_2} \int_{t-h_2}^{t-h_1} \dot{x}^T(s) \overline{S_2} \dot{x}(s) \, \mathrm{d}s \\
& = -e^{-2\alpha h_2} \int_{t-h_2}^{t-h(t)} \dot{x}^T(s) \overline{S_2} \dot{x}(s) \, \mathrm{d}s \\
& \quad - e^{-2\alpha h_2} \int_{t-h(t)}^{t-h_1} \dot{x}^T(s) \overline{S_2} \dot{x}(s) \, \mathrm{d}s \\
& \quad + 2\zeta^T(t) \mathcal{M} \Big[x(t - h_1) - x(t - h(t)) \\
& \qquad\qquad - \int_{t-h(t)}^{t-h_1} \dot{x}(s) \, \mathrm{d}s \Big] \\
& \quad + 2\zeta^T(t) \mathcal{N} \Big[x(t - h(t)) - x(t - h_2) \\
& \qquad\qquad - \int_{t-h_2}^{t-h(t)} \dot{x}(s) \, \mathrm{d}s \Big],
\end{aligned} \tag{13}$$

where

$$\mathcal{M} = \begin{bmatrix} \overline{M_1}^T & \overline{M_2}^T & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$
$$\mathcal{N} = \begin{bmatrix} \overline{N_1}^T & \overline{N_2}^T & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$
$$\begin{aligned}
\zeta(t) = \big[& x^T(t) \quad x^T(t - h(t)) \quad x^T(t - h_1) \\
& x^T(t - h_2) \quad \dot{x}^T(t) \quad \Theta^T(t) \big]^T.
\end{aligned}$$

From (6) it follows that

$$\begin{aligned}
2\dot{x}^T(t) P^{-1} \Big[& [A_0 - BK]x(t) + A_1 x(t - h(t)) \\
& + BK\Theta(t) - \dot{x}(t) \Big] = 0.
\end{aligned} \tag{14}$$

From (8)–(14), we have

$$\begin{aligned}
& \dot{V} + 2\alpha V \\
& \leq \zeta^T(t) \Xi \zeta(t) \\
& \quad - \int_{t-h(t)}^{t-h_1} \Big[2\zeta^T(t) \mathcal{M} \dot{x}(s) + \dot{x}^T(s) e^{-2\alpha h_2} \overline{S_2} \dot{x}(s) \Big] \mathrm{d}s \\
& \quad - \int_{t-h_2}^{t-h(t)} \Big[2\zeta^T(t) \mathcal{N} \dot{x}(s) + \dot{x}^T(s) e^{-2\alpha h_2} \overline{S_2} \dot{x}(s) \Big] \mathrm{d}s \\
& = h_{12}^{-1} \int_{t-h(t)}^{t-h_1} \Big\{ \zeta^T(t) \Xi \zeta(t) - 2h_{12} \zeta^T(t) \mathcal{M} \dot{x}(s) \\
& \qquad\qquad - h_{12} \dot{x}^T(s) e^{-2\alpha h_2} \overline{S_2} \dot{x}(s) \Big\} \mathrm{d}s \\
& \quad + h_{12}^{-1} \int_{t-h_2}^{t-h(t)} \Big\{ \zeta^T(t) \Xi \zeta(t) - 2h_{12} \zeta^T(t) \mathcal{N} \dot{x}(s) \\
& \qquad\qquad - h_{12} \dot{x}^T(s) e^{-2\alpha h_2} \overline{S_2} \dot{x}(s) \Big\} \mathrm{d}s
\end{aligned}$$

$$
= h_{12}^{-1} \int_{t-h(t)}^{t-h_1} \begin{bmatrix} \zeta(t) \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} \Xi & -h_{12}\mathcal{M} \\ -h_{12}\mathcal{M}^T & -h_{12}e^{-2\alpha h_2}\overline{S_2} \end{bmatrix}
$$

$$
\times \begin{bmatrix} \zeta(t) \\ \dot{x}(s) \end{bmatrix} \, \mathrm{d}s
$$

$$
+ h_{12}^{-1} \int_{t-h_2}^{t-h(t)} \begin{bmatrix} \zeta(t) \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} \Xi & -h_{12}\mathcal{N} \\ -h_{12}\mathcal{N}^T & -h_{12}e^{-2\alpha h_2}\overline{S_2} \end{bmatrix}
$$

$$
\times \begin{bmatrix} \zeta(t) \\ \dot{x}(s) \end{bmatrix} \, \mathrm{d}s,
$$

$$(15)$$

where

$$
\Xi = \begin{bmatrix}
\Xi_{11} & \Xi_{12} & \Xi_{13} & -\overline{N_1} & \Xi_{15} & P^{-1}BK \\
* & \Xi_{22} & \overline{M_2} & -\overline{N_2} & A_1^T P^{-1} & A_1^T P^{-1} \\
* & * & \Xi_{33} & 0 & 0 & 0 \\
* & * & * & \Xi_{44} & 0 & 0 \\
* & * & * & * & \Xi_{55} & P^{-1}BK \\
* & * & * & * & * & \Xi_{66}
\end{bmatrix},
$$

with

$$
\begin{aligned}
\Xi_{11} &= P^{-1}\big[A_0 - BK\big] + \big[A_0 - BK\big]^T P^{-1} \\
&\quad + 2\alpha P^{-1} + \overline{Q_1} + \overline{Q_2} - e^{-2\alpha h_1}\overline{S_1}, \\
\Xi_{12} &= \overline{N_1} - \overline{M_1} + P^{-1}A_1, \\
\Xi_{13} &= \overline{M_1} + e^{-2\alpha h_1}\overline{S_1}, \\
\Xi_{15} &= A_0^T P^{-1} - K^T B^T P^{-1}, \\
\Xi_{22} &= \overline{N_2} + \overline{N_2}^T - \overline{M_2} - \overline{M_2}^T, \\
\Xi_{33} &= -e^{-2\alpha h_1}\overline{Q_1} - e^{-2\alpha h_1}\overline{S_1}, \\
\Xi_{44} &= -e^{-2\alpha h_2}\overline{Q_2}, \\
\Xi_{55} &= h_1^2\overline{S_1} + h_{12}\overline{S_2} - 2P^{-1}, \\
\Xi_{66} &= P^{-1}A_0 + A_0^T P^{-1} + 2\alpha P^{-1} \\
&\quad - P^{-1}LC - C^T L^T P^{-1}.
\end{aligned}
$$

Setting

$$
\Upsilon_1 = \begin{bmatrix} \Xi & -h_{12}\mathcal{M} \\ -h_{12}\mathcal{M}^T & -h_{12}e^{-2\alpha h_2}\overline{S_2} \end{bmatrix} \tag{16}
$$

and

$$
\Upsilon_2 = \begin{bmatrix} \Xi & -h_{12}\mathcal{N} \\ -h_{12}\mathcal{N}^T & -h_{12}e^{-2\alpha h_2}\overline{S_2} \end{bmatrix}, \tag{17}
$$

we have

$$
\begin{aligned}
\dot{V}&(t, x_t, \Theta(t)) + 2\alpha V(t, x_t, \Theta(t)) \\
&\leq h_{12}^{-1} \int_{t-h(t)}^{t-h_1} \begin{bmatrix} \zeta(t) \\ \dot{x}(s) \end{bmatrix}^T \Upsilon_1 \begin{bmatrix} \zeta(t) \\ \dot{x}(s) \end{bmatrix} \, \mathrm{d}s \\
&\quad + h_{12}^{-1} \int_{t-h_2}^{t-h(t)} \begin{bmatrix} \zeta(t) \\ \dot{x}(s) \end{bmatrix}^T \Upsilon_2 \begin{bmatrix} \zeta(t) \\ \dot{x}(s) \end{bmatrix} \, \mathrm{d}s.
\end{aligned} \tag{18}
$$

We have to demonstrate that $\Upsilon_1 < 0$ is equivalent to $\Omega_1 < 0$ and $\Upsilon_2 < 0$ is equivalent to $\Omega_2 < 0$. Indeed, we

pre- and post-multiply both sides of Υ_1, Υ_2 with

$$
\mathrm{diag}\{P, P, P, P, P, P, P\},
$$

and, using the Schur complement lemma and the condition (7), we have that $\Upsilon_1 < 0$ is equivalent to $\Omega_1 < 0$ and $\Upsilon_2 < 0$ is equivalent to $\Omega_2 < 0$. Therefore,

$$
\dot{V}(t, x_t, \Theta(t)) + 2\alpha V(t, x_t, \Theta(t)) \leq 0, \quad \forall t \geq 0. \tag{19}
$$

Integrating both the sides of (19) from 0 to t, we obtain

$$
V(t, x_t, \Theta(t)) \leq V(0, x_0, \Theta(0))e^{-2\alpha t}, \quad \forall t \in \mathbb{R}^+.
$$

By simple computation, we have

$$
\begin{aligned}
V&(t, x_t, \Theta(t)) \\
&\geq \lambda_{\min}(P^{-1})\|x(t)\|^2 + \lambda_{\min}(P^{-1})\|\Theta(t)\|^2 \\
&\geq \lambda_{\min}(P^{-1})\|x(t)\|^2 = \lambda\|x(t)\|^2, \quad \forall t \geq 0,
\end{aligned}
$$

and

$$
V(0, x_0, \Theta(0)) \leq \Lambda\|\phi\|_{\mathcal{C}^1}^2.
$$

Hence

$$
\begin{aligned}
\lambda\|x(t,\phi)\|^2 &\leq V(t, x_t, \Theta(0)) \leq V(0, x_0, \Theta(0))e^{-2\alpha t} \\
&\leq \Lambda e^{-2\alpha t}\|\phi\|_{\mathcal{C}^1}^2,
\end{aligned}
$$

and the solution $x(t, \phi)$ of the system satisfies

$$
\|x(t, \phi)\| \leq \sqrt{\frac{\Lambda}{\lambda}} e^{-\alpha t}\|\phi\|_{\mathcal{C}^1}, \quad \forall t \geq 0,
$$

which implies that the closed-loop system is α-stable. This completes our proof. ∎

Next, we provide a numerical example to show the effectiveness of our result.

Example 1. Consider the system (1), where

$$
A_0 = \begin{bmatrix} -3 & 0 \\ 0.5 & -0.9 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0.1 & 0 \\ -0.5 & 0.3 \end{bmatrix},
$$

$$
B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad\qquad C = \begin{bmatrix} 1 & 0.1 \end{bmatrix}
$$

with

$$
\begin{cases}
h(t) = 0.1 + 0.8\sin^2 t, \\
\qquad \text{if} \quad t \in \mathcal{I} = \cup_{k\geq 0}[2k\pi, (2k+1)\pi], \\
h(t) = 0.1 \quad \text{if} \quad t \in \mathbb{R}^+ \setminus \mathcal{I},
\end{cases}
$$

and $\phi(t) = \begin{bmatrix} \cos t & \sin t \end{bmatrix}^T$.

It is worth noticing that the delay function $h(t)$ is non-differentiable, and therefore the design methods using observer-based and dynamic output feedback controllers for time-delay systems in the existing papers (Baser and Kizilsac, 2007; Chen, 2007; Ivanescu *et al.*, 2000; Kwon *et al.*, 2006; Park, 2004) are not applicable to this example.

We have that $\alpha = 0.4$. By using the LMI Toolbox from MATLAB, the LMIs in Theorem 1 are satisfied with $h_1 = 0.1, h_2 = 0.9$, and

$$P = \begin{bmatrix} 0.8186 & -0.1397 \\ -0.1397 & 8.9918 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 0.1698 & -0.0874 \\ -0.0874 & 2.3185 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 0.2237 & -0.4782 \\ -0.4782 & 2.6813 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 2.9162 & -0.5388 \\ -0.5388 & 3.9691 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 0.2129 & -0.4144 \\ -0.4144 & 6.4924 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} -0.1300 & 0.0907 \\ 0.0244 & -0.7705 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0.1550 & -0.1804 \\ -0.1080 & 2.0655 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 0.0270 & -0.0994 \\ -0.0210 & -0.0484 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} -0.0839 & 0.1947 \\ 0.1989 & -1.1023 \end{bmatrix},$$

$$Y = \begin{bmatrix} -0.3274 & 1.3559 \end{bmatrix}.$$

Therefore, the resulting controller and observer gains for the system (1) are

$$K = YP^{-1} = \begin{bmatrix} -0.3752 & 0.1450 \end{bmatrix},$$
$$L = -\frac{1}{2}PC^T = \begin{bmatrix} -0.4023 \\ -0.3797 \end{bmatrix}. \tag{20}$$

This implies that the system (1) is 0.4-stable by the observer-based controller with controller and observer gains given in (20). Moreover, the solution of the closed-loop system satisfies

$$\|x(t, \phi)\| \leq 5.1005 e^{-0.4t} \|\phi\|_{C^1}.$$

Figure 1 shows the closed-loop trajectories of the state variables $x_1(t)$ and $x_2(t)$. The simulation shows that the state trajectories converge to zeros.

Further simulation studies have also been conducted for cases where the interval time-varying $h(t)$ is non-differentiable and varies in the interval $[0.1, 0.9]$. In all the cases studied, it was found that the trajectories of the state variables $x_1(t)$ and $x_2(t)$ all exponentially converged to zeros.
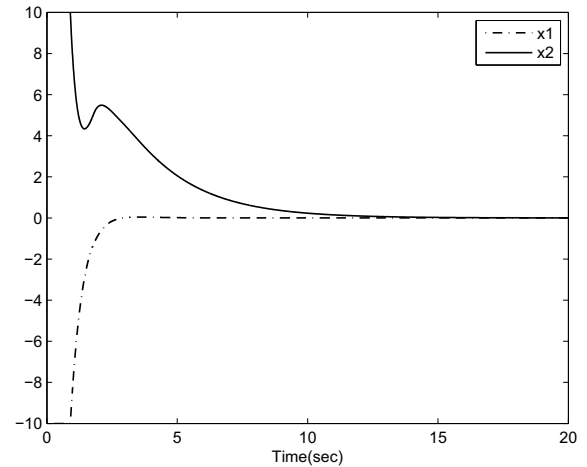

Fig. 1. Closed-loop trajectories of $x_1(t)$ and $x_2(t)$.

4. Conclusion

This paper has studied the problem of designing observer-based output feedback controllers for linear systems with an interval non-differentiable time-varying delay in the state vector. The design can be carried out by an efficient LMI-based algorithm. A numerical example has been given to demonstrate the simplicity of the design procedure.

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