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APPROXIMATE CONTROLLABILITY OF DELAYED DYNAMICAL SYSTEMS

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In the paper linear abstract retarded dynamical systems with lumped and distributed delays defined in infinite-dimensional Hilbert spaces are considered. Using frequency-domain methods, and spectral analysis for linear self-adjoint operators, the necessary and sufficient conditions for approximate relative controllability in finite time are formulated and proved. The method presented in the paper allows us to verify approximate relative controllability in finite time for abstract retarded dynamical systems by considering of approximate controllability in finite time of simpler suitably defined linear abstract dynamical systems without delays. Moreover, as an illustrative example approximate relative controllability in finite time for linear retarded distributed parameter dynamical systems with one constant delay is investigated. The results extend some relative controllability theorems, which are known in the literature, to more general classes of linear retarded dynamical systems.

1. Introduction

Retarded or hereditary dynamical systems are mathematical models of many processes in various areas of science and engineering. The main purpose of the present paper is to consider the so-called controllability problem for a linear abstract retarded dynamical system. Controllability is one of the fundamental concepts in mathematical control theory (Bensoussan *et al.*, 1993; Klamka, 1991) Roughly speaking, controllability generally means that it is possible to control a dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. In the literature there are many different definitions of controllability which depend on the class of dynamical systems (Bensoussan *et al.*, 1993; Klamka, 1991; 1993b; Nakagiri, 1987; Nakagiri and Yamamoto, 1989; O'Brien, 1979; Park *et al.*, 1990; Triggiani, 1975a; 1976; 1978). For infinite-dimensional dynamical systems it is necessary to distinguish between the notions of approximate and exact controllability (Bensoussan *et al.*, 1993; Klamka, 1982; 1991; 1992; 1993a; 1993b; O'Brien, 1979; Triggiani, 1975a; 1976; 1978). This follows directly from the fact that in infinite-dimensional spaces there exist linear subspaces which are not closed. Moreover, for retarded dynamical systems there are two fundamental concepts of controllability, namely relative controllability and absolute controllability (Bensoussan *et al.*, 1993;

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Klamka, 1991; 1993b; Nakagiri, 1987; Nakagiri and Yamamoto, 1989). Therefore, for retarded dynamical systems defined in infinite-dimensional spaces the following four kinds of controllability are considered: approximate relative controllability, exact relative controllability, approximate absolute controllability, and exact absolute controllability.

The present paper is devoted to a study of the approximate relative controllability for linear infinite-dimensional retarded dynamical systems. For such dynamical systems, direct verification of approximate relative controllability is a rather difficult and complicated task. Therefore, using frequency-domain methods (Bensoussan *et al.*, 1993; Kobayashi, 1992; Nakagiri and Yamamoto, 1989) it is shown that approximate controllability of linear retarded dynamical system can be checked by the approximate controllability condition for suitably defined and simpler infinite-dimensional dynamical system without delays. General results are then applied for approximate relative controllability investigations of a distributed-parameter dynamical system with one constant delay in the state variable.

The results presented in the paper extend to a more general class of abstract retarded dynamical systems controllability theorems given in (Bensoussan *et al.*, 1993; Klamka, 1982; 1991; Kobayashi, 1992; Nakagiri, 1987; Nakagiri and Yamamoto, 1989; O'Brien, 1979; Triggiani, 1976; 1978).

2. System Description and Basic Definitions

First we shall introduce the basic notation and terminology used throughout present paper. Let X be a separable Hilbert space. For a set $E \subset X$, $\mathrm{Cl}\, E$ denotes its closure. For a given real number $h > 0$, we denote by $L_2([-h,0],X)$ the separable Hilbert space of all strongly measurable and square integrable functions from $[-h,0]$ into X. Moreover, let us introduce the space (Bensoussan *et al.*, 1993; Klamka, 1991; Nakagiri, 1981; 1987; 1988; Nakagiri and Yamamoto, 1989) $M_2([-h,0],X) = X \times L_2([-h,0],X)$ denoted shortly by M_2 which is a separable Hilbert space with a standard scalar product

$$\langle g,f\rangle_{M_2} = \langle g^0,f^0\rangle_X + \langle g^1,f^1\rangle_{L_2} = \langle g^0,f^0\rangle_X + \int_{-h}^{0} \Big\langle g^1(s),f^1(s)\Big\rangle_X \,\mathrm{d}s$$

for $f = (f^0,f^1) \in M_2$ and $g = (g^0,g^1) \in M_2$.

Let $A_0 : X \supset D(A_0) \to X$ be a linear, generally unbounded, self-adjoint and positive-definite operator with a dense domain $D(A_0)$ in X and a compact resolvent $R(s;A_0)$ for all s in the resolvent set $\rho(A_0)$. Under these assumptions A_0 has the following properties (Chen and Russell, 1982; Tanabe, 1979; Triggiani, 1976; 1978):

1. A_0 has only pure discrete point spectrum $\sigma_p(A_0)$ consisting entirely of isolated real positive eigenvalues

$$0 < s_1 < s_2 < \ldots < s_i < \ldots, \qquad \lim_{i\to\infty} s_i = +\infty$$

Each eigenvalue s_i has finite multiplicity $n_i < \infty$, $i = 1, 2, \dots$ equal to the dimensionality of the corresponding eigenmanifold.

2. The eigenvectors $x_{ik} \in D(A_0)$, for $i = 1, 2, \dots$ and $k = 1, 2, \dots, n_i$, form a complete orthonormal set in the separable Hilbert space X.

3. A_0 has spectral representation

$$A_0 x = \sum_{i=1}^{i=\infty} s_i \sum_{k=1}^{k=n_i} \langle x, x_{ik} \rangle_X \, x_{ik} \quad \text{for} \quad x \in D(A_0)$$

4. The fractional powers A_0^α, $0 < \alpha < 1$, of the operator A_0 can be defined as follows:

$$A_0^\alpha x = \sum_{i=1}^{i=\infty} s_i^\alpha \sum_{k=1}^{k=n_i} \langle x, x_{ik} \rangle_X \, x_{ik} \quad \text{for} \quad x \in D(A_0^\alpha)$$

where $D(A_0^\alpha) = \left\{ x \in X : \sum_{i=1}^{i=\infty} s_i^{2\alpha} \sum_{k=1}^{k=n_i} \left| \langle x, x_{ik} \rangle_X \right|^2 < \infty \right\}.$

5. The operators A_0^α, $0 < \alpha \le 1$, are self-adjoint, positive-definite with dense domains in X and $-A_0^\alpha$ generate analytic semigroups on X. Particularly, $-A_0$ generates an analytic semigroup $T(t) : X \to X$ for $t \ge 0$.

We shall consider linear abstract retarded dynamical control systems described by the following functional differential equation (Nakagiri, 1981; 1986; 1987; 1988; Nakagiri and Yamamoto, 1989; O'Brien, 1979; Park *et al.*, 1990; Tanabe, 1992):

$$\dot{x}(t) = -A_0 x(t) + \sum_{k=1}^{k=p} c_k A_0^{\alpha_k} x(t - h_k) + \int_{-h}^{0} c_0(\tau) A_0^{\alpha_0} x(t + \tau) \, \mathrm{d}\tau + \sum_{j=1}^{j=m} b_j u_j(t) \tag{1}$$

with initial conditions

$$x(0) = g^0 \in X \quad \text{and} \quad x(t) = g^1(t) \in L_2\Big([-h, 0], X\Big) \tag{2}$$

where $0 < h_1 < h_2 < \dots < h_r < \dots < h_p \le h$ are constant delays, $c_r \in \mathbb{R}$ for $r = 1, 2, \dots, p$ are given constants, $c_0(\cdot)$, defined on $[-h, 0]$, is a real-valued Hölder-continuous function, $0 \le \alpha_r < 1$ for $r = 0, 1, \dots, p$ are fractional powers of the operator $A_0^{\alpha_r}$, $b_j \in X$ for $j = 1, 2, \dots, m$.

It is generally assumed that the admissible controls are such that $u_j(\cdot) \in L_2([0, \infty), \mathbb{R})$ for $j = 1, 2, \dots, m$.

It is well-known that the retarded system (1) with initial conditions (2), has for $t > 0$ a unique mild solution $x(\cdot\,; g, u)$ taking values in X (Nakagiri, 1981; Tanabe, 1992; Travis and Webb, 1974; 1976; Webb, 1976).

In the dynamical system (1) the space of control values is finite-dimensional and the control operator $B : \mathbb{R}^m \to X$ is given by

$$Bu = \sum_{j=1}^{j=m} b_j u_j(t) \tag{3}$$

Since X is a Hilbert space and $X = X^*$, the adjoint operator $B^* : X \to \mathbb{R}^m$ is defined as follows (Nakagiri and Yamamoto, 1989):

$$B^* x = \Big(\langle b_1, x\rangle_X , \langle b_2, x\rangle_X , \dots , \langle b_j, x\rangle_X , \dots , \langle b_m, x\rangle_X \Big) \tag{4}$$

To shorten the notation, let us introduce a self-adjoint operator $\eta(s)$ (Nakagiri, 1981; 1987; 1988; Nakagiri and Yamamoto, 1989; Tanabe, 1992):

$$\eta(s) = -\sum_{r=1}^{r=p} \chi_{(-\infty,-h_r]}(s) c_r A_0^{\alpha_r} - \int_{-h}^{0} c_0(\tau) A_0^{\alpha_0}\, \mathrm{d}\tau \quad \text{for } s \in [-h, 0] \tag{5}$$

where χ_E is the characteristic function of the interval E.

In what follows, we shall give short comments on the spectral decomposition of the retarded dynamical system (1). The detailed analysis of this problem can be found e.g. in (Nakagiri, 1987; 1988; Webb, 1976).

First of all, for each $z \in \mathbb{C}$ we introduce the densely-defined closed linear operator

$$\Delta(z; A_0, \eta) = zI + A_0 - \sum_{r=1}^{r=p} c_r \exp(-zh_r) A_0^{\alpha_r} - \int_{-h}^{0} c_0(\tau) \exp(z\tau) A_0^{\alpha_0}\, \mathrm{d}\tau \tag{6}$$

where I denotes the identity operator on X. The retarded resolvent set $\rho(A_0, \eta)$ is defined to be the set of all values $z \in \mathbb{C}$ for which the operator $\Delta(z; A_0, \eta)$ has a bounded inverse with dense domain in X. In this case $\Delta(z; A_0, \eta)^{-1}$ is the so-called retarded resolvent and is denoted by $R(z; A_0, \eta)$. The complement of $\rho(A_0, \eta)$ in the complex plane is called the retarded spectrum and denoted by $\sigma(A_0, \eta)$. It is well-known that the retarded resolvent set $\rho(A_0, \eta)$ is open in $\mathbb{C}$ and the retarded resolvent $R(z; A_0, \eta)$ is an analytic function for $z \in \rho(A_0, \eta)$. Moreover, let us denote by $\rho_0(A_0, \eta)$ the connected component of the resolvent set $\rho(A_0, \eta)$ which contains a right half-plane of the complex plane.

Let $x(t; g, 0)$ for $g \in M_2([-h, 0], X)$ be the mild solution of the homogeneous dynamical system (1). Define a family of bounded linear operators $S(t) : M_2 \to M_2$, for $t \geq 0$ by

$$S(t)g = \Big(x(t; g, 0),\ x_t(s; g, 0) \Big) \quad \text{for} \quad g \in M_2 \tag{7}$$

where $x_t(s; g, 0) = x(t + s; g, 0)$ for $s \in [-h, 0]$. Then $S(t)$ is a strongly continuous semigroup of bounded linear operators on M_2. Let A be the infinitesimal generator of a semigroup $S(t)$. Since the operator A_0 has compact resolvent, the spectrum $\sigma(A)$

is a pure discrete-point spectrum consisting entirely of a countable set of eigenvalues. In fact, we have

$$\sigma(A) = \bigcup_{i=1}^{i=\infty} \sigma_i \tag{8}$$

where

$$\sigma_i = \left\{ z \in \mathbb{C} : \Delta_i(z) = z + s_i - \sum_{r=1}^{r=p} c_r \exp(-zh_r) s_i^{\alpha_r} \right.$$
$$\left. - \int_{-h}^{0} c_0(\tau) \exp(z\tau) s_i^{\alpha_0} \, \mathrm{d}\tau = 0 \right\} \tag{9}$$

σ_i being non-empty for $i = 1, 2, \ldots$.

Now, we shall introduce various concepts of controllability for the retarded dynamical system (1). It is well-known that for retarded dynamical systems there exist two fundamental notions of controllability, namely relative controllability and absolute controllability. In the present paper, we shall concentrate on relative controllability. Since the dynamical system (1) is defined in an infinite-dimensional space X, it is necessary to distinguish between exact relative controllability and approximate relative controllability. However, since the control operator is finite-dimensional and therefore compact, the dynamical system (1) cannot be exactly relatively controllable for an infinite-dimensional space X (Triggiani, 1975b; 1977). Thus, in the sequel, we shall concentrate on approximate relative controllability. First of all, let R_t and R_∞, $t > 0$ denote attainable sets respectively given by

$$R_t = \Big\{ x(t; 0, u) \in X : u \in L_2\Big([0, t], \mathbb{R}^m\Big) \Big\} \quad \text{and} \quad R_\infty = \bigcup_{t>0} R_t \tag{10}$$

Definition 1. The dynamical system (1) is said to be *approximately relatively controllable in time* $t > 0$ if $\mathrm{Cl}\,(R_t) = X$.

Definition 2. The dynamical system (1) is said to be *approximately relatively controllable in finite time* if $\mathrm{Cl}\,(R_\infty) = X$.

Several others definitions of controllability for retarded dynamical systems can be found in the monographs (Bensoussan *et al.*, 1993; Klamka, 1991).

3. Approximate Controllability

In this section, we shall formulate and prove criteria for approximate relative controllability in finite time of the retarded dynamical system (1). First of all, we shall introduce the following notation (Bensoussan *et al.*, 1993; Klamka, 1991; Triggiani, 1976):

$$B_i = \begin{vmatrix} \langle b_1, x_{i1}\rangle_X & \langle b_2, x_{i1}\rangle_X & \cdots & \langle b_j, x_{i1}\rangle_X & \cdots & \langle b_m, x_{i1}\rangle_X \\ \langle b_1, x_{i2}\rangle_X & \langle b_2, x_{i2}\rangle_X & \cdots & \langle b_j, x_{i2}\rangle_X & \cdots & \langle b_m, x_{i2}\rangle_X \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle b_1, x_{ik}\rangle_X & \langle b_2, x_{ik}\rangle_X & \cdots & \langle b_j, x_{ik}\rangle_X & \cdots & \langle b_m, x_{ik}\rangle_X \\ \langle b_1, x_{in_i}\rangle_X & \langle b_2, x_{in_i}\rangle_X & \cdots & \langle b_j, x_{in_i}\rangle_X & \cdots & \langle b_m, x_{in_i}\rangle_X \end{vmatrix} \quad \text{for } i = 1, 2, \ldots \tag{11}$$

Now, let us recall a modified version of some necessary and sufficient conditions for approximate relative controllability in finite time.

Lemma 1. (Nakagiri and Yamamoto, 1989) *The dynamical system (1) is approximately relatively controllable in finite time if and only if*

$$\bigcap_{z \in \rho_0(A_0, \eta)} \operatorname{Ker} B^* R(z; A_0, \eta) = \{0\} \tag{12}$$

Theorem 1. *The dynamical system (1) is approximately relatively controllable in finite time if and only if*

$$\operatorname{rank} B_i = n_i \quad \textit{for} \quad i = 1, 2, \ldots \tag{13}$$

Proof. (Necessity) To obtain a contradiction suppose that there exists at least one index $i_0 \geq 1$ such that

$$\operatorname{rank} B_{i_0} < n_{i_0} \tag{14}$$

Therefore, since the rows of B_{i_0} are linearly dependent, there exist real coefficients γ_k, $k = 1, 2, \ldots, n_{i_0}$, $\sum_{k=1}^{k=n_{i_0}} \gamma_k^2 > 0$ such that

$$\begin{aligned} \sum_{k=1}^{k=n_{i_0}} \gamma_k \langle b_j, x_{i_0 k}\rangle_X &= \sum_{k=1}^{k=n_{i_0}} \langle b_j, \gamma_k x_{i_0 k}\rangle_X \\ &= \left\langle b_j, \sum_{k=1}^{k=n_{i_0}} \gamma_k x_{i_0 k} \right\rangle_X = \langle b_j, x^0\rangle_X = 0 \quad \text{for } j = 1, 2, \ldots, m \end{aligned} \tag{15}$$

where $x^0 = \sum_{k=1}^{k=n_{i_0}} \gamma_k x_{i_0 k}$ is a non-zero element. Therefore, by formulae (4), (6), (9) and (15) we deduce that there exist an eigenvalue $z_0 \in \sigma_{i_0}$ and a non-zero element $x^0 \in \operatorname{Ker} \Delta(z_0; A_0, \eta)$ such that

$$B^* x^0 = \Big(\langle b_1, x^0\rangle_X, \langle b_2, x^0\rangle_X, \ldots, \langle b_j, x^0\rangle_X, \ldots, \langle b_m, x^0\rangle_X \Big) = 0 \tag{16}$$

Let $z \in \rho(A_0, \eta)$. Since all the operators A_0^α for $0 \leq \alpha \leq 1$ are self-adjoint, by (6) the operator $\Delta(z; A_0, \eta)$ is normal and, moreover, its inverse $\Delta(z; A_0, \eta)^{-1} = R(z; A_0, \eta)$

is also normal for all $z \in \rho(A_0,\eta)$. Furthermore, by (6) and (9) for a given $z \in \rho(A_0,\eta)$ the eigenvalues of the retarded resolvent $R(z;A_0,\eta)$ are equal to $\Delta_i(z)^{-1} \in \mathbb{C}$, for $i=1,2,\dots$. Therefore, for $x \in X$ we have

$$\begin{aligned} R(z;A_0,\eta)x &= \left(zI + A_0 - \sum_{r=1}^{r=p} c_r \exp(-zh_r)A_0^{\alpha_r} - \int_{-h}^{0} c_0(\tau)\exp(z\tau)A_0^{\alpha_0}\,\mathrm{d}\tau\right)^{-1} \\ &= \left(zI + A_0 - \sum_{r=1}^{r=p} c_r \exp(-zh_r)A_0^{\alpha_r} - \int_{-h}^{0} c_0(\tau)\exp(z\tau)A_0^{\alpha_0}\,\mathrm{d}\tau\right)^{-1} \\ &\quad\times \sum_{i=1}^{i=\infty}\sum_{k=1}^{k=n_i} \langle x, x_{ik}\rangle_X\, x_{ik} \\ &= \sum_{i=1}^{i=\infty}\left(z + s_i - \sum_{r=1}^{r=p} c_r \exp(-zh_r)s_i^{\alpha_r} - \int_{-h}^{0} c_0(\tau)\exp(z\tau)s_i^{\alpha_0}\,\mathrm{d}\tau\right)^{-1} \\ &\quad\times \sum_{k=1}^{k=n_i} \langle x, x_{ik}\rangle_X\, x_{ik} = \sum_{i=1}^{i=\infty}\Big(\Delta_i(z)\Big)^{-1}\sum_{k=1}^{k=n_i}\langle x, x_{ik}\rangle_X\, x_{ik} \end{aligned} \tag{17}$$

Therefore, from (15)–(17), it follows directly that

$$\begin{aligned} B^*R(z;A_0,\eta)x^0 &= B^*\Big((\Delta_{i_0}(z))\Big)^{-1} x_0 \\ &= \Big((\Delta_{i_0}(z))\Big)^{-1} B^*x^0 = 0 \quad \text{for each } z \in \rho(A_0,\eta) \end{aligned} \tag{18}$$

This contradicts (12) and therefore, by Lemma 1, the dynamical system is not approximately relatively controllable in finite time. Hence the necessity follows.

(Sufficiency) Since the operator $-A_0$ generates an analytic semigroup $T(t)$ for $t>0$, (12) is the necessary and sufficient condition for approximate controllability in any time interval for the dynamical system without delays (Klamka, 1991; 1993b; Triggiani, 1975a; 1976; 1978)

$$\dot{x}(t) = -A_0x(t) + \sum_{j=1}^{j=m} b_j u_j(t) \tag{19}$$

Since attainable sets for the dynamical systems (1) and (19) are the same for $t \in [0,h_1]$, Definitions 1 and 2 imply directly approximate relative controllability in finite time for the dynamical system (1), and the proof is complete. ■

Corollary 1. *Suppose that all the eigenvalues s_i, $i = 1, 2, \dots$ are simple, i.e. $n_i = 1$ for $i = 1, 2, \dots$. Then the dynamical system (1) is approximately relatively controllable in finite time interval if and only if*

$$\sum_{j=1}^{j=m} \langle b_j, x_i \rangle_X^2 \neq 0 \quad for \ \ i = 1, 2, \dots \tag{20}$$

Proof. From Theorem 1 it follows immediately that for the case when multiplicities $n_i = 1$ for $i = 1, 2, \dots$ the dynamical system (1) is approximately relatively controllable in finite time if and only if the m-dimensional row vectors

$$B_i = \left| \langle b_1, x_i \rangle_X \quad \langle b_2, x_i \rangle_X \quad \cdots \quad \langle b_j, x_i \rangle_X \quad \cdots \quad \langle b_m, x_i \rangle_X \right| \neq 0 \tag{21}$$

for $i = 1, 2, \dots$. Since the relations (20) and (21) are equivalent, Corollary 1 follows immediately. ■

Corollary 2. *The dynamical system (1) is approximately relatively controllable in finite time if and only if the dynamical system without delays*

$$\dot{x}(t) = -A_0^{\beta} x(t) + \sum_{j=1}^{j=m} b_j u_j(t), \qquad 0 < \beta < \infty \tag{22}$$

is approximately controllable in finite time for some β.

Proof. Comparing approximate controllability results given in (Bensoussan *et al.*, 1993; Klamka, 1991; Triggiani, 1976) with equalities (13) in Theorem 1, we conclude that the retarded dynamical system (1) is approximately relatively controllable in finite time if and only if the dynamical system without delays (22) is approximately controllable for $\beta = 1$. On the other hand, according to (O'Brien, 1979), approximate controllability of the dynamical system (22) for $\beta = 1$ is equivalent to its approximate controllability for each $\beta \in (0, \infty)$. This complete the proof. ■

4. Example

Let us consider a retarded dynamical system with distributed parameters described by the following partial differential equation:

$$w_t(t, y) = -a w_{yyyy}(t, y) + c w_{yy}(t - h, y) + \int_{-h}^{0} w_{yy}(t + \tau, y) \, \mathrm{d}\tau + \sum_{j=1}^{j=m} b_j(y) u_j(t) \tag{23}$$

defined for $t > 0$, $y \in [0, L]$, with homogeneous boundary conditions

$$w(t, 0) = w(t, L) = w_{yy}(t, 0) = w_{yy}(t, L) = 0 \tag{24}$$

and initial conditions

$$w(0, y) = g^0(y) \in L_2\Big([0, L], \mathbb{R}\Big) = X \quad \text{and} \quad w(t, y) = g^1(t, y) \in L_2\Big([-h, 0], X\Big) \tag{25}$$

where $b_j(y) = b_j \in L_2([0,L],\mathbb{R}) = X$, $j = 1,2,\ldots,m$ are given functions, $u_j(t) \in L_2([0,\infty),\mathbb{R})$, $j = 1,2,\ldots,m$ are scalar control functions, $h > 0$ is a constant delay, a is a given positive constant, c is a given non-zero real constant.

The retarded linear partial differential equation (23) can be expressed in an abstract form (1) by substituting $w(t,y) = x(t) \in X$ and using the unbounded linear differential operator $A_0 : X \supset D(A_0) \to X$ defined as follows:

$$A_0 x = A_0 w(y) = a w_{yyyy}(y) \tag{26}$$

$$D(A_0) = \Big\{ x = w(y) \in H^4\Big([0,L],\mathbb{R}\Big) : w(0) = w(L) = w_{yy}(0) = w_{yy}(L) = 0 \Big\} \tag{27}$$

where $H^4([0,L],\mathbb{R})$ denotes the fourth-order Sobolev space.

The unbounded linear differential operator A_0 has the following properties (Bensoussan *et al.*, 1993; Klamka, 1991; Triggiani, 1976):

1. A_0 is self-adjoint and positive-definite with dense domain $D(A_0)$ in X.
2. There exists a compact inverse A_0^{-1} and, consequently, the resolvent $R(s;A_0)$ of A_0 is a compact operator for all $s \in \rho(A_0)$.
3. A_0 has the spectral representation

$$A_0 x = A_0 w(y) = \sum_{i=1}^{i=\infty} s_i \langle x, x_i \rangle_X x_i \quad \text{for } x \in D(A_0)$$

where $s_i > 0$ and $x_i(y) \in D(A_0)$, $i = 1,2,\ldots$ are simple $(n_i = 1)$ eigenvalues and the corresponding eigenfunctions of A_0, respectively. Moreover

$$s_i = a\left(\frac{\pi i}{L}\right)^4, \quad x_i(y) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi i y}{L}\right) \quad \text{for } y \in [0,L]$$

and the set $\{x_i(y),\ i = 1,2,\ldots\}$ forms a complete orthonormal system in X.

4. Fractional powers A_0^α, $0 < \alpha \le 1$ can be defined by

$$A_0^\alpha x = A_0^\alpha w(y) = \sum_{i=1}^{i=\infty} s_i^\alpha \langle x, x_i \rangle_X x_i = \sum_{i=1}^{i=\infty} s_i^\alpha \left(\int_0^L w(y) x_i(y)\,\mathrm{d}y \right) x_i(y)$$

for $x \in D(A_0^\alpha)$, and each of them is also a self-adjoint and positive-definite operator with a dense domain in X. Moreover, it should be noted that the fact that A_0 is a differential operator does not ensure that A_0^α is also a differential operator. However, particularly for $\alpha = 1/2$, we have

$$\begin{aligned} A_0^{1/2} x &= A_0^{1/2} w(y) = -\sqrt{a}\, w_{yy}(y) \\ D(A_0^{1/2}) &= \Big\{ x = w(y) \in X : w(0) = w(1) = 0 \Big\} \end{aligned} \tag{28}$$

Therefore, the unbounded linear differential operator defined by (26) and (27) satisfies all the assumptions stated in Section 3 and hence eqn. (23) has the following abstract representation:

$$\dot{x}(t) = -A_0 x(t) - \frac{c}{\sqrt{a}} A_0^{1/2} x(t-h) - \frac{1}{\sqrt{a}} \int_{-h}^{0} A_0^{1/2} x(t+\tau)\,\mathrm{d}\tau + \sum_{j=1}^{j=m} b_j(y) u_j(t) \tag{29}$$

Thus, by using general results of Section 3 it is possible to formulate a necessary and sufficient condition for approximate relative controllability in finite time of the linear retarded distributed-parameter dynamical system (23).

Theorem 2. *The linear retarded partial differential dynamical system (23) is approximately relatively controllable in finite time if and only if*

$$\sum_{j=1}^{j=m} \left(\int_0^L \sqrt{\frac{2}{L}} b_j(y) \sin\left(\frac{\pi i y}{L}\right) \mathrm{d}y \right)^2 \neq 0 \quad \textit{for} \quad i = 1, 2, \dots \tag{30}$$

Proof. Let us observe that the dynamical system (23) satisfies all the assumptions of Corollary 1. Therefore, taking into account the analytic formula for the eigenvectors $x_i(y) \in L_2([0,L], \mathbb{R})$, $i = 1, 2, \dots$ and the form of the inner product in the separable Hilbert space $L_2([0,L], \mathbb{R})$, from (20) we obtain directly (30). ■

5. Final Remarks

In the present paper relative controllability problems for linear abstract retarded dynamical systems with lumped and distributed delays have been considered. Using only the resolvent methods and spectral analysis of unbounded linear operators the necessary and sufficient conditions for approximate relative controllability in finite time have been formulated and proved. These conditions allow us to investigate approximate relative controllability for abstract retarded dynamical systems by checking approximate controllability of simplified linear abstract dynamical systems without delays.

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