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OPTIMIZATION AND SENSITIVITY OF STEPPED COLUMNS UNDER CIRCULATORY LOAD

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This paper deals with the optimal design and sensitivity analysis of a stepped column subjected to a nonconservative circulatory loading. The analysis is focused on maximization of the critical force under the constraint of constant volume. A considerable increase in the critical load is achieved by introducing, in the middle of the column span, an infinitesimal segment. It follows from the sensitivity analysis that the stability of the system is extremely sensitive with respect to the direction of loading application, while an increase in the stiffness of the infinitesimal segment improves the reliability and safety of the whole system. A generalized condition of multimodal design optimization is proposed in the paper. The analysis is based on the transfer matrix method.

1. Introduction

Optimization theory under stability and frequency constraints was developed by Prager and Taylor (1968), while Życzkowski and Gajewski (1971) were probably the first researchers who considered the optimization problem in cantilevers under circulatory forces. The research was then continued by many others, cf. (Błachut and Gajewski, 1980; Bogacz *et al.*, 1979; 1986; Claudon, 1975; Gutkowski *et al.*, 1991; Hanaoka and Washizu, 1980; Tada *et al.*, 1985; 1989). Some results related to the column optimization (maximization of the critical force) are shown in Fig. 1. A majority of the presented results are obtained by the finite-element method applied to a column divided into ten elements of constant height and linear mass distribution. For such discretisation, the highest value of critical load, equal to $P_{cr} = 92.56\,EJ/h^2$, was obtained by Gutkowski *et al.*, (1991) by applying a method based on the Kuhn-Tucker necessary conditions for optimality. A considerable rise in the critical force value, up to $136.5\,EJ/l^2$ was reported by Tada *et al.*, (1985) who used the inverse variational principle. The authors increased the number of finite elements and applied the adaptive mesh technique, in which a shorter length was assigned to a larger gradient of the cross-sectional area.

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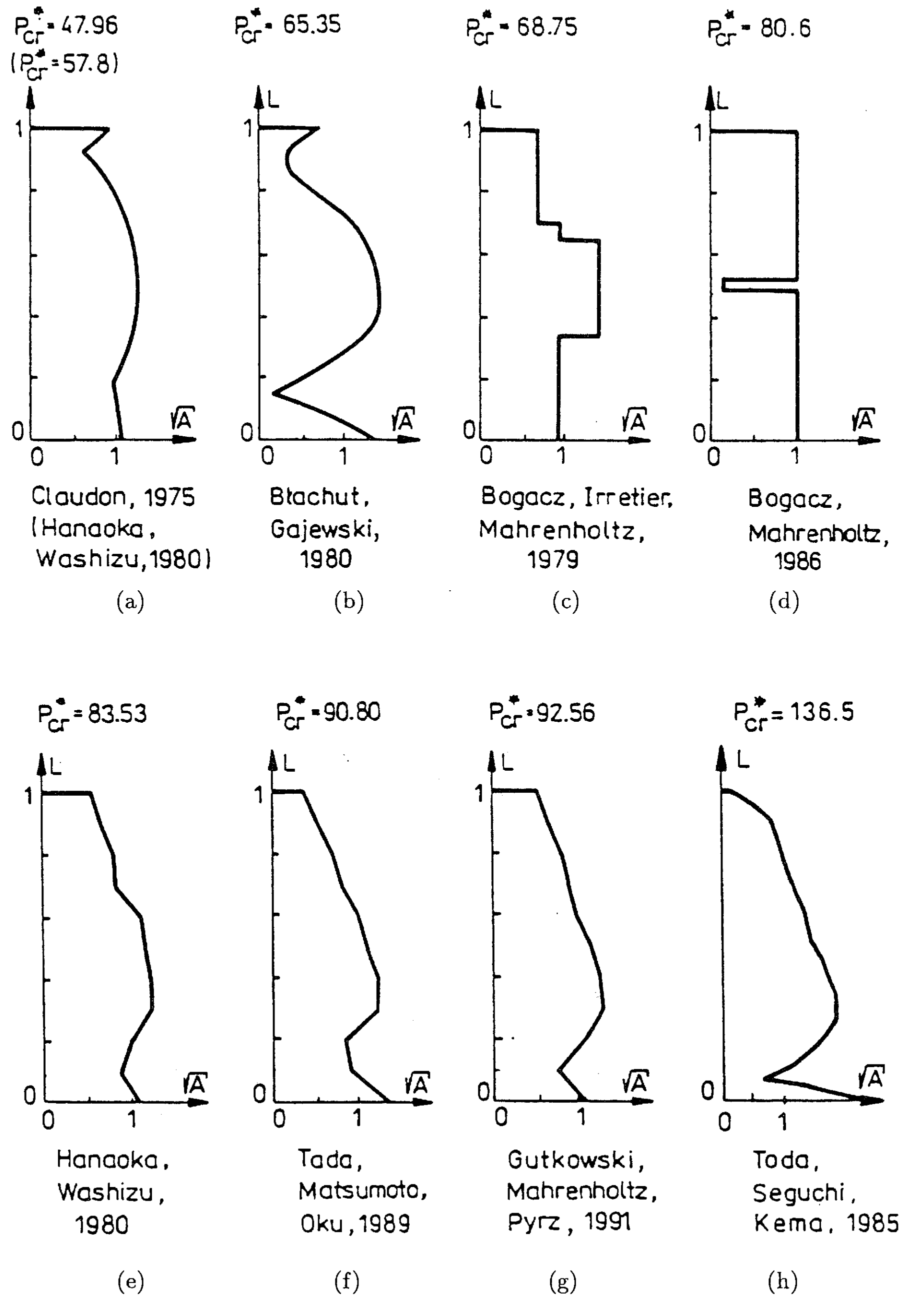


Fig. 1. Results of column optimization under circulatory load (maximization of the critical load).

In practical applications the stepped columns and beams have wider applications than the optimal continuous forms. The optimization result for the stepped column consisting of four segments obtained by Bogacz *et al.* (1979) is shown in Fig. 1(c).

A proper choice of the starting point in the space of design variables, i.e. the initial dimensions of the column, is essential for the result of optimization. When looking for an initial shape, the authors take into account the fact that the localized loss of structural stiffness can stabilize nonconsevative systems. This phenomenon was studied for the columns with different models of the stiffness discontinuity, namely a hinge-joint (Bogacz and Mahrenhololtz, 1986; Bogacz and Niespodziana, 1987), transverse-slidable joint (Bogacz and Imiełowski, 1990; Imiełowski, 1989) and generalized rotationally-slidable joint (Bogacz and Imiełowski, 1994). One of the conclusions of this research reveals that a fourfold increase in the critical force value is possible for a column with locally reduced stiffness placed in the middle of the column span, see Fig. 1(d) and Fig. 2(a). This shape is taken as a starting point for the optimization procedure. Since the authors seek the simplest possible shape of a column, the stepped shape is chosen for analysis.

It is noteworthy that for the optimal shape the error of the approximation method appears. For example, the results of force maximization for the stepped column composed of two segments, obtained by the variational approach, yield the critical force of 6% higher than that predicted by the transfer-matrix technique (Bogacz *et al.*, 1979). Variations of about 20% in the value of critical force are obtained in (Claudon, 1975) and (Hanaoka and Washizu, 1980) by using the same optimality criteria, see Fig. 1(a).

In the present paper, an analytical solution is obtained by applying the transfer-matrix method while the gradient-projection method is used as an optimization procedure. The study is completed by the sensitivity analysis, in which the influence of the infinitesimal segment stiffness, the direction of the acting force and dimensions of segments is taken into account. The results of numerical calculations are then compared with those published in the literature.

2. Problem Formulation

2.1. Mechanical Model

A model of the structure under consideration is shown in Fig. 2(c). The stepped column consists of n segments of cross-section A_i, length l_i, mass m_i and stiffness EJ_i. A single infinitesimal segment is assumed in the form of an elastic hinge-joint. It is located at the point x_S, in the centre of the structure. The column is subjected to a tangential compressive force P.

The equation of motion for the i-th beam segment for small harmonic vibrations is of the form

$$\frac{\partial^2}{\partial x^2}\left(EI_i\frac{\partial^2 y_i}{\partial x^2}\right) + P\frac{\partial^2 y_i}{\partial x^2} + \rho A_i\frac{\partial^2 y_i}{\partial t^2} = 0 \tag{1}$$

where ρA_i denotes the mass per unit length.

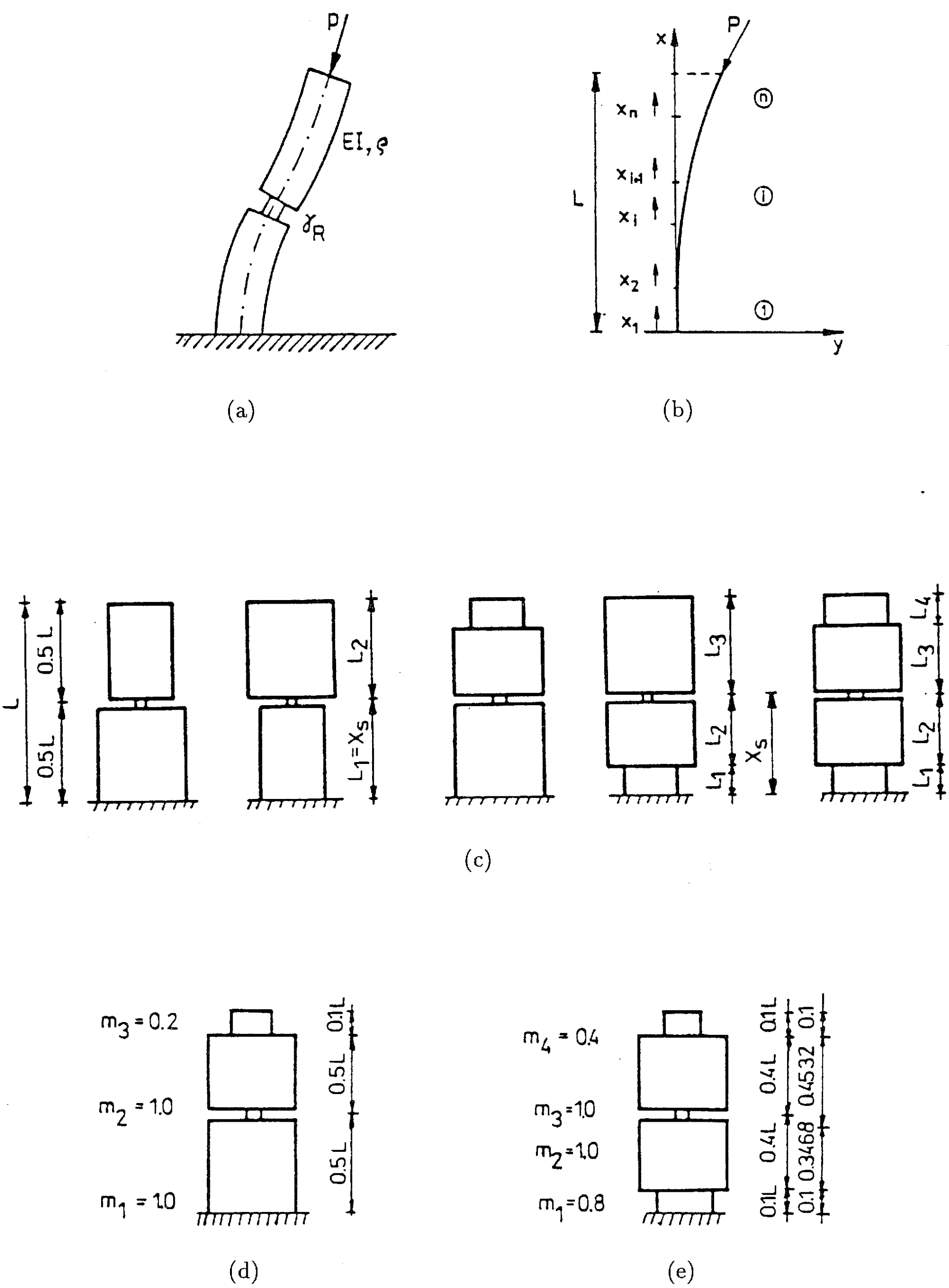


Fig. 2. Column with infinitesimal segment (a), segmentation of the structure (b), and forms of the column under consideration (c)–(e).

We assume the separation of variables typical for a harmonic motion

$$y_i(x,t) = w_i(x)\exp(i\omega t) \tag{2}$$

where ω is the angular frequency. The exact solution for the segment of uniform mass and stiffness distribution is given by

$$w_i(x) = A_1 \sinh \lambda_1 x + A_2 \cosh \lambda_1 x + A_3 \sin \lambda_2 x + A_4 \cos \lambda_2 x \tag{3}$$

where

$$\lambda_{1/2} = \sqrt{\pm\frac{P}{2EI_i} + \sqrt{\left(\frac{P}{2EI_i}\right)^2 + \frac{\rho A_i \omega^2}{EI_i}}} \tag{4}$$

The conditions of force equilibrium and displacement continuity for the infinitesimal segment follow from the analysis reported in (Bogacz and Imiełowski, 1990; 1994; Bogacz and Niespodziana, 1987; Imiełowski, 1989). The adequate conditions are

$$\begin{aligned} &w_i(x_s) = w_{i+1}(x_s), \qquad w_i^{II}(x_s) = w_{i+1}^{II}(x_s), \\ &w_i^{III}(x_s) = w_{i+1}^{III}(x_s), \quad w_i^{I}(x_s) - w_i^{I}(x_s) = \gamma_R w_{1+1}^{II}(x_s) \end{aligned} \tag{5}$$

where $(\cdot)^I = \mathrm{d}(\cdot)/\mathrm{d}x$ and γ_R is the flexibility of the infinitesimal segment. The boundary conditions for the Beck problem are of the form

$$w_1(0) = w_1^I(0) = w_n^{II}(L) = w_n^{III}(L) = 0 \tag{6}$$

In what follows, all cross-sections are assumed to be geometrically similar:

$$A_i = m_i A_0, \qquad J_i = m_i^2 J_0 \tag{7}$$

where A_0 and J_0 are reference values, and m_i is a dimensionless design variable, which is constant in the i-th segment. From now on, we use the following dimensionless quantities:

$$P^* = PL^2/EJ_0, \qquad \omega^{*2} = \omega^2\rho A_0 L^4/EJ_0, \qquad u = w/L, \qquad \xi = x/L \tag{8}$$

Equations (1), (5) and (6) can now be rewritten in the final form

$$\Big[m_i^2 u_i^{II}(\xi_i)\Big]^{II} + P^* u_i^{II}(\xi_i) - w^{*2} m_i u_i(\xi_i) = 0, \qquad 0 \le \xi_i \le 1 \tag{9}$$

$$u_i^I(\xi_S) - u_i^I(\xi_S) = \gamma_R^* u_{i+1}^{II}(\xi_S) \tag{10}$$

$$u_1(0) = u_1^I(0) = m^2 u_n^{II}(1) = \Big[m^2 u_n^{II}(1)\Big]^I = 0 \tag{11}$$

where $\gamma_R^* = \gamma_R EJ/L$ is the dimensionless parameter of the joint flexibility. The stiffness parameter $\kappa_R^* = 1/\gamma_R^*$ is taken for simplification.

The problem is solved with the use of the transfer-matrix method, in which all the generalized displacements and forces of an arbitrary cross-section are defined as components of the state vector $\boldsymbol{Z}_i$, with the following constitutive form:

$$\boldsymbol{Z}_i = [y,\, \varphi,\, M, Q]_i^T = [w, w', -EIw'', -EIw''']^T \tag{12}$$

The relation between the state vectors $\boldsymbol{Z}_i$ and $\boldsymbol{Z}_{i+1}$ on both boundaries of the i-th segment is represented by $\boldsymbol{Z}_{i+1} = \boldsymbol{T}_i \boldsymbol{Z}_i$, where $\boldsymbol{T}_i$ is the partial transfer matrix. The global description having the form of a relation between the state vectors $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_{n+1}$ is then obtained by multiplying the matrices $\boldsymbol{T}_i$ for successive segments:

$$\boldsymbol{Z}_{n+1} = \boldsymbol{T}_n \boldsymbol{T}_{n-1} \cdots \boldsymbol{T}_2 \boldsymbol{T}_1 \boldsymbol{Z}_1 = \boldsymbol{T} \boldsymbol{Z}_1 \tag{13}$$

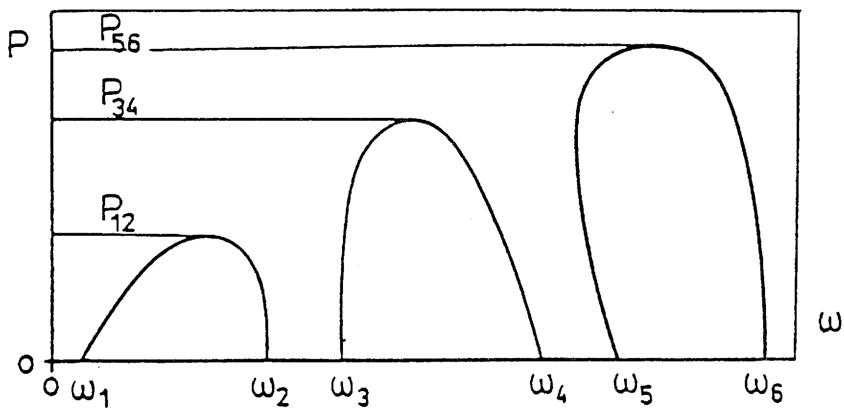
The partial transfer matrix $\boldsymbol{T}_i$ for a beam segment is derived from the solution (3) and given in (Bogacz *et al.*, 1979). For the elastic hinge-joint it follows from relations (5) and (10). The reader is referred to (Bogacz *et al.*, 1979; Pestel and Leckie, 1963) for details. There are four equations in the set (13) with eight unknown components of the vectors $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_{n+1}$. The problem, however, can be solved since four unknowns are to be obtained from the boundary conditions (11). The characteristic equation from which the singularity of $\boldsymbol{T}$ follows, establishes the relation between the force and frequency:

$$\Phi(P^*, \omega^*) = 0 \tag{14}$$

The points corresponding to the roots of (14) which are the eigenvalues of the problem, obtained for the successive values of P^*, constitute the so-called characteristic curves (eigencurves) on the force-frequency plane. A typical configuration of eigencurves for the column of a constant cross-section is sketched in Fig. 3(a) and by a dotted line in Fig. 5(a). The critical state is defined as the smallest critical force for which the successive eigenfrequencies coincide to form an imaginary conjugate couple. On the force-frequency plane $P^* = P^*(\omega^*)$, the critical force is represented by a maximum value, see Fig. 3(a).

2.2. Optimality Criteria

When solving the problems of nonconservative systems we have to overcome mathematical difficulties which are related to the fact that such systems have no potential and, therefore, the governing differential equations are not self-adjoint. Some of the methods reported in the literature are discussed in Section 1. In the present paper, we adopt a multimodal analysis with dynamical stability constraints. In this approach the objective function $P^*_{cr} = P^*_{cr}(\alpha)$, where α stands for a set of design variables, is not defined explicitly.



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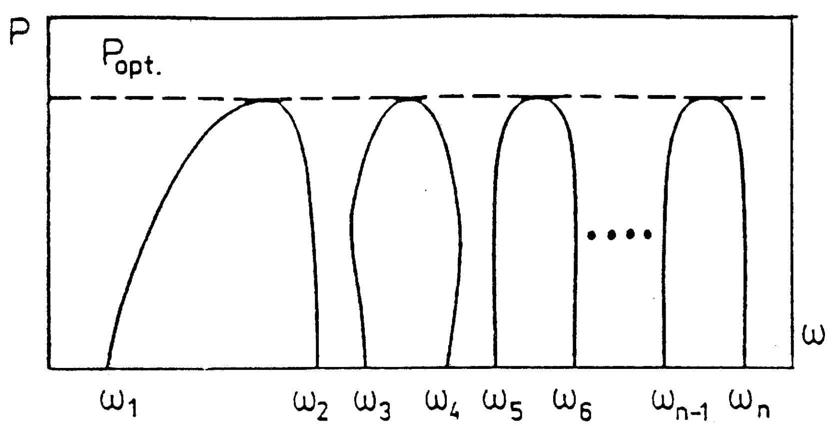


(b)

Fig. 3. Definition of the critical forces (a) and the configuration of the characteristic curves for the optimal shape (b).

The appropriate optimization conditions are imposed in the frequency domain as limitations on variation in the shapes of characteristic curves. The definition of an optimum state is adopted from (Tada *et al.*, 1985). The optimal point represents here the state for which all pairs of eigenvalues become double roots with the same value of critical force as shown in Fig. 3(b). Taking this into account, the optimization process consists in the determination of a mass distribution, such that the successive double roots represent equal values of critical forces:

$$P_{12} = P_{34} = P_{56} = \cdots = P_{ij} = \cdots \tag{15}$$

Here P_{ij} denotes the critical load corresponding to the i-th and j-th frequency branches in the force-frequency plane. Claudon (1975) increased the value of P_{12} until it satisfied the condition $P_{12} = P_{34}$. Hanaoka and Washizu (1980) improved this result by taking both P_{12} and P_{34} into account and maximizing them simultaneously. In (Bogacz *et al.*, 1979; Gutkowski *et al.*, 1991; Tada *et al.*, 1989) six first eigenfrequencies were considered. In (Bogacz *et al.*, 1979; Tada *et al.*, 1989) the condition of equality of three successive values of P_{ij}, i.e. $P_{12} = P_{34} = P_{56}$, is finally satisfied.

In this paper the problem is reduced to increasing the value of P_{12} under the constraint

$$P_{\rm opt} = P_{12} \leq P_{i,j} \tag{16}$$

Notice that in (16) we define the optimum critical force as that occurring with the first and second natural form, i.e. $P_{\rm opt} = P_{12}$. This assumption is very useful in calculations since the necessity for keeping two or three first values of the critical force within an acceptable accuracy range can be eliminated.

The next constraint is introduced to preserve a high value of the critical force against the shape perturbation. Due to possible interactions between the successive characteristic curves a discontinuous decrease in the critical force value can appear. The phenomenon is explained in Section 3, cf. (Bogacz and Imiełowski, 1994; Claudon, 1975; Hanaoka and Washizu, 1980; Mahrenholtz and Bogacz, 1981). The assumption of the minimum distance between two successive eigencurves prevents such interactions. The formula for a sufficient distance between two curves is as follows:

$$\omega_{i+1}^* - \omega_i^* \geq c \tag{17}$$

where c is a positive number and i denotes the i-th frequency branch. The condition formulated in this manner was introduced in (Tada, 1989). However, the condition (17) is here applied only for $P^* < P_{\rm opt} = P_{12}$, and a switch-over of characteristic curves resulting in the determination of a critical force larger than P_{12} is permitted.

3. Numerical Example: Maximization of the Critical Force

We look for a mass distribution which maximizes the critical load under the constraint of a constant total mass of the column. For the stepped column this condition is written as

$$\sum_{i=1}^{n} m_i l_i = 1 \tag{18}$$

where n denotes the number of segments. A structure composed of at most four segments with an infinitesimal segment located in the centre of the column is considered.

Looking for the initial shape, we check all possible combinations of the design variables within the range: $l_i = 0.1, 0.2, 0.3, 0.4$ and $m_j = 1.0, 0.8, 0.6, 0.4, 0.2$, $i,j = 1,2,3,4$. For two of them a considerable increase in the critical forces up to values $P_{14} = 140EJ/L^2$ and $P_{23} = 60EJ/L^2$ is observed. The appropriate segmentations are shown in Figs. 2(d) and 2(e). Notice that the corresponding configuration of characteristic forces, shown with dashed lines in Fig. 4(a), differs qualitatively from those typical for the Beck column, which are depicted in Fig. 3(a). In order to restore the required configuration (by lowering joint position n) we cause a qualitative change in the shapes of characteristic curves. A discontinuous increase in the critical force value follows a variation of this design variable as shown in Fig. 2(e). This phenomenon is explained in Fig. 4(a). The dashed, dotted and solid lines are sketched for $x_S = 0.5$, $x_S = 0.44778$ and $x_S = 0.4468$, respectively. For the highest joint position the column losses its stability and oscillates with the second and third eigenfrequencies, whereas for the lower position the instability occurs with the first and second natural form. The switch-over is depicted with a dotted line. Notice that the "jump" phenomenon (a discontinuous increase in the critical force) was here directly applied to the preliminary optimization.

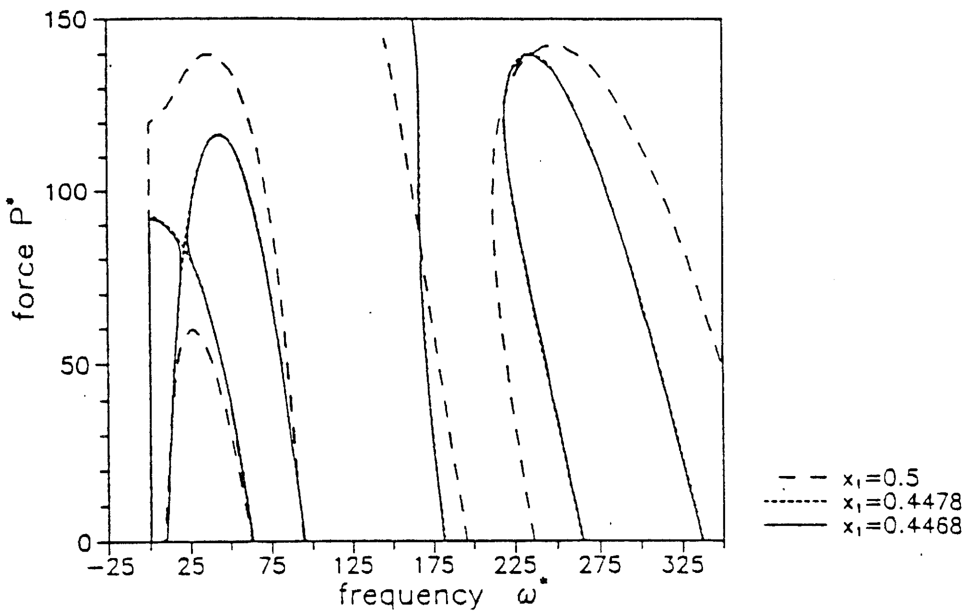
The shape obtained is shown in Fig. 2(e) with the lengths of segments marked on the right-hand side. It is taken as the initial guess for the gradient procedure. The procedure selects the design parameter $\Delta\alpha \in \{m_i, l_i, x_s\}$ by analyzing the configuration of characteristic curves and satisfying the conditions (16), (17) and (18). Notice that the condition (16) guarantees only a non-negative value of the objective function gradient and not the largest one. A detailed procedure is given in (Mahrenholtz and Bogacz, 1981). Discontinuous changes in the critical force are prevented by the condition (17), however a feasible switching of eigencurves resulting in determination of a critical force larger then the value of P_{12} is observed, see Fig. 4(b). Notice that the condition (18) can be satisfied for a final shape because the scale effect does not influence qualitatively the configuration of eigencurves.

The value of the critical force of the obtained segmentation is equal to $P_{\rm cr} = 133.38EJ/L^2$. The column is shown in Fig. 5(b) and the corresponding configuration of the characteristic curves is depicted with a solid line in Fig. 5(a). The eigencurves of the column of a constant cross-section are drawn by using dotted lines. The value of the critical force is compared with the best result reported by Tada *et al.* (1985), obtained for a complicated shape consisting of twenty trapezoidal elements of different lengths, see Fig. 1(h).

It is evident that the result obtained does not represent the optimal value even for a class of stepped columns. When higher frequencies are taken into consideration and a continuous mass distribution is allowed, this result can be improved. However, this requires greater computational efforts. On the other hand, due to a high sensitivity, such optimality is questionable from the viewpoint of the structure reliability and safety.

4. Sensitivity Analysis

In the sensitivity analysis we determine the dependence of the structure response upon the design variables. The critical load and the corresponding critical frequency are taken here as the response. The design variables considered are: the dimensions of segments, the stiffness of the infinitesimal segment and the direction of the acting force, defined by the tangency coefficient η as shown in Fig. 6(c).



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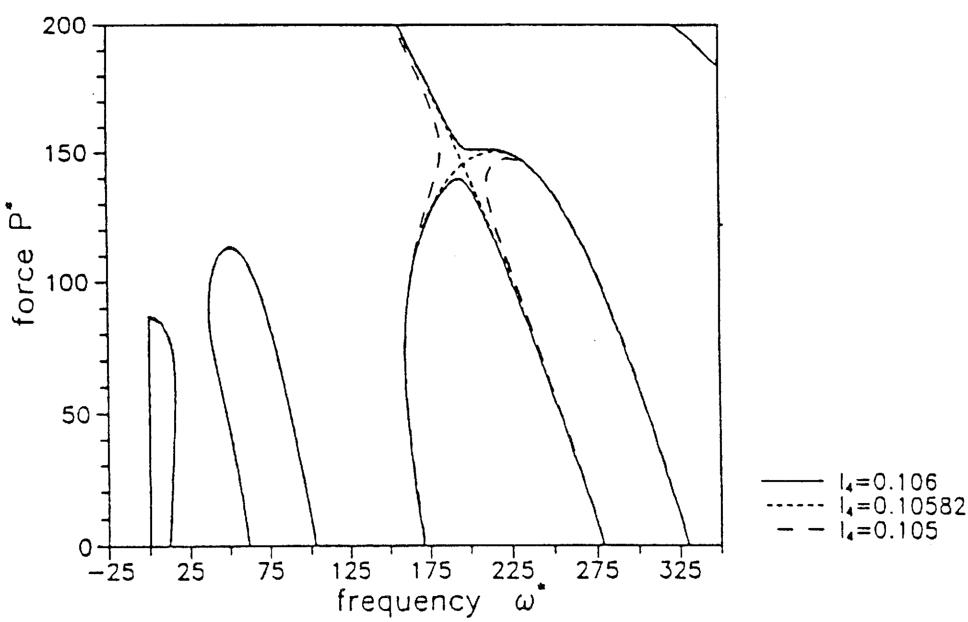
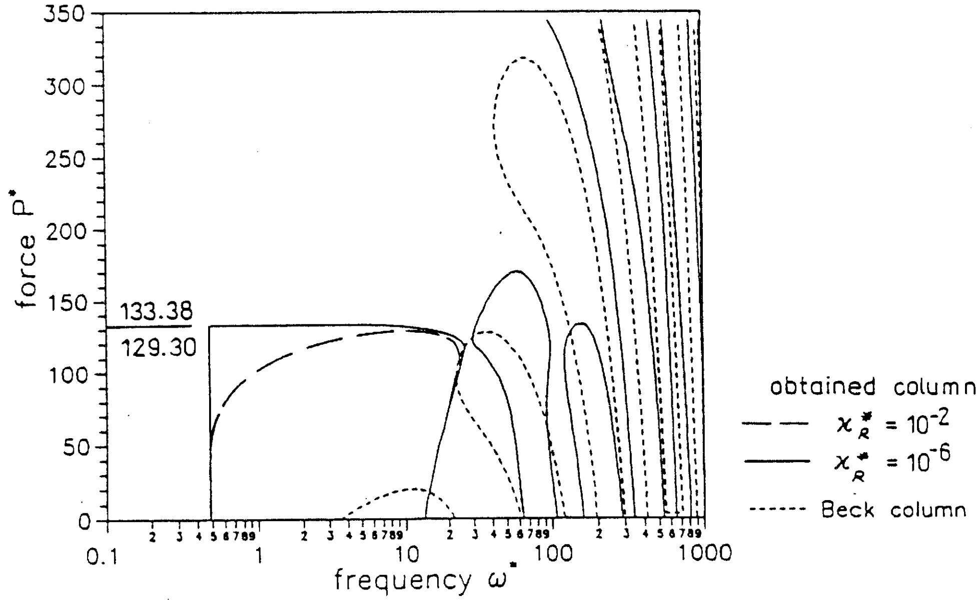


(b)

Fig. 4. Switch-over of the characteristic curves: configuration of curves for the initial column shape ($x_S = 0.4468$) (a) and permitted switch-over of the characteristic curves (b).



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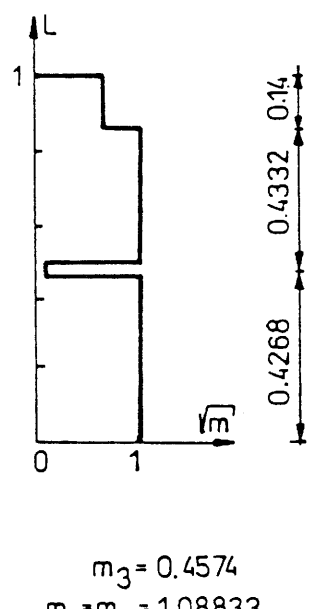


(b)

Fig. 5. Configuration of the eigencurves for the obtained columns and for Beck column (a) and the shape of the obtained stepped column (b).

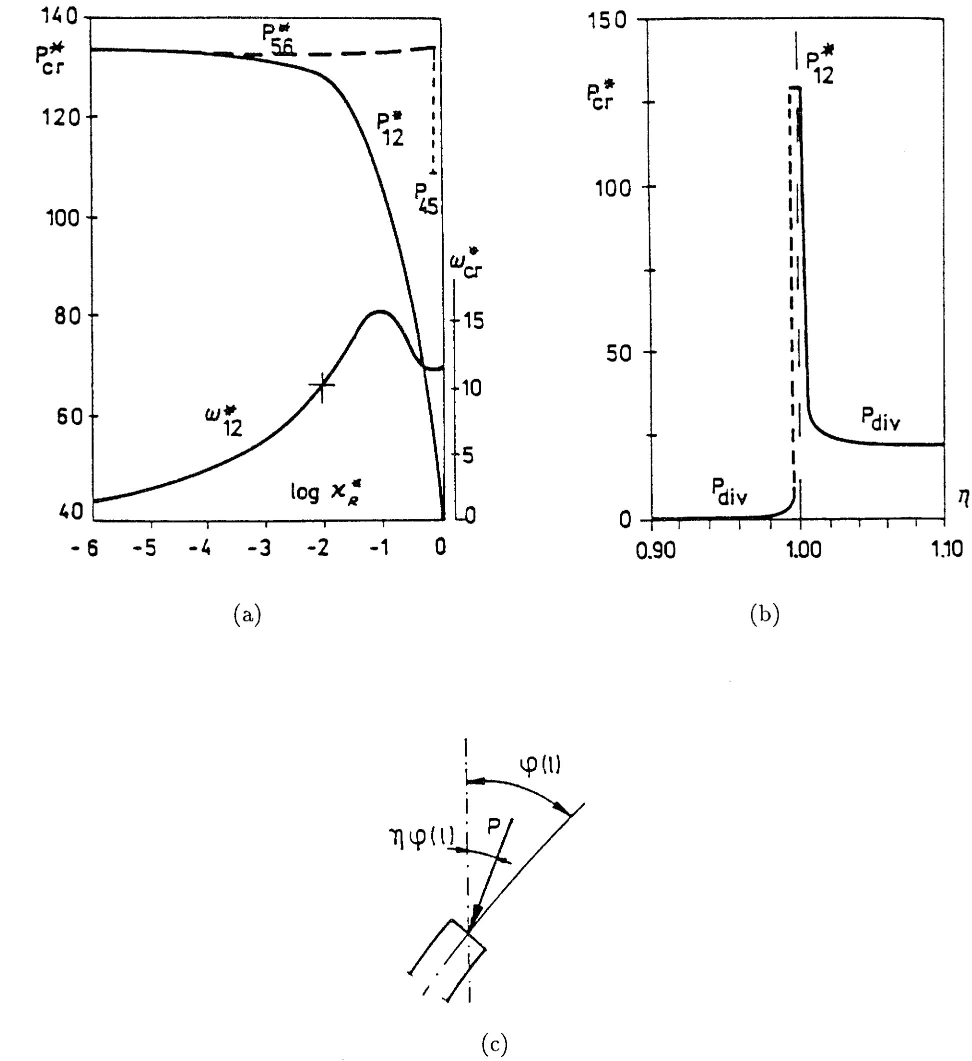


Fig. 6. Influence of the joint stiffness on the values of critical force and critical frequency (a), the dependence of the critical force on the direction of the acting force (b), and the definition of the tangency coefficient (c).

An important question from the point of view of the structure reliability and safety is the determination of the infinitesimal segment stiffness. The shape obtained in Section 3 was calculated for a very small value equal to $\kappa_R^* = 10^{-6}$. It follows from the discussion presented in (Bogacz and Imiełowski, 1994) that in the limiting case, i.e. for $\kappa_R^* \to 0$, the system becomes unstable. It is obvious that an increase in the joint stiffness will stabilize the system. The diagrams of $P_{\rm cr}^* = P_{\rm cr}^*(\kappa_R^*)$ and $\omega_{\rm cr}^* = \omega_{\rm cr}^*(\kappa_R^*)$ are shown in Fig. 6(a). Notice that the decrease in the critical force is accompanied by an increase in the corresponding critical frequency. The drop in the critical force, represented by the vertical dotted line from P^*_{56} to P^*_{45}, destabilizes the structure.

This analysis allows for a correction of the optimization result. We assume that the critical frequency of the optimal strucutre should preserve a value for the uniform Beck column, i.e. $\omega^*_{\rm cr} = 11.01$. From the graph we put the stiffness equal to $\kappa^*_R = 10^{-2}$. The respective critical force value is equal to $P_{\rm cr} = 129,305EJ/L^2$. The corresponding configuration of characteristic curves is presented in Fig. 5(a) by a dashed line.

Let us turn into the influence of tangency coefficient η. The diagram of $P^*_{\rm cr} = P^*_{\rm cr}(\eta)$ is shown in Fig. 6(b). As can be seen, even a slight disturbance of η involves a discontinuous decrease in the critical force value, which changes the character of stability loss from flutter to divergence. For $\kappa^*_R = 10^{-2}$ this range extends. This corresponds to the results obtained in (Bogacz and Niespodziana, 1987) for the column of a constant cross-section with a single elastic hinge-joint.

Consider the influence of segment dimensions. The diagrams of $P^*_{\rm cr} = P^*_{\rm cr}(m_3)$, $P^*_{\rm cr} = P^*_{\rm cr}(l_3)$ and $P^*_{\rm cr} = P^*_{\rm cr}(l_1 = x_s)$, for two values of joint stiffness $\kappa^*_R = 10^{-6}$ and $\kappa^*_R = 10^{-2}$, are depicted in Fig. 7. Let us note that an increase in the joint stiffness results in an extension of the range for which discontinuous changes in the critical force value do not appear. Such a result is expected from the point of view of the structure reliability and safety.

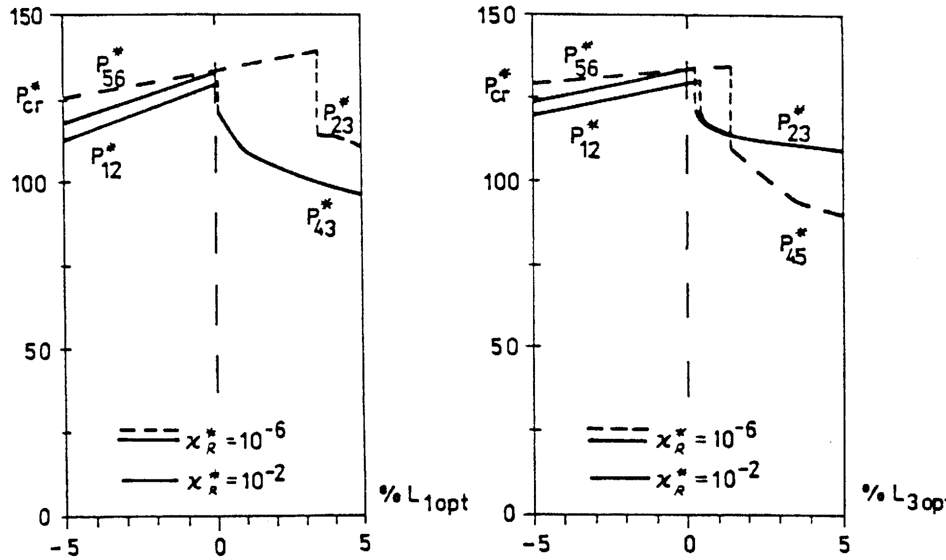
It is interesting to observe the variation of the critical load $\Delta P^*_{\rm cr}$ which corresponds to the $\pm 1\%$ variation of selected design variables. Such an analysis was made in (Gutkowski and Pyrz, 1991) when considering a continuous variation of the column shape. The results for the stepped column are presented in Table 1. Notice that only for the case of discontinuous critical force change the system turns out to be very sensitive to the design variable variation. In general, an increase in the joint stiffness involves a lower sensitivity of the system.

Tab. 1. Variation of $\Delta P^*_{\rm cr}$ corresponding to variations of selected design variables.

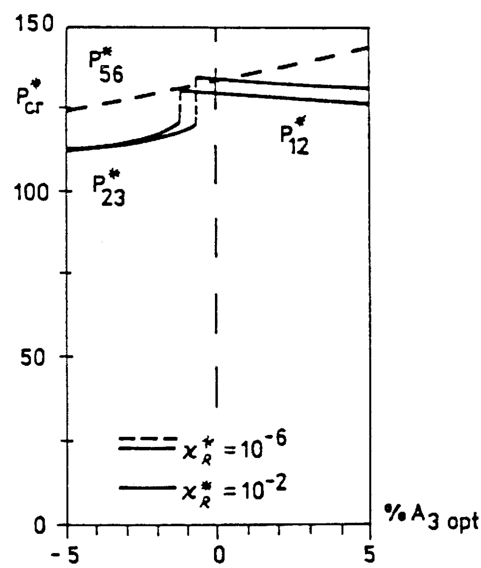
	$P^*_{\rm cr}$	$\Delta\ P^*_{\rm cr}\ (m_3)$		$\Delta\ P^*_{\rm cr}\ (l_3)$		$\Delta\ P^*_{\rm cr}\ (l_1)$	
		-1%	+1%	-1%	+1%	-1%	+1%
$\kappa^*_R = 10^{-6}$	133.380	9.57	0.40	1.50	13.69	2.46	18.43
$\kappa^*_R = 10^{-2}$	129.305	0.45	0.49	1.21	10.66	2.86	15.58

5. Conclusions

This paper deals with the optimal design and sensitivity analysis of a stepped column subjected to a nonconservative circulatory loading. The analysis is focused on maximization of the critical force under the constraint of preserving a constant volume. A considerable increase in the critical load up to 129.3 EJ/L^2 is achieved for a column consisting of four segments one of which is an infinitesimal segment situated in the middle of the column span. We proposed a generalized condition for the optimal design suitable for multimodal optimization.



(a)



(b)

Fig. 7. Influence of the segment dimensions on the critical force value.

As can be seen from the presented analysis, both the optimization results obtained and those taken from the literature represent only local minima, depending strongly on the initial shape of the structure, i.e. on the initial set of design variables. Moreover, the problem considered, being a problem of repeated eigevalues, turns out to be very sensitive with respect to variation of design variables. The optimal value is represented by a point on a discontinuous surface in the space of design variables. A drop in the critical force value may occur due to slight disturbances of the design variable. A variation of the direction of the loading force turns out to be extremely dangerous from the point of view of stability, whereas an increase in the infinitesimal segment stiffness results in an improvement of the reliability and safety of the system. The results obtained can be improved for a column with locally reduced stiffness and with a continuous variation of the cross-section. Hovewer, due to a considerable sensitivity, such a model would be questionable from the viewpoint of structure reliability and safety.

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