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ON THE DYNAMICS OF A CHEMOSTAT MODEL WITH DELAYED NUTRIENT RECYCLING †

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This paper studies the dynamics of a chemostat model with n populations competing for one nutrient which can be recycled due to decomposition of dead biomass. Several kinds of results about local and global stability of non-negative equilibria, uniform persistence and control of populations are obtained.

Keywords: chemostat model, delayed nutrient recycling, stability, uniform persistence, population control

1. Introduction

It is well-known that nutrient recycling processes in aquatic ecosystems of chemostat-type models can be mathematically modelled by means of distributed delay terms (Beretta *et al.*, 1990; 1995; Fergola *et al.*, 1995; 1997; Freedman and Xu, 1993; He and Ruan, 1998; Ruan and Wolkowicz, 1995). Here we study a chemostat model with n density-dependent populations competing for one critical nutrient which can be recycled from the dead biomass, without interaction terms. In our paper, we prove several kinds of results about local and global stability of non-negative equilibria, uniform persistence and control of populations. Our model reduces to the one studied in (Beretta *et al.*, 1990; He and Ruan, 1998) where stability results were proved for a model including only one population in the absence of density dependence effects. In (Freedman and Xu, 1993) a similar problem was studied for two interacting populations. By supposing that both populations have the same memory functions, uniform persistence results were obtained. Again, persistence and stability results can be found in (Beretta *et al.*, 1995; Ruan and Wolkowicz, 1995), where chain chemostat models with one nutrient and two populations and delay nutrient recycling were considered. In both these models the kernels are different but density dependence effects were not considered.

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The scheme of this paper is as follows. In Section 2, we present the model and notations used. Section 3 is devoted to the analysis of non-negative equilibria and their stability properties. Local stability conditions for positive equilibria of system (1) have been obtained by using a suitable Lyapunov-Krasovskii functional (Fergola *et al.*, 1995). By specializing the kernels and by constructing a new Lyapunov functional we are able to give sufficient conditions for global stability of a positive equilibrium. The case of a single population has also been considered and the obtained stability results have been compared with those of (Beretta *et al.*, 1990). In Section 4, we study the problem of survival of populations and we obtain results about uniform persistence and control of populations. Finally, a brief conclusion is given in Section 5.

2. Model and Notation

Consider the following chemostat model with delayed nutrient recycling:

$$\begin{cases} \dot{R} = D(R^0 - R) - \sum_{i=1}^{n} \frac{m_i}{y_{R_i}} U_i(R) N_i + \sum_{i=1}^{n} b_i \gamma_i I_i, \\ \dot{N}_i = N_i \big[m_i U_i(R) - (D + \gamma_i) - \delta_i N_i \big], \quad 1 \le i \le n, \end{cases} \tag{1}$$

where all constants are positive and $R(t)$ is the nutrient concentration at time t, $N_i(t)$ is the biomass concentration of the i-th population at time t, D is the constant dilution rate of the chemostat, R^0 is the constant input concentration of the limiting nutrient, m_i is the maximum specific growth of the i-th population, y_{R_i} is the constant yield of the i-th species per unit resource R consumed, b_i is the fraction of the death biomass recycled as nutrient for the i-th population, γ_i is the death rate of the i-th species, δ_i is the intraspecific competition of biotic species, $U_i(R)$ is the uptake function for limiting nutrient by the i-th planktonic species, which satisfies the conditions

1. $U_i : \mathbb{R}_{+0} = [0, +\infty) \to [0, 1), \quad U_i \in C^1(\mathbb{R}_{+0})$,
2. $U_i'(R) > 0, \quad U_i''(R) < 0 \;$ for $R \in \mathbb{R}_{+0}$,
3. $U_i(0) = 0, \quad \lim_{R \to +\infty} U_i(R) = 1$.

Furthermore,

$$I_i \triangleq \int_0^{+\infty} f_i(s) N_i(t - s) \, \mathrm{d}s$$

with the delay kernels $f_i(s)$ being non-negative, bounded functions for $s \in [0, +\infty)$ such that

$$\int_0^{+\infty} f_i(s) \, \mathrm{d}s = 1, \quad 1 \le i \le n.$$

The initial-value condition of (1) is

$$R(t_0) = R_0, \quad N_i(t_0 + \theta) = \phi_i(\theta), \quad \theta \in (-\infty, 0], \quad 1 \le i \le n, \tag{2}$$

where R_0 is a non-negative constant and ϕ_i is a bounded, continuous and non-negative function on $(-\infty, 0]$. The solution to (1) with initial data (2) should be denoted by

$$\begin{aligned} x_t &= x(t+\theta; t_0, R_0, \phi) \\ &= \big(R(t+\theta; t_0, R_0, \phi), N_1(t+\theta; t_0, R_0, \phi,), \dots, N_n(t+\theta; t_0, R_0, \phi,)\big), \\ x_{t_0} &= \big(R_0, \phi_1(\theta), \dots, \phi_n(\theta)\big), \end{aligned}$$

where $\phi = (\phi_1, \dots, \phi_n)^T$ is an initial vector function.

Existence, uniqueness and non-negativeness of the solutions can be shown by means of the same techniques as in (Beretta *et al.*, 1995; Fergola *et al.*, 1995). So there exists a unique solution $x(t) = x(t; t_0, R_0, \phi)$ which is defined for all $t \in [t_0, t_0 + L)$, which is the maximum existence interval of the solutions to (1). It is known from Lemma 1 in (Fergola *et al.*, 1997) that the solutions (1) with bounded initial values (2) are bounded, so a unique solution to (1) with initial value (2) exists for all $t \geq t_0$.

3. Non-Negative Equilibria and Their Stability Property

3.1. The Trivial Equilibrium of System (1)

It is obvious that $E_0 \triangleq (R^0, 0, \dots, 0)$ is a trivial equilibrium of system (1). In this subsection, we study the stability of E_0.

Theorem 1. *If the inequalities* $m_i U_i(R^0) < D + \gamma_i$, $1 \leq i \leq n$ *are satisfied, then* E_0 *is locally asymptotically stable.*

Proof. By the change of variables

$$r = R - R^0, \qquad n_i = N_i$$

the linearized system of (1) around E_0 is

$$\begin{cases} \dot r = -Dr - \displaystyle\sum_{i=1}^{n} \frac{m_i}{y_{R_i}} U_i(R^0) n_i + D \sum_{i=1}^{n} b_i \gamma_i \int_0^{+\infty} f_i(s) n_i(t-s)\,\mathrm{d}s, \\ \dot n_i = \big[m_i U_i(R^0) - (D+\gamma_i)\big] n_i, \quad 1 \leq i \leq n. \end{cases} \tag{3}$$

The characteristic equation of E_0 is

$$\begin{vmatrix} \lambda + D & \dfrac{m_1}{y_{R_1}} U_1(R^0) - b_1\gamma_1 F_1(\lambda) & \cdots & \dfrac{m_n}{y_{R_n}} U_n(R^0) - b_n\gamma_n F_n(\lambda) \\ 0 & \lambda - \big(m_1 U_1(R^0) - (D+\gamma_1)\big) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda - \big(m_n U_n(R^0) - (D+\gamma_n)\big) \end{vmatrix} = 0,$$

where $F_i(\lambda)$ is the Laplace transform of the kernel function $f_i(s)$, $i = 1, \ldots, n$. Because of our hypothesis, the eigenvalues

$$\lambda_0 = -D, \quad \lambda_i = m_i U_i(R^0) - (D + \gamma_i), \quad 1 \le i \le n$$

are negative, and therefore E_0 is locally asymptotically stable. This completes the proof. ■

Theorem 2. *If $m_i \le (D + \gamma_i)$, $1 \le i \le n$, then E_0 is globally stable.*

Proof. It is easy to prove that $\lim_{t\to+\infty} N_i(t) = 0$. Now we claim that $\lim_{t\to+\infty} R(t) = R^0$. But first, we show that $\lim_{t\to+\infty} \int_0^{+\infty} f_i(s) N_i(t-s)\,\mathrm{d}s = 0$, $1 \le i \le n$.

From Lemma 1 in (Fergola *et al.*, 1997), for any positive numbers A_0 and A_i ($i = 1, \ldots, n$), if the initial value of (2) satisfies

$$R_0 \le A_0, \phi_i(\theta) \le A_i, \quad i = 1, \ldots, n \quad \text{for } \theta \in (-\infty, 0],$$

then there exist positive constants $B_0 \ge A_0$ and $B_i \ge A_i$, $i = 1, \ldots, n$ such that

$$R(t; t_0, R_0, \phi) \le B_0, \quad N_i(t; t_0, R_0, \phi) \le B_i, \qquad t \ge t_0. \tag{4}$$

For any $\epsilon > 0$, we can choose a time T_1 large enough such that

$$B_i \int_{T_1}^{+\infty} f_i(s)\,\mathrm{d}s < \frac{\epsilon}{2}.$$

From (4), we have

$$\int_{T_1}^{+\infty} f_i(s) N_i(t-s)\,\mathrm{d}s < \frac{\epsilon}{2}.$$

Since $\lim\limits_{t\to+\infty} N_i(t) = 0$, we can choose a time T_2 large enough such that

$$N_i(t; t_0, R_0, \phi) < \frac{\epsilon}{2(\int_0^{T_1} f_i(s)\,\mathrm{d}s + 1)}$$

for $t > t_0 + T_2$. So we have

$$\int_0^{T_1} f_i(s) N_i(t-s)\,\mathrm{d}s < \frac{\epsilon}{2}, \qquad t > t_0 + T_1 + T_2.$$

Hence, for any solution to (1), we have

$$\int_0^{+\infty} f_i(s) N_i(t-s)\,\mathrm{d}s < \epsilon, \qquad t > t_0 + T_1 + T_2.$$

Therefore

$$\lim_{t\to+\infty}\int_0^{+\infty} f_i(s)N_i(t-s)\,\mathrm{d}s = 0, \quad 1\le i\le n.$$

From the first equation of (1), we get

$$R(t) = e^{-D(t-t_0)}\left\{R_0 + \int_{t_0}^{t} e^{D(\tau-t_0)}\Big[DR_0 - \sum_{i=1}^{n}\frac{m_i}{y_{R_i}}U_i(R)N_i(\tau) + \sum_{i=1}^{n} b_i\gamma_i I_i\Big]\,\mathrm{d}\tau\right\}.$$

By means of the de l'Hospital rule, we can show that $\lim_{t\to+\infty} R(t) = R_0$. Noticing the arbitrariness of A_0 and A_i, $i=1,\dots,n$, E_0 is globally attractive. It follows from Theorem 1 that E_0 is globally stable. The proof is thus completed. ■

3.2. Positive Equilibria of System (1)

The positive equilibrium of (1) must satisfy

$$\begin{cases} D(R^0-R) - \displaystyle\sum_{i=1}^{n}\frac{m_i}{y_{R_i}}U_i(R)N_i + \sum_{i=1}^{n} b_i\gamma_i I_i = 0, \\ m_iU_i(R) - (D+\gamma_i) - \delta_i N_i = 0. \end{cases}$$

Noticing that N_i, $i=1,\dots,n$ are positive constants at the positive equilibrium and $\int_0^{+\infty} f_i(s)\,\mathrm{d}s = 1$, we have $I_i = N_i$. Therefore the above equations are equivalent to the following system:

$$\begin{cases} D(R^0-R) - \displaystyle\sum_{i=1}^{n}\frac{m_i}{y_{R_i}}U_i(R)N_i + \sum_{i=1}^{n} b_i\gamma_i N_i = 0, \\ m_iU_i(R) - (D+\gamma_i) - \delta_i N_i = 0. \end{cases} \tag{5}$$

From the second equation of (5), we have

$$N_i = \big[m_iU_i(R) - (D+\gamma_i)\big]/\delta_i, \quad 1\le i\le n. \tag{6}$$

Substituting (6) into the first equation of (5) yields

$$\psi(R) = 0, \tag{7}$$

where

$$\begin{aligned}\psi(R) = {} & \sum_{i=1}^{n}\frac{m_i^2}{\delta_i y_{R_i}}U_i^2(R) - \sum_{i=1}^{n}\Big[\frac{m_i(D+\gamma_i)}{\delta_i y_{R_i}} + \frac{m_ib_i\gamma_i}{\delta_i}\Big]U_i(R) \\ & + \sum_{i=1}^{n}\frac{b_i\gamma_i(D+\gamma_i)}{\delta_i} - D(R^0-R), \end{aligned} \tag{8}$$

$$\psi(0) = \sum_{i=1}^{n} \frac{b_i\gamma_i(D+\gamma_i)}{\delta_i} - DR^0, \qquad \lim_{R\to+\infty} \psi(R) = +\infty.$$

Therefore, if

$$\sum_{i=1}^{n} \frac{b_i\gamma_i(D+\gamma_i)}{\delta_i} < DR^0,$$

then there must be a constant $R^* > 0$ such that $\psi(R^*) = 0$. Furthermore, if

$$m_i U_i(R^*) > (D+\gamma_i)$$

for $1 \le i \le n$, then there exists a positive equilibrium $E^* \triangleq (R^*, N_1^*, \ldots, N_n^*)$, where

$$N_i^* = \left[m_i U_i(R^*) - (D+\gamma_i)\right]/\delta_i, \quad 1 \le i \le n.$$

Now we investigate the locally asymptotic stability of E^*.

Theorem 3. *The positive equilibrium E^*, if it exists, is locally asymptotically stable if*

$$\begin{cases} D > \dfrac{1}{2}\displaystyle\sum_{i=1}^{n} b_i\gamma_i - \sum_{i=1}^{n} \frac{m_i}{y_{R_i}} U_i'(R^*)N_i^*, \\ \dfrac{\delta_i U_i(R^*)}{y_{R_i} U_i'(R^*)} > \dfrac{1}{2} b_i\gamma_i, \quad 1 \le i \le n. \end{cases} \tag{9}$$

Proof. The linearized system of (1) at E^* is

$$\begin{cases} \dot{r} = -\left[D + \displaystyle\sum_{i=1}^{n} \frac{m_i}{y_{R_i}} U_i'(R^*)N_i^*\right] r - \sum_{i=1}^{n} \frac{m_i}{y_{R_i}} U_i(R^*) n_i \\ \qquad + \displaystyle\sum_{i=1}^{n} b_i\gamma_i \int_0^{+\infty} f_i(s) n_i(t-s)\,\mathrm{d}s, \\ \dot{n}_i = m_i U_i'(R^*)N_i^* r - \delta_i N_i^* n_i, \quad 1 \le i \le n. \end{cases} \tag{10}$$

Define the Lyapunov-Krasovskii functional (Fergola *et al.*, 1995)

$$V_1 = r^2 + \sum_{i=1}^{n} w_i n_i^2 + \sum_{i=1}^{n} b_i\gamma_i \int_0^{+\infty} f_i(s) \int_{t-s}^{t} n_i^2(u)\,\mathrm{d}u\,\mathrm{d}s,$$

where w_i's, are positive constants to be chosen. Calculating the derivative of V_1 along the solutions to (10) yields

$$\begin{aligned}
\dot{V}_1 &= 2r\Big\{-\Big[D+\sum_{i=1}^{n}\frac{m_i}{y_{R_i}}U_i'(R^*)N_i^*\Big]r-\sum_{i=1}^{n}\frac{m_i}{y_{R_i}}U_i(R^*)n_i \\
&\quad+\sum_{i=1}^{n}b_i\gamma_i\int_0^{+\infty}f_i(s)n_i(t-s)\,\mathrm{d}s\Big\}+2\sum_{i=1}^{n}w_in_i\Big[m_iU_i'(R^*)N_i^*r-\delta_iN_i^*n_i\Big] \\
&\quad+\sum_{i=1}^{n}b_i\gamma_in_i^2(t)-\sum_{i=1}^{n}b_i\gamma_i\int_0^{+\infty}f_i(s)n_i^2(t-s)\,\mathrm{d}s \\
&\le -2\Big[D+\sum_{i=1}^{n}\frac{m_i}{y_{R_i}}U_i'(R^*)N_i^*\Big]r^2-2\sum_{i=1}^{n}\frac{m_i}{y_{R_i}}U_i(R^*)rn_i \\
&\quad+\sum_{i=1}^{n}b_i\gamma_ir^2+\sum_{i=1}^{n}b_i\gamma_i\int_0^{+\infty}f_i(s)n_i^2(t-s)\,\mathrm{d}s-2\sum_{i=1}^{n}w_i\delta_iN_i^*n_i^2 \\
&\quad+2\sum_{i=1}^{n}w_im_iU_i'(R^*)N_i^*rn_i+\sum_{i=1}^{n}b_i\gamma_i{n_i}^2-\sum_{i=1}^{n}b_i\gamma_i\int_0^{+\infty}f_i(s)n_i^2(t-s)\,\mathrm{d}s.
\end{aligned}$$

Let

$$w_i=\frac{U_i(R^*)}{y_{R_i}N_i^*U_i'(R^*)},\qquad 1\le i\le n.$$

From Assumption 2 on $U_i(R)$, we have $w_i>0$, $i=1,\ldots,n$. Therefore

$$\begin{aligned}
\dot{V}_1 &\le -\Big[2\Big(D+\sum_{i=1}^{n}\frac{m_i}{y_{R_i}}U_i'(R^*)N_i^*\Big)-\sum_{i=1}^{n}b_i\gamma_i\Big]r^2 \\
&\quad-\sum_{i=1}^{n}\Big[2\frac{\delta_iU_i(R^*)}{y_{R_i}U_i'(R^*)}-b_i\gamma_i\Big]{n_i}^2. \qquad (11)
\end{aligned}$$

By (9), the right-hand side of (11) is negative definite, and therefore E^* is locally asymptotically stable (Kolmannovskii and Nosov, 1986). The proof is completed. ■

In order to obtain sufficient conditions for global stability of E^*, we specify the kernel functions as follows:

$$f_i(s)=\frac{\beta_i^{p_i+1}}{p_i!}s^{p_i}e^{-\beta_is},\quad 1\le i\le n, \qquad (12)$$

where p_i is a non-negative integer and β_i is a positive constant. For convenience of discussion, we introduce a lemma about the right upper derivative.

Lemma 1. (Ma, 1996) *Let $g(x)$ be a differentiable function on interval Γ. Then the right upper derivative of $|g(x)|$ exists and*

$$D^{+}|g(x)| = \operatorname{sgn}[g(x)]\frac{\mathrm{d}}{\mathrm{d}x}g(x)$$

on interval Γ.

Theorem 4. *The positive equilibrium E^*, if it exists, is globally stable provided the following inequalities hold:*

$$\begin{cases} D > \sum_{i=1}^{n} m_i U_i'(0), \\ \delta_i > \dfrac{m_i}{y_{R_i}} U_i(R^*) + b_i\gamma_i, \quad 1 \le i \le n. \end{cases} \tag{13}$$

Proof. Rewrite (1) in the following equivalent form:

$$\begin{cases} \dot{R} = -D(R-R^*) - \sum_{i=1}^{n} \dfrac{m_i}{y_{R_i}}\left[U_i(R) - U_i(R^*)\right]N_i \\ \qquad - \sum_{i=1}^{n} \dfrac{m_i}{y_{R_i}} U_i(R^*)(N_i - N_i^*) \\ \qquad + \sum_{i=1}^{n} b_i\gamma_i \dfrac{\beta_i^{p_i+1}}{p_i!}\displaystyle\int_0^{+\infty} s^{p_i}e^{-\beta_i s}\left(N_i(t-s) - N_i^*\right)\mathrm{d}s, \\ \dot{N}_i = N_i\left[m_i\left(U_i(R) - U_i(R^*)\right) - \delta_i(N_i - N_i^*)\right], \quad 1 \le i \le n. \end{cases} \tag{14}$$

Define the Lyapunov functional

$$\begin{aligned} V_2(x_t) &= |R(t) - R^*| + \sum_{i=1}^{n} |\ln N_i(t) - \ln N_i^*| \\ &\quad + \sum_{i=1}^{n} b_i\gamma_i \sum_{j=0}^{p_i} \frac{\beta_i^j}{j!}\left|\int_0^{+\infty} s^j e^{-\beta_i s}\left(N_i(t-s) - N_i^*\right)\mathrm{d}s\right|. \end{aligned} \tag{15}$$

From

$$\begin{aligned} &\int_0^{+\infty} s^j e^{-\beta_i s}(N_i(t-s) - N_i^*)\,\mathrm{d}s \\ &\qquad = \int_{-\infty}^{t} (t-s)^j e^{-\beta_i(t-s)}(N_i(s) - N_i^*)\,\mathrm{d}s, \quad 0 \le j \le p_i, \end{aligned}$$

we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty} s^j e^{-\beta_i s}\big(N_i(t-s)-N_i^*\big)\,\mathrm{d}s &= -\beta_i\int_0^{+\infty} s^j e^{-\beta_i s}\big(N_i(t-s)-N_i^*\big)\,\mathrm{d}s\\ &\quad + j\int_0^{+\infty} s^{j-1} e^{-\beta_i s}\big(N_i(t-s)-N_i^*\big)\,\mathrm{d}s,\\ &\qquad\qquad 1\le j\le p_i,\\ \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty} e^{-\beta_i s}\big(N_i(t-s)-N_i^*\big)\,\mathrm{d}s &= -\beta_i\int_0^{+\infty} e^{-\beta_i s}\big(N_i(t-s)-N_i^*\big)\,\mathrm{d}s\\ &\quad + \big(N_i(t)-N_i^*\big).\end{aligned}$$

Recalling Lemma 1 and the properties of $U_i(R)$, the right upper derivative of $V_2(x_t)$ along the solutions to (14) satisfies

$$\begin{aligned}D^+V_2(x_t) &\le -D|R-R^*| - \sum_{i=1}^n \frac{m_i}{y_{R_i}}\big|U_i(R)-U_i(R^*)\big|N_i\\ &\quad + \sum_{i=1}^n \frac{m_i}{y_{R_i}} U_i(R^*)|N_i-N_i^*|\\ &\quad + \sum_{i=1}^n b_i\gamma_i \frac{\beta_i^{p_i+1}}{p_i!}\left|\int_0^{+\infty} s^{p_i} e^{-\beta_i s}\big(N_i(t-s)-N_i^*\big)\,\mathrm{d}s\right|\\ &\quad + \sum_{i=1}^n m_i\big|U_i(R)-U_i(R^*)\big| - \sum_{i=1}^n \delta_i|N_i-N_i^*|\\ &\quad + \sum_{i=1}^n b_i\gamma_i \sum_{j=1}^{p_i}\frac{\beta_i^j}{j!}\left[-\beta_i\left|\int_0^{+\infty} s^j e^{-\beta_i s}\big(N_i(t-s)-N_i^*\big)\,\mathrm{d}s\right|\right.\\ &\quad \left. + j\left|\int_0^{+\infty} s^{j-1} e^{-\beta_i s}\big(N_i(t-s)-N_i^*\big)\,\mathrm{d}s\right|\right]\\ &\quad + \sum_{i=1}^n b_i\gamma_i\left[-\beta_i\left|\int_0^{+\infty} e^{-\beta_i s}\big(N_i(t-s)-N_i^*\big)\,\mathrm{d}s\right| + \big|N_i(t)-N_i^*\big|\right]\\ &\le -D|R-R^*| + \sum_{i=1}^n \frac{m_i}{y_{R_i}} U_i(R^*)|N_i-N_i^*|\\ &\quad + \sum_{i=1}^n b_i\gamma_i \frac{\beta_i^{p_i+1}}{p_i!}\left|\int_0^{+\infty} s^{p_i} e^{-\beta_i s}\big(N_i(t-s)-N_i^*\big)\,\mathrm{d}s\right|\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n} m_i U_i'(0)|R-R^*| - \sum_{i=1}^{n} \delta_i |N_i - N_i^*| \\
&-\sum_{i=1}^{n} b_i\gamma_i \sum_{j=1}^{p_i} \frac{\beta_i^{j+1}}{j!}\left| \int_0^{+\infty} s^j e^{-\beta_i s}\left(N_i(t-s)-N_i^*\right) \mathrm{d}s \right| \\
&+\sum_{i=1}^{n} b_i\gamma_i \sum_{j=1}^{p_i} \frac{\beta_i^{j}}{(j-1)!}\left| \int_0^{+\infty} s^{j-1} e^{-\beta_i s}\left(N_i(t-s)-N_i^*\right) \mathrm{d}s \right| \\
&-\sum_{i=1}^{n} b_i\gamma_i\beta_i \left| \int_0^{+\infty} e^{-\beta_i s}\left(N_i(t-s)-N_i^*\right) \mathrm{d}s \right| + \sum_{i=1}^{n} b_i\gamma_i |N_i(t)-N_i^*| \\
\leq\ & -\Big[D - \sum_{i=1}^{n} m_i U_i'(0)\Big]|R-R^*| \\
&-\sum_{i=1}^{n}\Big(\delta_i - \frac{m_i}{y_{R_i}} U_i(R^*) - b_i\gamma_i\Big)|N_i - N_i^*|. \qquad (16)
\end{aligned}
$$

In view of (13), $D^+V_2(x_t)$ is negative definite. Therefore the positive equilibrium E^* is globally stable (Kuang, 1993). The proof is thus completed. ■

3.3. Positive Equilibria for a Single Population System

In this subsection, we restrict our attention to system (1) with a single population, i.e. $n = 1$. We follow the same procedure as the one in (Beretta *et al.*, 1990) to study the locally asymptotic stability of positive equilibria of the following system:

$$
\begin{cases}
\dot{R} = D(R^0 - R) - \dfrac{m}{y_R} U(R)N + b\gamma \displaystyle\int_0^{+\infty} f(s)N(t-s)\,\mathrm{d}s, \\
\dot{N} = N\big[mU(R) - (D+\gamma) - \delta N\big].
\end{cases} \qquad (17)
$$

The linearized system of (17) at the positive equilibrium $E^* \triangleq (R^*, N^*)$ is

$$
\begin{cases}
\dot{r} = -\Big[D + \dfrac{m}{y_R} U'(R^*)N^*\Big] r - \dfrac{m}{y_R} U(R^*) n_1 + b\gamma \displaystyle\int_0^{+\infty} f(s) n_1(t-s)\,\mathrm{d}s, \\
\dot{n}_1 = mU'(R^*)N^* r - \delta N^* n_1.
\end{cases} \qquad (18)
$$

After almost the same analysis of the eigenvalues at E^* as in Theorem 7 of (Beretta *et al.*, 1990), we get the following result.

Theorem 5. *Let*

$$\begin{cases} \delta\Big[D+2\dfrac{m}{y_R}U'(R^*)N^*\Big] > m\Big[b\gamma-\dfrac{D+\gamma}{y_R}\Big]U'(R^*), \\[2ex] D^2+\Big[\delta-\dfrac{m}{y_R}U'(R^*)\Big]^2 N^{*2} > 2\dfrac{m\gamma}{y_R}U'(R^*)N^*. \end{cases} \tag{19}$$

Then the positive equilibrium E^*, if it exists, is locally asymptotically stable.

Remark 1. If we set $\delta = 0$ in Theorem 5, system (17) reduces to that without logistic term considered in (Beretta *et al.*, 1990) and Theorem 5 reduces to Theorem 7 in (Beretta *et al.*, 1990), so the condition (19) is more general than that of Theorem 7 in (Beretta *et al.*, 1990).

We can also apply Theorem 3 to get the condition of the local stability to system (17). But since system (17) is a two-dimensional model, we use the eigenvalue method to investigate the local stability of (17) at the positive equilibrium. Theorem 5 is true for any form of kernel functions. In order to simplify condition (19), we assume further the following special kernel function in (17):

$$f(s) = \beta e^{-\beta s}, \tag{20}$$

where β is a positive constant.

Theorem 6. *Let*

$$\delta\Big[D+2\frac{m}{y_R}U'(R^*)N^*\Big] > m\Big[b\gamma-\frac{D+\gamma}{y_R}\Big]U'(R^*). \tag{21}$$

Then the positive equilibrium E^*, if it exists, is locally asymptotically stable.

Proof. The characteristic equation of (18) around E^* is

$$\lambda^3 + l_1\lambda^2 + l_2\lambda + l_3 = 0,$$

where

$$\begin{aligned} l_1 &= \beta + D + \delta N^* + \frac{m}{y_R}U'(R^*)N^*, \\ l_2 &= \beta\big(D+\delta N^* + \frac{m}{y_R}U'(R^*)N^*\big) \\ &\quad + \Big[\delta D + \frac{m(D+\gamma)}{y_R}U'(R^*)\Big]N^* + 2\delta\frac{m}{y_R}U'(R^*)N^{*2}, \\ l_3 &= \beta\Big\{\Big[\delta D + \frac{m(D+\gamma)}{y_R}U'(R^*)\Big]N^* + 2\delta\frac{m}{y_R}U'(R^*)N^{*2}\Big\} \\ &\quad - \beta m b\gamma U'(R^*)N^*. \end{aligned}$$

From (21), we have $l_3 > 0$. The Routh-Hurwitz criterion states that $\mathrm{Re}\,(\lambda) < 0$ iff

$$l_1 l_2 - l_3 > 0.$$

In our case, the inequality $l_1 l_2 - l_3 > 0$ is satisfied. The proof of Theorem 6 is completed. ■

Remark 2. If we set $\delta = 0$, Theorem 6 reduces to Theorem 8 in (Beretta *et al.*, 1990). So condition (21) includes the one of Theorem 8 in (Beretta *et al.*, 1990) as its special case.

4. Survival of Populations

4.1. Uniform Persistence of Populations

The objective of this subsection is to derive sufficient conditions for uniform persistence of populations of system (1). We refer to the definition of uniform persistence of populations given in (Freedman and Xu, 1993). Because of Theorem 2, here we suppose that

$$m_i > D + \gamma_i, \quad i = 1, \ldots, n.$$

Let ϵ be a any given small positive constant and

$$\bar{N}_i = \frac{m_i - (D + \gamma_i)}{\delta_i}, \quad 1 \le i \le n. \tag{22}$$

From (1) that it follows

$$\dot{N}_i(t) < -\delta_i \epsilon N_i \quad \text{if} \quad N_i(t) > \bar{N}_i + \epsilon.$$

Hence for each solution to (1), there exists a time $t_1 > t_0$ such that

$$N_i(t) \le \bar{N}_i + \epsilon, \quad 1 \le i \le n \tag{23}$$

holds for all $t \ge t_1$. We write

$$R_b = R^0 - \frac{1}{D} \sum_{i=1}^{n} \frac{m_i}{y_{R_i}} \bar{N}_i. \tag{24}$$

Theorem 7. *The populations of (1) are uniformly persistent if* $R_b > 0$ *and*

$$m_i U_i(R_b) > D + \gamma_i, \quad 1 \le i \le n. \tag{25}$$

Proof. By (23) and the first equation of (1), we have

$$\dot{R} \ge D(R^0 - R) - \sum_{i=1}^{n} \frac{m_i}{y_{R_i}} (\bar{N}_i + \epsilon), \quad t \ge t_1.$$

By the comparison principle, we have

$$R(t) \geq R_0 e^{-D(t-t_0)} + (1 - e^{-D(t-t_0)})\Big[R^0 - \frac{1}{D}\sum_{i=1}^{n} \frac{m_i}{y_{R_i}}(\bar{N}_i + \epsilon)\Big], \quad t \geq t_1.$$

Taking the lower limit yields

$$\liminf_{t\to+\infty} R(t) \geq R^0 - \frac{1}{D}\sum_{i=1}^{n} \frac{m_i}{y_{R_i}}(\bar{N}_i + \epsilon).$$

This being for each $\epsilon > 0$, we have

$$\liminf_{t\to+\infty} R(t) \geq R_b. \tag{26}$$

Since $U_i(R)$ is a continuous function, we can choose ϵ and $\eta > 0$ small enough such that

$$m_i U_i(R_b - \epsilon) > D + \gamma_i + \eta, \quad 1 \leq i \leq n. \tag{27}$$

Since $U_i(R)$ is also an increasing function, by (26) and (27) for each solution to (1) there exists a time $t_2 \geq t_1$ such that

$$m_i U_i\big(R(t)\big) > D + \gamma_i + \eta, \quad 1 \leq i \leq n \tag{28}$$

for all $t \geq t_2$.

From the i-th equation of (1), we get

$$\begin{aligned} N_i(t) \;=\; & N_i(t_2)\Big\{\exp\int_{t_2}^{t}\big[(D+\gamma_i) - m_i U_i(R)\big]\,\mathrm{d}\tau \\ & + \delta_i N_i(t_2)\int_{t_2}^{t}\exp\int_{\tau}^{t}\big((D+\gamma_i) - m_i U_i(R)\big)\,\mathrm{d}s\,\mathrm{d}\tau\Big\}^{-1}. \end{aligned}$$

Application of (28) yields

$$N_i(t) \geq \frac{N_i(t_2)}{\exp[-\eta(t-t_2)] + \frac{\delta_i}{\eta}N_i(t_2)(1-\exp[-\eta(t-t_2)]}$$

for $t \geq t_2$. Taking the lower limits on both the sides of this inequality, we get

$$\liminf_{t\to+\infty} N_i(t) \geq \frac{\eta}{\delta_i} > 0, \quad 1 \leq i \leq n. \tag{29}$$

Therefore the populations of (1) are uniformly persistent. The proof is completed. ■

4.2. Control of Populations

This subsection is devoted to control of populations. A procedure is presented to adjust some parameters of system (1) such that the size of populations stays eventually in a desired set.

Now we introduce the following notation:

$$q_i(\epsilon) = m_i\big(1 - U_i(R_b - \epsilon)\big), \quad \alpha_i(\epsilon) = \frac{q_i(\epsilon)}{\delta_i}.$$

We set

$$q_i \triangleq q_i(0) = m_i\big(1 - U_i(R_b)\big), \quad \alpha_i \triangleq \alpha_i(0) = \frac{q_i(0)}{\delta_i}.$$

Theorem 8. *Under the condition of Theorem 7, for a sufficiently small $\epsilon > 0$, the solutions to (1) will finally enter the set*

$$\Omega = \big\{N_1, \dots, N_n |\ N_i \in \mathbb{R}_{+0},\ |N_i - \bar{N}_i| < \alpha_i + \epsilon,\ \ 1 \le i \le n\big\},$$

where $\bar{N}_i$ is defined in (22).

Proof. Rewrite the equations for N_i of (1) in the following equivalent form:

$$\dot{N}_i = N_i\Big[\big(m_i - (D + \gamma_i) - \delta_i N_i\big) - m_i\big(1 - U_i(R)\big)\Big]. \tag{30}$$

Define the Lyapunov functions

$$W_i = \frac{1}{2}(N_i - \bar{N}_i)^2, \quad 1 \le i \le n.$$

The derivative of W_i along the solution to (30) satisfies

$$\begin{aligned} \dot{W}_i &= N_i\Big[-\delta_i(N_i - \bar{N}_i)^2 - m_i\big(1 - U_i(R)\big)(N_i - \bar{N}_i)\Big] \\ &\le N_i\Big[-\delta_i|N_i - \bar{N}_i|^2 + m_i\big(1 - U_i(R)\big)|N_i - \bar{N}_i|\Big]. \end{aligned} \tag{31}$$

Suppose that $\bar{\epsilon} > 0$ is a sufficiently small number. By (26) and (29), for each solution to (1) there exists a time t_3 such that

$$R(t) \ge R_b - \bar{\epsilon}, \quad N_i(t) \ge \frac{\eta}{2\delta_i}, \quad t \ge t_3. \tag{32}$$

By (31) and (32), we have

$$\begin{aligned} \dot{W}_i &\le N_i\Big[-\delta_i|N_i - \bar{N}_i|^2 + m_i\big(1 - U_i(R_b - \bar{\epsilon})\big)|N_i - \bar{N}_i|\Big] \\ &= \delta_i N_i |N_i - \bar{N}_i|^2\Big[-1 + \frac{q_i(\bar{\epsilon})/\delta_i}{|N_i - \bar{N}_i|}\Big], \quad \text{if } |N_i - \bar{N}_i| > 0. \end{aligned} \tag{33}$$

Choose $\bar{\epsilon}$ small enough so that

$$\alpha_i + \epsilon > \alpha_i(\bar{\epsilon}). \tag{34}$$

If $|N_i - \bar{N}_i| > \alpha_i + \epsilon$, by (32)–(34) we get

$$\begin{aligned}\dot{W}_i &\leq \delta_i N_i |N_i - \bar{N}_i|^2 \Big[-1 + \frac{\alpha_i(\bar{\epsilon})}{\alpha_i + \epsilon} \Big] \\ &\leq -\frac{1}{2}\eta |N_i - \bar{N}_i|^2 \Big[1 - \frac{\alpha_i(\bar{\epsilon})}{\alpha_i + \epsilon} \Big], \quad t \geq t_3.\end{aligned}$$

Therefore each solution to (1) satisfies

$$|N_i - \bar{N}_i| \leq \alpha_i + \epsilon, \quad 1 \leq i \leq n$$

for sufficiently large t. The proof is completed. ■

Remark 3. Combining (23), Theorems 7 and 8, we have shown that the populations of system (1) are uniformly persistent and will finally enter the set

$$\Sigma = \big\{(N_1, \ldots, N_n) |\, N_i \in \mathbb{R}_{+0}\,,\ \bar{N}_i - \alpha_i - \epsilon < N_i < \bar{N}_i + \epsilon, \ \ 1 \leq i \leq n\big\}$$

for a sufficient large time t. Therefore we have presented a procedure by which one can control the parameters of (1) so that the size of populations eventually stays in a desired set.

5. Conclusion

In this paper, we have studied a chemostat model with n populations competing for one nutrient with time delay. The work of this paper consists of two parts. The first one is mainly about the stability of the positive equilibrium. Lyapunov functionals are constructed to obtain both local and global stabilities. The second one is about the uniform persistence and control problem of populations. A procedure is presented by which one can adjust the parameters of the model so that the size of populations enters and stays eventually in a desired set.

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