THE ENERGY METHOD FOR ELASTIC PROBLEMS WITH NON-HOMOGENEOUS BOUNDARY CONDITIONS

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In this paper we propose the weighted energy method as a way to study estimates of solutions of boundary-value problems with non-homogeneous boundary conditions in elasticity. First, we use this method to study spatial decay estimates in two-dimensional elasticity when we consider non-homogeneous boundary conditions on the boundary. Some comments in the case of harmonic vibrations are considered as well. We also extend the arguments to a class of three-dimensional problems in a cylinder. A section is devoted to the study of an ill-posed problem. Some remarks are presented in the last section of the paper.

Keywords: weighted energy method, decay estimates, Navier equations, non-homogeneous boundary conditions

1. Introduction

The energy method is an appropriate tool in the study of the behaviour of solutions of partial differential equations. There is an important amount of literature on this method with references to the case of problems with homogeneous boundary conditions. This is not the case when the boundary conditions are not homogeneous. If we restrict our attention to the study of spatial estimates of solutions of elliptic partial differential equations, we only know a few contributions (Ames and Payne, 2000; Horgan and Payne, 1992; Knops and Payne, 1998; Quintanilla, 1997a; 1997b; 1998). If we take a look at the history of these studies, we can recall the paper of Lin and Payne on two ill-posed parabolic problems (Lin and Payne, 1993), see also (Franchi and Straughan, 1994). In that paper, an idea was outlined that inspired the contribution in the reference (Quintanilla, 1997a). The main thought was to consider estimates on smaller domains in several directions. When the boundary conditions were known, an alternative method (Horgan and Payne, 1992) was proposed in their studies concerning the stability with respect to the geometry of the cross-section.

Some contributions to the Laplace equation and the biharmonic equation were obtained by Ames and Payne in the recent work (Ames and Payne, 2000). Some contributions to the elasticity system were obtained in the references (Knops and Payne, 1998; Quintanilla, 1997b; 1998). In (Quintanilla, 2000), the author proposed an approach to this kind of questions also based on the energy methods in order to deal with non-homogeneous boundary conditions. The main idea was to introduce a weight function in the energy function. This kind of procedure resembles the one used by Straughan (1982), and Galdi and Rionero (1985) in the study of unbounded domains, when we allow for unbounded behaviour at the infinity. It is worth noticing that our weight functions concern only bounded directions. Here, we try to extend these methods to the system of elasticity. In this situation things seem more difficult than for the Laplace equation or the heat equation. We have to restrict our attention to a particular family of isotropic and homogeneous materials. It is known that considerations of positive definite energy restrict the range of Poisson's ratio to $-1 < \nu < 1/2$, but our method only applies when $\nu < 1/4$.

As the results that we present here are related to the Saint-Venant principle, it is worth recalling the references (Horgan, 1989; 1996; Horgan and Knowles, 1983), where the history and the state of the art of this study are well described.

It is worth noticing that the results hold for solutions having *a priori* suitable behaviour at the spatial infinity (e.g. going to zero or having derivative going to zero), and to eliminate this restrictions seems a (fundamental) open problem.

In Section 2 we recall some preliminaries related to inequalities of Poincaré's type. The evolution of the solutions of a non-homogeneous ordinary differential equation is also recalled. Section 3 is devoted to the study of the solutions of the Navier equations in the case of a strip, when we assume non-homogeneous boundary conditions in a great part of the boundary. In Section 4 we consider a similar question for the solutions of the amplitude terms of the steady-state vibrations. The extension to the case of a cylinder is developed in Section 5. The last section of the paper considers the case where we have no information on a part of the boundary.

2. Preliminaries

Summation and differentiation conventions will be used throughout this paper. We recall that summation over repeated indices is implied and that the suffix '$,i$' denotes $\partial/\partial x_i$.

We recall that the function $\sin \pi x$ satisfies the clamped eigenvalue problem

$$\Delta\phi + \lambda\phi = 0, \quad (0,1), \tag{1}$$

$$\phi = 0, \quad \{0,1\}. \tag{2}$$

We shall denote by $\phi_{[0,1]}$ the eigenfunction that satisfies $\sup_{[0,1]} \phi = 1$. We know that if $x \neq 0, 1$, then $\phi_{[0,1]}(x) > 0$. Thus, for all $0 < \epsilon < 1$ we may define the subdomain

$$[0,1]^{(\epsilon)} = \{x \in [0,1], \phi_{[0,1]}(x) \geq \epsilon\}. \tag{3}$$

In the next sections we will obtain estimates of the form

$$E(z) \leq -A^{-1}\frac{\mathrm{d}E}{\mathrm{d}z} + R(z), \tag{4}$$

where $R(z)$ is a given function. If we want to study the asymptotic behaviour of the function $E(z)$, we may use

$$\exp(-Az)\frac{\mathrm{d}}{\mathrm{d}z}\big(\exp(Az)E(z)\big) \leq AR(z). \tag{5}$$

After a quadrature, it follows that

$$E(z) \leq \Big(E(0) + A\int_0^z \exp(A\xi)R(\xi)\,\mathrm{d}\xi\Big)\exp(-Az), \quad z \geq 0. \tag{6}$$

Equation (4) will appear (in several points) in the case where there exists a function $r(\tau)$ such that

$$R(z) = \int_z^\infty r(\tau)\,\mathrm{d}\tau. \tag{7}$$

After integration by parts, we obtain the equality

$$A\int_0^z \exp(A\xi)R(\xi)\,\mathrm{d}\xi\exp(-Az) = R(z) - \Big(R(0) - \int_0^z \exp(A\xi)r(\xi)\,\mathrm{d}\xi\Big)\exp(-Az). \tag{8}$$

From (6) and (8) we see that

$$E(z) \leq E(0)\exp(-Az) + R(z) - \Big(R(0) - \int_0^z \exp(A\xi)r(\xi)\,\mathrm{d}\xi\Big)\exp(-Az), \quad z \geq 0. \tag{9}$$

In this paper we will use several inequalities of Poincaré's type. Let us recall that there exists a positive constant λ_1 such that the estimate

$$\int_0^1 xu^2\,\mathrm{d}x \leq \lambda_1^{-1}\int_0^1 x(u')^2\,\mathrm{d}x \tag{10}$$

is satisfied for any function u that vanishes when $x = 1$, and u and its derivative are bounded at $x = 0$. We may recall that λ_1 is the first eigenvalue of the Sturm-Liouville singular problem

$$(xu')' + \lambda xu = 0, \quad (0,1),$$

$$u(1) = 0, u(0) \text{ bounded and } xu'(x) \to 0 \text{ as } x \to 0.$$

This first eigenvalue agrees with the square of the first zero of the Bessel function $J_0(x)$ (Weinberger, 1995, pp. 176–180). Approximations to this constant are well known in the literature. We have $\sqrt{\lambda_1} = 2.4048\ldots$

We also need another differential inequality of this kind. We know that there exists a positive constant such that the estimate

$$\int_0^1 u^2\,\mathrm{d}x \leq \mu_1^{-1}\int_0^1 x(u')^2\,\mathrm{d}x \tag{11}$$

is satisfied for every function u that vanishes when $x = 1$, and u and its derivative are bounded at $x = 0$. It is well known that this constant corresponds to the first eigenvalue of the singular Sturm-Liouville eigenvalue problem

$$\begin{gathered}(xu')' + \mu u = 0, \quad (0,1),\\ u(0), u'(0) \text{ bounded, and } u(1) = 0.\end{gathered} \tag{12}$$

3. Problem in a Strip

We consider a problem modelled by the system of the homogeneous and isotropic linear elasticity (Navier's system):

$$u_{i,jj} + \alpha u_{j,ji} = 0, \tag{13}$$

in the semi-infinite strip $(0,\infty)\times(0,1)$. Here u_i are the components of the displacement with respect to a given Cartesian coordinates system and α is a positive constant. We assume that $0 \leq \alpha < 2$, but it is possible to extend the arguments to some cases when $\alpha < 0$. We recall the relation

$$\alpha = (1-2\nu)^{-1}. \tag{14}$$

Here ν is the Poisson ratio. Thus the range of the applicability of our approach requires the Poisson ratio to satisfy $\nu < 1/4$.

We assume the boundary conditions

$$u_i(x_1,0) = f_i(x_1), \quad u_i(x_1,1) = g_i(x_1), \tag{15}$$

and

$$u_i(0,x_2) = h_i(x_2). \tag{16}$$

The functions f_i, g_i, h_i are data. We assume that

$$f_i(0) = h_i(0), \quad g_i(0) = h_i(1).$$

For later use, we recall that if ϕ is a function that depends only on the variable x_2 and (u_i) is an arbitrary solution of the two-dimensional version of the system (13), the relation

$$\begin{aligned}&\Big[\phi(2u_{i,j} + \alpha\delta_{ij}u_{k,k} + \alpha u_{j,i})u_i\Big]_{,j}\\ &\quad - \alpha\Big[(\phi u_1 u_2)_{,2} - \phi u_{1,2}u_2 - \phi u_1 u_{2,2}\Big]_{,1}\\ &= \phi F(u_{i,j}, u_{i,j}) - \phi'' u_i u_i - \alpha\phi'' u_2^2\\ &\quad + \Big[\phi'(u_i u_i + \alpha u_2^2),\Big]_{,2}\end{aligned} \tag{17}$$

is satisfied.

It is worth noticing that whenever $0 \le \alpha < 2$, the function

$$F(u_{i,j}, u_{i,j}) = 2u_{i,j}u_{i,j} + \alpha u_{i,i}u_{j,j} + \alpha u_{i,j}u_{j,i} \tag{18}$$

satisfies

$$F(u_{i,j}, u_{i,j}) \ge (2-\alpha)u_{i,j}u_{i,j} + \alpha u_{i,i}u_{j,j}, \tag{19}$$

which is positive. Thus it can be used to define a measure on the solutions.

In this section we assume that the energy

$$\begin{aligned}E(0) = &\int_0^\infty\int_0^1 \phi_{[0,1]}\big[F(u_{i,j}, u_{i,j})\\ &+ \pi^2(u_i u_i + \alpha u_2^2)\big]\,\mathrm{d}a\end{aligned} \tag{20}$$

is bounded and that the asymptotic condition

$$\begin{aligned}\lim_{x_1\to\infty}\int_0^1 &\phi_{[0,1]}(2u_{i,1}u_i + \alpha u_{k,k}u_1 + \alpha u_{1,i}u_i\\ &+\alpha u_{1,2}u_2 + \alpha u_1 u_{2,2})\,\mathrm{d}x_2 = 0\end{aligned} \tag{21}$$

is satisfied.

If we define

$$\begin{aligned}E(z) = &\int_z^\infty\int_0^1 \phi_{[0,1]}\big[F(u_{i,j}, u_{i,j})\\ &+ \pi^2(u_i u_i + \alpha u_2^2)\big]\,\mathrm{d}a,\end{aligned} \tag{22}$$

the use of the divergence theorem allows us to obtain the relation

$$\begin{aligned}E(z) = &-\int_0^1 \phi_{[0,1]}(2u_{i,1}u_i + \alpha u_{k,k}u_1 + \alpha u_{1,i}u_i\\ &+ \alpha u_{1,2}u_2 + \alpha u_1 u_{2,2})\,\mathrm{d}x_2\\ &+ \pi\int_z^\infty\big((f_1^2+g_1^2)+(1+\alpha)(f_2^2+g_2^2)\big)\mathrm{d}\xi.\end{aligned} \tag{23}$$

We also have

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}z} = &-\int_0^1 \phi_{[0,1]}\big[F(u_{i,j}, u_{i,j})\\ &+ \pi^2(u_i u_i + \alpha u_2^2)\big]\,\mathrm{d}x_2.\end{aligned} \tag{24}$$

In the next step we estimate $E(z)$ in terms of its spatial derivative and the boundary conditions. It will be useful to consider the integrals

$$I_1 = -2\int_0^1 \phi_{[0,1]}u_{i,1}u_i\,\mathrm{d}x_2, \tag{25}$$

$$I_2 = -\alpha\int_0^1 \phi_{[0,1]}\alpha u_{k,k}u_1\,\mathrm{d}x_2, \tag{26}$$

$$I_3 = -\alpha\int_0^1 \phi_{[0,1]}u_{1,i}u_i\,\mathrm{d}x_2, \tag{27}$$

$$I_4 = -\alpha\int_0^1 \phi_{[0,1]}u_{1,2}u_2\,\mathrm{d}x_2, \tag{28}$$

and

$$I_5 = -\alpha\int_0^1 \phi_{[0,1]}u_1 u_{2,2}\,\mathrm{d}x_2. \tag{29}$$

The Hölder inequality and the arithmetic-geometric mean inequality imply

$$\begin{aligned}I_1 &\le 2\Big(\int_0^1\phi_{[0,1]}u_{i,1}u_{i,1}\,\mathrm{d}x_2\Big)^{1/2}\Big(\int_0^1\phi_{[0,1]}u_i u_i\,\mathrm{d}x_2\Big)^{1/2}\\ &\le \Big(\epsilon_1\int_0^1 \phi_{[0,1]}u_{i,1}u_{i,1}\,\mathrm{d}x_2\\ &\qquad + \frac{1}{\epsilon_1}\int_0^1 \phi_{[0,1]}u_i u_i\,\mathrm{d}x_2\Big),\end{aligned} \tag{30}$$

$$I_2 \leq \alpha\Big(\int_0^1 \phi_{[0,1]} u_{i,i}u_{j,j}\,\mathrm{d}x_2\Big)^{1/2}\Big(\int_0^1 \phi_{[0,1]} u_1^2\,\mathrm{d}x_2\Big)^{1/2}$$
$$\leq \alpha\Big(\frac{\epsilon_2}{2}\int_0^1 \phi_{[0,1]} u_{i,i}u_{j,j}\,\mathrm{d}x_2 + \frac{1}{2\epsilon_2}\int_0^1 \phi_{[0,1]} u_1^2\,\mathrm{d}x_2\Big), \quad (31)$$

$$I_3 \leq \alpha\Big(\int_0^1 \phi_{[0,1]} u_{1,i}u_{1,i}\,\mathrm{d}x_2\Big)^{1/2}\Big(\int_0^1 \phi_{[0,1]} u_i u_i\,\mathrm{d}x_2\Big)^{1/2}$$
$$\leq \alpha\Big(\frac{\epsilon_3}{2}\int_0^1 \phi_{[0,1]} u_{1,i}u_{1,i}\,\mathrm{d}x_2 + \frac{1}{2\epsilon_3}\int_0^1 \phi_{[0,1]} u_i u_i\,\mathrm{d}x_2\Big), \quad (32)$$

$$I_4 \leq \alpha\Big(\int_0^1 \phi_{[0,1]} u_{1,2}u_{1,2}\,\mathrm{d}x_2\Big)^{1/2}\Big(\int_0^1 \phi_{[0,1]} u_2^2\,\mathrm{d}x_2\Big)^{1/2}$$
$$\leq \alpha\Big(\frac{\epsilon_4}{2}\int_0^1 \phi_{[0,1]} u_{1,2}u_{1,2}\,\mathrm{d}x_2 + \frac{1}{2\epsilon_4}\int_0^1 \phi_{[0,1]} u_2^2\,\mathrm{d}x_2\Big), \quad (33)$$

and

$$I_5 \leq \alpha\Big(\int_0^1 \phi_{[0,1]} u_{2,2}^2\,\mathrm{d}x_2\Big)^{1/2}\Big(\int_0^1 \phi_{[0,1]} u_1^2\,\mathrm{d}x_2\Big)^{1/2}$$
$$\leq \alpha\Big(\frac{\epsilon_5}{2}\int_0^1 \phi_{[0,1]} u_{2,2}^2\,\mathrm{d}x_2 + \frac{1}{2\epsilon_5}\int_0^1 \phi_{[0,1]} u_1^2\,\mathrm{d}x_2\Big). \quad (34)$$

Here $\epsilon_i,\ i = 1,\dots,5$ are arbitrary positive constants. One would like to optimize these quantities in order to make a comparison with (22). It does not seem an easy task, because it involves solving nonlinear equations (polynomials). Thus we obtain an estimate by taking some values for the parameters ϵ_i.

For instance, if we take $\epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = 1$ and $\epsilon_1 = \alpha\epsilon_4/2$, it follows that

$$\sum_{1\leq i\leq 5} I_i \leq \alpha\int_0^1 \phi_{[0,1]} u_{i,j}u_{i,j}\,\mathrm{d}x_2 + \frac{\alpha}{2}\int_0^1 \phi_{[0,1]} u_{i,i}u_{j,j}\,\mathrm{d}x_2 + \left(\frac{3\alpha}{2}+\frac{2}{\alpha}\right)\int_0^1 \phi_{[0,1]} u_i u_i\,\mathrm{d}x_2. \quad (35)$$

As $\alpha \geq 0$, we have

$$\sum_{1\leq i\leq 5} I_i \leq -M\frac{\partial E}{\partial z}, \quad (36)$$

where

$$M = \max\left(\alpha(2-\alpha)^{-1}, \frac{1}{2}, \pi^{-1}\Big(\frac{3\alpha}{2}+\frac{2}{\alpha}\Big)\right). \quad (37)$$

From (23) and (36), we obtain

$$E(z) \leq -M\frac{\partial E}{\partial z} + S(z), \quad (38)$$

where

$$S(z) = \pi\int_z^\infty \Big(f_1^2 + g_1^2 + (1+\alpha)(f_2^2+g_2^2)\Big)\,\mathrm{d}\xi. \quad (39)$$

As (38) is an estimate of the type (4), we deduce the estimate

$$E(z) \leq E(0)\exp(-M^{-1}z) + S(z) - \Big(S(0) - \int_0^z \exp(M^{-1}\xi)s(\xi)\,\mathrm{d}\xi\Big)\exp(-M^{-1}z), \quad (40)$$

where

$$s(\xi) = \pi\Big(f_1^2(\xi)g_1^2(\xi) + (1+\alpha)\big(f_2^2(\xi)+g_2^2(\xi)\big)\Big). \quad (41)$$

Thus we have proved the following result:

Theorem 1. *Let* (u_i) *be a solution to the problem defined by the system (13), boundary conditions (15) and asymptotic conditions (19), (20). Then the energy function defined in (22) satisfies the estimate (40).*

If we assume that there exist two positive constants K, and ω such that $|s(\xi)| \leq K\exp(-\omega\xi)$, we conclude a decay of exponential type for the function $E(z)$.

If we write

$$E(\epsilon, z) = \int_z^\infty\int_{\{\phi_{[0,1]}\geq\epsilon\}} \Big[F(u_{i,j},u_{i,j}) + \pi^2(u_iu_i + \alpha\pi^2u_2^2)\Big]\,\mathrm{d}a, \quad (42)$$

we see that

$$E(\epsilon, z) = \epsilon^{-1}E(z). \quad (43)$$

Estimates (40) and (43) give an estimate for the decay uniform in the domains of the form $[x_1,\infty)\times\{\phi_{[0,1]}\geq\epsilon\}$.

As the estimate used in (19) could be improved, we may conclude that the estimate (40) could be also improved. We do not consider this analysis to save cumbersome calculations.

If we assume that $g_i = 0$, we could consider an alternative approach. In this case we can use the weight function $\phi = x_2$. We have $\phi'' = 0$, and relation (17) reduces. Assuming suitable asymptotic conditions, we define

$$E^*(z) = \int_z^\infty \int_0^1 x_2 F(u_{i,j}, u_{i,j})\,\mathrm{d}a. \tag{44}$$

We can adapt the arguments proposed previously in this situation, but we need to use some kind of the Poincaré inequality. In this sense, we can recall the estimate (10). We could obtain an estimate of the type (40) after obtaining the fit constants in this case.

It is also worth considering another measure on the solutions. We write

$$W(z) = \int_z^\infty \int_0^1 u_i u_i\,\mathrm{d}a. \tag{45}$$

Using estimates (11) and (19), we obtain

$$W(z) \le (2-\alpha)^{-1}\mu_1^{-1} E^*(z). \tag{46}$$

The results obtained by Horgan and Payne (1992) apply to the problem considered here. Nevertheless, in our approach the measure considered is different from that used in (Horgan and Payne, 1992).

4. Steady-State Vibrations in a Strip

Now, we look at a problem of the steady-state vibrations of the form

$$u_i(t,x,y) = v_i(x,y)\exp(i\varsigma t), \tag{47}$$

where ς is a strictly positive constant. The amplitude term (v_i) satisfies the system

$$v_{i,jj} + \alpha v_{j,ji} + C^2 v_i = 0. \tag{48}$$

Here C can be obtained in terms of the Lamé constant μ, the mass density ρ and ς. In fact, we have

$$C^2 = \varsigma^2 \rho \mu^{-1}.$$

In this section, we consider a problem determined by system (48) and boundary conditions (15) and (16).

From now on, we assume that $2C^2 < \pi^2$. Let us consider the weight function $\phi_{[0,1]}$, analysed in Section 3.

In this section we assume that the energy

$$E_C(0) = \int_0^\infty \int_0^1 \phi_{[0,1]}\Big[F(v_{i,j}, v_{i,j}) + (\pi^2 - 2C^2)v_i v_i + \alpha\pi^2 v_2^2\Big]\,\mathrm{d}a \tag{49}$$

is bounded and that the asymptotic condition

$$\lim_{x_1\to\infty} \int_0^1 \phi_{[0,1]}(2v_{i,1}v_i + \alpha v_{k,k}v_1 + \alpha v_{1,i}v_i + \alpha v_{1,2}v_2 + \alpha v_1 v_{2,2})\,\mathrm{d}x_2 = 0 \tag{50}$$

is satisfied. If we define the function

$$E_C(z) = \int_z^\infty \int_0^1 \phi_{[0,1]}\Big[F(v_{i,j}, v_{i,j}) + (\pi^2 - 2C^2)v_i v_i + \alpha\pi^2 v_2^2\Big]\,\mathrm{d}a, \tag{51}$$

we have

$$E_C(z) = -\int_0^1 \phi_{[0,1]}(2v_{i,1}v_i + \alpha v_{k,k}v_1 + \alpha v_{1,i}v_i + \alpha v_{1,2}v_2 + \alpha v_1 v_{2,2})\,\mathrm{d}x_2 + \pi\int_z^\infty \big((f_i f_i + g_i g_i) + \alpha(f_2^2 + g_2^2)\big)\,\mathrm{d}\xi \tag{52}$$

and

$$\frac{\mathrm{d}E_C}{\mathrm{d}z} = -\int_0^1 \phi_{[0,1]}\Big[F(v_{i,j}, v_{i,j}) + (\pi^2 - 2C^2)v_i v_i + \alpha\pi^2 v_2^2\Big]\,\mathrm{d}x_2. \tag{53}$$

We can reproduce the arguments developed in the previous section. Doing so, we obtain

$$E_C(z) \le E_C(0)\exp(-M_C^{-1}z) + S(z) - \Big(S(0) - \int_0^z \exp(M_C^{-1}\xi)s(\xi)\,\mathrm{d}\xi\Big)\exp(-M_C^{-1}z), \tag{54}$$

where

$$s(\xi) = \pi\left(f_i(\xi)f_i(\xi) + g_i(\xi)g_i(\xi) + \alpha\big(f_2^2(\xi) + g_2^2(\xi)\big)\right),$$

$$S(z) = \int_z^\infty s(\xi)\,\mathrm{d}\xi.$$

Here the constant M_C is defined as in (31), but changing the constant π by $\sqrt{\pi^2 - 2C^2}$.

If we set

$$E_C(\epsilon, z) = \int_z^\infty \int_{\{\phi_{[0,1]}\ge\epsilon\}} \big[F(v_{i,j}, v_{i,j}) + (\pi^2 - 2C^2)v_i v_i + \alpha\pi^2 v_2^2\big]\,\mathrm{d}a, \tag{55}$$

we see that

$$E_C(\epsilon, z) = \epsilon^{-1}E_C(z). \tag{56}$$

Estimates (54) and (56) give an estimate for the decay uniform in domains of the form $[x_1, \infty) \times \{\phi_{[0,1]} \ge \epsilon\}$.

If we assume that $g_i = 0$, we can consider an alternative approach. Let the weight function be $\phi = \sin\sqrt{2}Cx_2$. We may use similar results as in the previous section, but in this case we need to work with the first eigenvalues (λ_1^ϕ and μ_1^ϕ) of the singular problems

$$\big(\phi(x)u'\big)' + \lambda\phi(x)u = 0, \quad (0,1),$$

$$u(1) = 0, u(0) \text{ bounded, and } xu'(x) \to 0 \text{ as } x \to 0,$$

and

$$\begin{gathered}\big(\phi(x)u'\big)' + \mu\phi'(x)u = 0, \ \ (0,1), \\ u(0), u'(0) \text{ bounded, and } u(1) = 0,\end{gathered} \tag{57}$$

respectively.

This also allows us to obtain decay estimates in the L^2 measure of the solutions.

5. The Case of the Cylinder

It is not difficult to extend our arguments to three dimensions in some cases, but the geometry of the cross-section produces some difficulties in many situations. We consider a problem determined by the three-dimensional version of the system of equations (13) in the semi-infinite cylinder $(0,\infty)\times D$, where D is a two-dimensional region (not necessarily bounded) such that we can apply the divergence theorem. We assume that the boundary of D can be expressed as the union of two subsets D_1 and D_2, where $D_1 \cap D_2 = \emptyset$. The boundary conditions are

$$u_i(x_1,x_2,x_3) = \begin{cases} f_i(x_1,x_2,x_3) & \text{if } (x_2,x_3)\in D_1, \\ 0 & \text{if } (x_2,x_3)\in D_2, \end{cases} \tag{58}$$

and

$$u_i(0,x_2,x_3) = h_i(x_2,x_3). \tag{59}$$

Here we assume that

$$\begin{aligned} f_i(0,x_2,x_3) &= h_i(x_2,x_3) \quad \text{if } (x_2,x_3)\in D_1, \\ h_i(x_2,x_3) &= 0 \qquad\qquad\quad \text{if } (x_2,x_3)\in D_2. \end{aligned}$$

If ϕ is a function that depends only on the variables (x_2,x_3) and (u_i) is an arbitrary solution of the three-dimensional version of the system (13), the following relation holds:

$$\begin{aligned} &\Big[\phi(2u_{i,j} + \alpha\delta_{ij}u_{k,k} + \alpha u_{j,i})u_i\Big]_{,j} \\ &\qquad - \alpha\Big[(\phi u_1 u_\beta)_{,\beta} - \phi u_{1,\beta}u_\beta - \phi u_1 u_{\beta,\beta}\Big]_{,1} \\ &= \phi F(u_{i,j},u_{i,j}) - \Delta\phi u_i u_i - \alpha\phi_{,\beta\gamma}u_\beta u_\gamma \\ &\qquad + \frac{\partial}{\partial x_\beta}(\phi_{,\beta}u_i u_i) + \alpha\frac{\partial}{\partial x_\beta}(\phi_{,\gamma}u_\beta u_\gamma). \end{aligned} \tag{60}$$

Here δ_{ij} is the Kronecker symbol, and indices β and γ are restricted to values 2 and 3. It is worth noticing that, when ϕ depends only on the variable x_2, this equality reduces to

$$\begin{aligned} &\Big[\phi(2u_{i,j} + \alpha\delta_{ij}u_{k,k} + \alpha u_{j,i})u_i\Big]_{,j} \\ &\qquad - \alpha\Big[(\phi u_1 u_2)_{,\beta} - \phi u_{1,2}u_2 - \phi u_1 u_{2,2}\Big]_{,1} \\ &= \phi F(u_{i,j},u_{i,j}) - \Delta\phi_{,22}u_i u_i - \alpha\phi_{,22}u_2^2 \\ &\qquad + \frac{\partial}{\partial x_2}(\phi_{,2}u_i u_i) + \alpha\frac{\partial}{\partial x_\beta}(\phi_{,2}u_\beta u_2). \end{aligned} \tag{61}$$

Our goal in this section is to develop our study in a similar way to the one followed in Section 3. Due to (60), we have to make some changes in the approach.

In the sequel we are going to work with non-negative functions $\phi(x_2,x_3)$ that satisfy the following conditions:
(i) $\phi(x_2,x_3) = 0$ if and only if $(x_2,x_3)\in D_1$,
(ii) there exists a positive constant $\zeta(\alpha)$ such that

$$\begin{aligned} &\int_D \Big[\phi(2-\alpha)(\xi_{\beta,\gamma}\xi_{\beta,\gamma}) - \Delta\phi(\xi_2^2+\xi_3^2) - \alpha\phi_{,\beta\gamma}\xi_\beta\xi_\gamma\Big]\,\mathrm{d}a \\ &\qquad \geq \zeta(\alpha)\int_D \phi(\xi_{\beta,\gamma}\xi_{\beta,\gamma} + \xi_2^2 + \xi_3^2)\,\mathrm{d}a \end{aligned} \tag{62}$$

for every vector field (ξ_2,ξ_3) that vanishes in D_2.

Condition (ii) on the function ϕ is imposed to guarantee that the function

$$\int_D \Big(\phi F(u_{i,j},u_{i,j}) - \Delta\phi u_i u_i - \alpha\phi_{,\gamma\beta}u_\beta u_\gamma\Big)\,\mathrm{d}a \tag{63}$$

can be seen as a measure on the solutions of the three-dimensional version of the system (13), satisfying the boundary conditions (58) and (59).

Example 1. Let us assume that D is the unit square $(0,1)^2$ and D_1 is the point of the form (x_2,x_3), $0 \leq x_3 \leq 1$ and $x_2 = 0$ or 1. We may consider the function $\phi(x_2,x_3) = \sin\pi x_2$. The conditions are satisfied for every $\alpha < 2$. When D_1 is the subset of points of the form $(0,x_3)$, the function $\phi(x_2,x_3) = x_2$ works. ♦

Example 2. In case $D = (0,1)\times(-\infty,\infty)$ and $D_1 = \partial D$, we may consider again the function $\phi(x_2,x_3) = \sin\pi x_2$. Furthermore, if $\hat{S}$ is a subset in the interior of $(0,1)\times(-\infty,\infty)$, $D = (0,1)\times(-\infty,\infty) - \hat{S}$ and D_1 is the set of point (x_2,x_3), $x_2 = 0$ or 1, we may consider the same function. It is worth remarking that in this case the cross-section is unbounded. Again, if D_1 is the set of points of the form $(0,x_3)$, the function $\phi(x_2,x_3) = x_2$ works. ♦

Example 3. Let $0 < a < b$ be two arbitrary positive constants and $D = \{(x_2, x_3), a < r < b\}$, where $r^2 = (x_2^2 + x_3^2)$, and $D_1 = \partial D$. If we consider the function

$$\phi = (r-a)(b-r), \tag{64}$$

we have

$$\phi_{,\beta\gamma} = -2\delta_{\beta\gamma} + \frac{\delta_{\beta\gamma} r^2 - x_\beta x_\gamma}{r^3}(a+b). \tag{65}$$

Thus

$$\Delta\phi = -4 + \frac{a+b}{r}, \tag{66}$$

and the matrix $(\Delta\phi\delta_{\beta\gamma} + \alpha\phi_{,\beta\gamma})$ is

$$\hat{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \tag{67}$$

where

$$\begin{aligned} m_{11} &= -4 - 2\alpha + \frac{(a+b)}{r}\left(1 + \alpha\frac{x_2^2}{r^2}\right), \\ m_{12} &= -\alpha(a+b)\frac{x_1x_2}{r^3}, \\ m_{21} &= -\alpha(a+b)\frac{x_1x_2}{r^3}, \\ m_{22} &= -4 - 2\alpha + \frac{(a+b)}{r}\left(1 + \alpha\frac{x_1^2}{r^2}\right). \end{aligned}$$

Whenever this matrix is negative definite, condition (ii) is satisfied. As the trace of $\hat{M}$ is $r^{-1}(2+\alpha)(a+b-4r)$ and the determinant is

$$(4-2\alpha)^2 + \left(\frac{a+b}{r}\right)^2(1+\alpha) - (4-2\alpha)(2+\alpha)\frac{a+b}{r},$$

the matrix $\hat{M}$ is negative definite whenever $b < 3a$ and

$$(4-2\alpha)^2 + \left(\frac{a+b}{b}\right)^2(1+\alpha) - (4-2\alpha)(2+\alpha)\frac{a+b}{a}.$$

In order to illustrate the possibilities of the example, we consider some particular cases. When $\alpha = 1/3$, we have

$$\left(\frac{10}{3}\right)^2 + \frac{4}{3}\left(\frac{a+b}{b}\right)^2 - \frac{70}{9}\frac{a+b}{a},$$

which is always positive if a/b is greater than the unique positive solution of the equation

$$4x^3 + 8x^2 + 14x - \frac{70}{3} = 0.$$

This solution is

$$-\frac{2}{3} + \frac{409 + 15(763)^{1/3}}{32^{2/3}} - \frac{13}{3(2(409+15\sqrt{763}))^{1/3}} \cong 0.934491.$$

It is clear that we can extend this process whenever $\alpha < 1/2$, because when $\alpha = 1/2$, the corresponding equation is

$$3x^3 + 6x^2 + 6x - 15 = 0.$$

One thinks that alternative selections of the function ϕ could open many other possibilities. In this case the region D is not simply connected. ♦

Now, we extend the arguments of Section 3 to the three-dimensional case.

We assume that the energy

$$E_\phi(0) = \int_0^\infty \int_D \Big(\phi F(u_{i,j}, u_{i,j}) - \Delta\phi u_i u_i - \alpha\phi_{,\gamma\beta} u_\beta u_\gamma\Big)\,\mathrm{d}v \tag{68}$$

is bounded and that the asymptotic condition

$$\lim_{x_1\to\infty} \int_D \phi\Big[2u_{i,1}u_i + \alpha u_{k,k}u_1 + \alpha u_{1,i}u_i + \alpha u_{1,\beta}u_\beta + \alpha u_1 u_{\beta,\beta}\Big]\,\mathrm{d}a = 0 \tag{69}$$

is satisfied. If we define the function

$$E_\phi(z) = \int_z^\infty \int_D \Big(\phi F(u_{i,j}, u_{i,j}) - \Delta\phi u_i u_i - \alpha\phi_{,\gamma\beta} u_\beta u_\gamma\Big)\,\mathrm{d}v, \tag{70}$$

the use of the divergence theorem and the boundary conditions allow us to see that

$$\begin{aligned} E_\phi(z) = &-\int_D \phi\Big[2u_{i,1}u_i + \alpha u_{k,k}u_1 + \alpha u_{1,i}u_i + \alpha u_{1,\beta}u_\beta + \alpha u_1 u_{\beta,\beta}\Big]\,\mathrm{d}a \\ &-\int_z^\infty \int_{D_1} \Big(\phi_{,\beta} n_\beta f_i f_i + \alpha\phi_{,\gamma} n_\beta f_\beta f_\gamma\Big)\,\mathrm{d}a, \end{aligned} \tag{71}$$

where n_β are the components of the outward normal $\mathbf{n}$ to the boundary of D.

In this situation, it is not very difficult to reproduce the arguments of Section 3. If we write

$$I_1 = -\int_D 2\phi u_{i,1} u_i\,\mathrm{d}a, \tag{72}$$

$$I_2 = -\alpha\int_D \phi\alpha u_{k,k} u_1\,\mathrm{d}a, \tag{73}$$

$$I_3 = -\alpha\int_D \phi u_{1,i} u_i\,\mathrm{d}a, \tag{74}$$

$$I_4 = -\alpha\int_D \phi u_{1,\beta} u_\beta\,\mathrm{d}a, \tag{75}$$

and

$$I_5 = -\alpha \int_D \phi u_1 u_{\beta,\beta} \, \mathrm{d}a, \tag{76}$$

then, after some calculations similar to those followed in Section 3, we see that

$$\sum_{i=1,5} I_i \leq N_\phi \frac{\partial E_\phi}{\partial z}, \tag{77}$$

where N_ϕ is a constant that is easily computable. We obtain

$$E_\phi(z) = -N_\phi \frac{\mathrm{d}E_\phi}{\mathrm{d}z} + P(z), \tag{78}$$

where

$$P(z) = -\int_z^\infty \int_{D_1} \Big(\phi_{,\beta} n_\beta f_i f_i + \alpha \phi_{,\gamma} n_\beta f_\beta f_\gamma\Big) \mathrm{d}a. \tag{79}$$

Thus we have proved the following result:

Theorem 2. *Let (u_i) be a solution to the problem determined by the system (13), boundary conditions (58), (59) and asymptotic conditions (69). Then the energy function defined in (70) satisfies the estimate*

$$E_\phi(z) \leq E_\phi(0) \exp(-N_\phi^{-1} z) + P(z)\Big(P(0) - \int_0^z \exp(N_\phi^{-1}\xi) p(\xi) \, \mathrm{d}\xi\Big) \exp(-N_\phi^{-1} z), \tag{80}$$

where

$$p(\xi) = -\int_{D_1} \Big(\phi_{,\beta} n_\beta f_i f_i + \alpha \phi_{,\gamma} n_\beta f_\beta f_\gamma\Big) \, \mathrm{d}l. \tag{81}$$

Defining the domains

$$D(\epsilon) = \big\{\mathbf{x} \in D, \phi(\mathbf{x}) \geq \epsilon\big\}, \tag{82}$$

and setting

$$E_\phi(\epsilon, z) = \int_z^\infty \int_{D(\epsilon)} \Big(\phi F(u_{i,j}, u_{i,j}) - \Delta\phi u_i u_i - \alpha \phi_{,\gamma\beta} u_\beta u_\gamma\Big) \, \mathrm{d}v, \tag{83}$$

we obtain

$$E_\phi(\epsilon, z) \leq \epsilon^{-1} E_\phi(z). \tag{84}$$

Estimates (79) and (84) give a uniform decay in the domains of the form $[x_1, \infty) \times D(\epsilon)$.

In the remainder of this paper, we consider the case where $D = (0,1)^2$ and D_1 is the set of the points of the form $(0, x_3)$, where $0 \leq x_3 \leq 1$. If we define

$$W(z) = \int_z^\infty \int_D u_i u_i \, \mathrm{d}v, \tag{85}$$

estimates (11) and (19) allow us to obtain

$$W(z) \leq (2-\alpha)^{-1} \mu_1^{-1} E_{x_2}(z). \tag{86}$$

Then (79) and (85) allow us to obtain the estimate

$$W(z) \leq (2-\alpha)^{-1} \mu_1^{-1} \Big(E_{x_2}(0) \exp(-M_{x_2}^{-1} z) + P_{x_2}(z) - \Big(P_{x_2}(0) - \int_0^z \exp(M_{x_2}^{-1}\xi) \times p_{x_2}(\xi) \mathrm{d}\xi\Big) \exp(-M_{x_2}^{-1} z)\Big). \tag{87}$$

The constant M_{x_2} that arises in this estimate can be obtained as the one determined in Section 3 for the decay of $E^*(z)$ and

$$p_{x_2}(\xi) = \int_{D_1} (f_i f_i + \alpha f_2 f_2) \, \mathrm{d}l.$$

This estimate is uniform on the whole cross-section D.

It seems possible to extend the arguments used to study the steady-state vibrations in the case of a cylinder.

Remark 1. In order to possess a more explicit knowledge of the estimates (75) and (82), it is suitable to obtain an upper bound for the term $E_{x_2}(0)$ in terms of the boundary conditions. We do it whenever we assume that $h_i(0, x_3) = f_i(0, 0, x_3) = 0$ for all $0 \leq x_3 \leq 1$.

We see that

$$E_{x_2}(0) \leq (2+\alpha) \int_0^\infty \int_D (u_{i,j} u_{i,j} + \alpha u_{i,i} u_{j,j}) \, \mathrm{d}v. \tag{88}$$

The integral on the right-hand side of (88) was studied in (Quintanilla, 1997b). In this situation the solution to the problem determined by conditions (58) and (59) is the sum of the solutions $\tilde{u}_i$ and $\hat{u}_i$ that correspond to the case when $f_i = 0$ and $h_i = 0$, respectively. Thus we see that

$$\begin{aligned} \int_0^\infty \int_D (u_{i,j} u_{i,j} + \alpha u_{i,i} u_{j,j}) \, \mathrm{d}v &= \int_0^\infty \int_D \Big((\tilde{u}_{i,j} + \hat{u}_{i,j})(\tilde{u}_{i,j} + \hat{u}_{i,j}) + \alpha(\tilde{u}_{i,i} + \hat{u}_{i,i})(\tilde{u}_{j,j} + \hat{u}_{j,j})\Big) \, \mathrm{d}v \\ &\leq 2\Big(\int_0^\infty \int_D (\tilde{u}_{i,j} \tilde{u}_{i,j} + \alpha \tilde{u}_{i,i} \tilde{u}_{j,j}) \, \mathrm{d}v + \int_0^\infty \int_D (\hat{u}_{i,j} \hat{u}_{i,j} + \alpha \hat{u}_{i,i} \hat{u}_{j,j}) \, \mathrm{d}v\Big). \end{aligned} \tag{89}$$

To calculate these integrals, we can use the arguments presented in (Quintanilla, 1997b). We obtain

$$\int_0^\infty \int_D (\tilde{u}_{i,j}\tilde{u}_{i,j} + \alpha\tilde{u}_{i,i}\tilde{u}_{j,j})\,\mathrm{d}v \le (1+3\alpha)\int_D (h_{i,2}h_{i,2} + h_{i,3}h_{i,3})\,\mathrm{d}a \int_D h_i h_i\,\mathrm{d}a, \tag{90}$$

and

$$\int_0^\infty \int_D (\hat{u}_{i,j}\hat{u}_{i,j} + \alpha\hat{u}_{i,i}\hat{u}_{j,j})\,\mathrm{d}v \le (1+3\alpha)\Big(\int_0^\infty\int_0^1 (f_{i,1}f_{i,1} + f_{i,3}f_{i,3})\,\mathrm{d}a + \int_0^\infty\int_0^1 f_i f_i\,\mathrm{d}a\Big). \tag{91}$$

The combination of the estimates (88)–(91) gives the desired upper bound.

6. An Ill-Posed Problem

This section is devoted to the study of spatial estimates for an ill-posed problem determined by the three-dimensional version of the system of equations (13) and the boundary conditions

$$\begin{aligned} u_i(x_1, x_2, 1) &= 0, \\ u_i(x_1, 0, x_3) &= f_i(x_2, x_3), \\ u_i(x_1, 1, x_3) &= 0, \end{aligned} \tag{92}$$

but we have no information on the displacement on the part of the boundary consisting of the points of the form $(x_1, x_2, 0)$. This result will be an extension of the one obtained in (Quintanilla, 1997a), when we allow for non-homogeneous boundary conditions on a part of the lateral surface.

We assume that

$$\Sigma(0,0) = \int_0^\infty \int_D x_2 F(u_{i,j}, u_{i,j})\,\mathrm{d}v < \infty, \tag{93}$$

and the asymptotic conditions (68) are satisfied. If we define the function

$$\Sigma(z_1, z_3) = \int_{z_1}^\infty \int_{z_3}^1 \int_0^1 x_2 F(u_{i,j}, u_{i,j})\,\mathrm{d}v, \tag{94}$$

we obtain

$$\begin{aligned} \Sigma(z_1, z_3) = &-\int_{z_1}^\infty \int_0^1 x_2(2u_{i,3}u_i + \alpha u_{k,k}u_3 \\ &+ \alpha u_{3,i}u_3 + \alpha u_{3,2}u_2 + \alpha u_3 u_{2,2})\,\mathrm{d}a \\ &- \int_{z_3}^1 \int_0^1 x_2(2u_{i,1}u_i + \alpha u_{k,k}u_1 + \alpha u_{1,i}u_1 \\ &+ \alpha u_{3,i}u_3 + \alpha u_{1,2}u_2 + \alpha u_1 u_{2,2})\,\mathrm{d}a \\ &+ \int_{z_1}^\infty \int_{z_3}^1 (f_i f_i + \alpha f_2^2)\,\mathrm{d}a, \end{aligned} \tag{95}$$

$$\frac{\partial\Sigma}{\partial z_1} = -\int_{z_3}^1 \int_0^1 x_2 F(u_{i,j}, u_{i,j})\,\mathrm{d}a, \tag{96}$$

and

$$\frac{\partial\Sigma}{\partial z_3} = -\int_{z_1}^\infty \int_0^1 x_2 F(u_{i,j}, u_{i,j})\,\mathrm{d}a. \tag{97}$$

From (95)–(97) we can obtain an estimate of the form

$$\Sigma \le -M_{x_2}\left(\frac{\partial\Sigma}{\partial z_1} + \frac{\partial\Sigma}{\partial z_3}\right) + Q, \tag{98}$$

where

$$Q = \int_{z_1}^\infty \int_{z_3}^1 (f_i f_i + \alpha f_2^2)\,\mathrm{d}a. \tag{99}$$

If we integrate (98) along the lines of the form

$$z_1 - z_1^0 = z_3 - z_3^0, \tag{100}$$

we obtain the estimate

$$\begin{aligned} &\Sigma(z_1, z_1 + z_3^0 - z_1^0) \\ &\quad\le \Sigma(z_1^0, z_3^0)\exp\big(-M_{x_2}^{-1}(z_1 - z_1^0)\big) + M_{x_2}^{-1} \\ &\quad\times\left(\int_{z_1^0}^{z_1} \exp\big(M_{x_2}^{-1}(\xi - z_1^0)\big) Q\big(\xi, \xi + z_3^0 - z_1^0\big)\,\mathrm{d}\xi\right) \\ &\quad\times\exp\big(-M_{x_2}^{-1}(z_1 - z_1^0)\big),\ \ z_1 \ge z_1^0. \end{aligned} \tag{101}$$

Thus we have proved the following result:

Theorem 3. *Let (u_i) be a solution to the problem determined by the system (13), boundary conditions (92) and asymptotic conditions (69). Then the energy function defined in (94) satisfies the estimate (101).*

This result is a natural extension of that obtained in (Quintanilla, 1997a).

7. Some Remarks

In (Quintanilla, 2000) the author proposed to apply energy arguments when non-homogeneous conditions are imposed on the whole boundary. But we could not do it (in general) due to the term of the form $u_{j,ji}$ in the Navier system of equations (13). Furthermore, this is the reason why we have to restrict our attention to the cases of $\alpha < 2$. We have also seen by means of a remark that the condition on α should be more restrictive when the relative geometry of the cross-section and the subset of the boundary with non-homogeneous conditions are complex.

We can recall that in other contributions of this kind in elasticity the restriction is more relaxed (see, e.g., Flavin *et al.*, 1989; Horgan and Payne, 1992; Quintanilla, 1997a). Thus there are some natural open questions:

1. Extension of the energy arguments to the case where the non-homogeneous boundary conditions are imposed in the whole of the boundary.
2. Analysis when $\alpha \geq 2$.
3. The results hold for solutions having *a priori* suitable behaviour at the spatial infinity. For instance, it is assumed that the solutions tend to zero. A (fundamental) open problem is to eliminate this restriction.

It is worth remarking that the anti-plane deformations of an isotropic and homogeneous elastic solid are governed by the Laplace equation. This equation was studied in (Quintanilla, 2000) and it was proved there that we may obtain spatial decay estimates when the non-homogeneous conditions are imposed on the whole of the boundary. We also note that the weight functions used here concern only bounded directions.

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