POLYNOMIAL SYSTEMS THEORY APPLIED TO THE ANALYSIS AND DESIGN OF MULTIDIMENSIONAL SYSTEMS

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The use of a principal ideal domain structure for the analysis and design of multidimensional systems is discussed. As a first step it is shown that a lattice structure can be introduced for IO-relations generated by polynomial matrices in a signal space X (an Abelian group). It is assumed that the matrices take values in a polynomial ring $F[p]$ where F is a field such that $F[p]$ is a commutative subring of the ring of endomorphisms of X. After that it is analysed when a given $F[p]$ acting on X can be extended to its field of fractions $F(p)$. The conditions on the pair $(F[p], X)$ are quite restrictive, i.e. each non-zero $a(p) \in F[p]$ has to be an automorphism on X before the extension is possible. However, when this condition is met, say for operators $\{p_1, p_2, \ldots, p_{n-1}\}$, a polynomial ring $F[p_1, p_2, \ldots, p_n]$ acting on X can be extended to $F(p_1, p_2, \ldots, p_{n-1})[p_n]$, resulting in a principal ideal domain structure. Hence in this case all the rigorous principles of 'ordinary' polynomial systems theory for the analysis and design of systems is applicable. As an example, both an observer for estimating non-measurable outputs and a stabilizing controller for a distributed parameter system are designed.

Keywords: nD systems, module of fractions, partial differential equations, polynomial systems theory

1. Introduction

Polynomial systems theory for time-invariant linear differential and difference systems is a well-established and efficient tool for the analysis and design of control systems (Blomberg and Ylinen, 1983; Kučera, 1979; Rosenbrock, 1970; Wolovich, 1974). The methodology utilises the algebraic properties of polynomials with real or complex coefficients and the strong interplay between the ring of polynomials and the general theory of linear constant coefficient differential/difference equations. The key for the 'success' of this theory seems to be its computational nature, i.e. the ring $F[p]$ of polynomials over a field F in an operator p normally satisfies the assumptions of a division algorithm which can be used to find common factors and to manipulate polynomial matrices into suitable canonical forms in an algorithmic way. Thus all necessary computations inside the theory can be implemented on a computer.

Polynomial systems theory was originally developed for analysis and design of control systems. Therefore only input-output systems describing cause-effect relationships were considered. Later on, the theory has been generalized to so-called *behavioural systems theory*, where the variables of systems are not *a priori* divided into inputs and outputs (Willems, 1991; 1997; Valcher and Willems, 1999).

The extension of polynomial systems theory multidimensional systems (nD systems) has been done, e.g., in (Oberst, 1990). The resulting structure is a ring $F[p_1, p_2, \ldots, p_n]$ of polynomials over a field F in two or more operators $p_1, p_2, \ldots, p_n$ acting on a given signal space. This is not a principal ideal domain but a Noetherian domain which offers a much weaker methodology for manipulation on models. The theory relies on concepts and methods of module algebra and shows that on some assumptions about the signal space, computational techniques are again available for the analysis and design of multivariable systems. This computational theory, however, is quite complex. A good introduction to the theory from the behavioural point of view is given in (Wood, 2000). In (Napoli and Zampieri, 1999; Zampieri, 1998) the connections of the input-output representation and state space representation of 2D systems are studied. In particular, the causality with respect to given 'past' and 'future' is considered.

In this paper, our goal is to look for simpler algebraic structures for the analysis and design of multidimensional systems. Multidimensional systems are usually used as models of *dynamic* linear distributed parameter systems. This means that at least one, say p_n, of the operators is a differentiation or shift operator with respect to *time*. The multidimensional polynomial ring $F[p_1, p_2, \ldots, p_n]$ can be represented as the ring $F[p_1, p_2, \ldots, p_{n-1}][p_n]$ of ordinary polynomials in this operator with polynomial coefficients.

Under relatively weak conditions for the signal space the polynomial ring of coefficients can be extended to its field of fractions $F(p_1, p_2, \dots, p_{n-1})$. If this extension can be made, a ring of polynomials with rational coefficients is obtained. This is an Euclidean domain so that all methods developed for ordinary polynomial systems seem to be applicable. This kind of construction was applied to the classical theory of two-dimensional systems (Morf *et al.*, 1977). Unfortunately, there is a serious restriction for that. It is easy to show that rational forms can be considered as mappings from signals to signals only if their denominators are automorphisms of the signal space (Blomberg and Ylinen, 1983). This usually means that the original signal space has to be restricted or extended to satisfy this condition. On the other hand, this property is strongly motivated by the causality requirement.

The structure of the paper is such that the results related to polynomial systems over an arbitrary field are considered first. The most important one is the construction of a module of fractions over a ring of fractions. Given a denominator set, this construction can always be done but then the original signal space is also extended to a space of rational signals. The system models are extended correspondingly. The requirements for the denominators needed for identifying the original system with a subsystem of the extended system are given.

After the application of the results to nD systems, we suppose that operators $p_1, \dots, p_{n-1}$ are such that the non-zero elements of $F[p_1, \dots, p_{n-1}]$ are automorphisms. We can proceed without this assumption but then the results are applicable only to the extended system with rational signals. Basic methods for observer and controller design are presented.

The methology is applied to the design of a feedback controller for the system in which a metal wire is pulled out from a heating treatment with a constant velocity, and the control problem is to manipulate the temperature distribution of the pulled wire to a desired temperature. Also, a temperature estimator is designed. In the example we take a signal space which can be considered as a 'projection' of the space of infinitely differentiable 2D signals.

Finally, some agreements concerning the concepts and notation regarding functions (or mappings) and relations are made. We will basically identify functions and relations with their graphs, i.e. consider them as sets of ordered pairs. Thus functions are simply relations of a special type. Furthermore, a relation R has always an inverse relation R^{-1} as a relation usually called a *converse*, but a function f has the inverse f^{-1} as a function if and only if it is injective. The composite relation $R \circ S$ of two relations S and R as well as the composite function $f \circ g$ of two functions g and f is always defined regardless of its domains and ranges.

2. Polynomials of Endomorphisms and IO-Relations

In this section, we review some basic concepts of polynomial systems theory. See, e.g. (Blomberg and Ylinen, 1983; Ylinen, 1980) for a more detailed approach.

A polynomial system description consists of equations of the form

$$\underbrace{(a_0 + a_1 p + \cdots + a_n p^n)}_{a(p)}\, y = \underbrace{(b_0 + b_1 p + \cdots + b_m p^m)}_{b(p)}\, u, \tag{1}$$

where $u,\ y \in X$ and X is an additive Abelian group. We have $a_0, \dots a_n, b_0 \dots b_m,\ p \in \mathrm{End}(X)$. The coefficients a_i, b_i of the polynomials are assumed to belong to a commutative subfield F of the ring $\mathrm{End}(X)$. If $pf = fp$ for every $f \in F$, then $F[p]$ forms a commutative subring of $\mathrm{End}(X)$. In this case, X can be considered as a left-module over $F[p]$.

Suppose that X is so 'rich' that p is an *indeterminate* over F, i.e. for each $a(p) \in F[p]$ we have $a(p) = a_0 + a_1 p + \cdots + a_n p^n = 0$ if and only if $a_0 = a_1 = \cdots = a_n = 0$ Then the representation of each $a(p) \in F[p]$ is unique and the *degree function* $d(a(p)) = \max\{m \mid a_m \neq 0\}$, $d(0) = -\infty$, is well defined. This implies further that the polynomial ring $F[p]$ satisfies the *division algorithm*, i.e. for arbitrary non-zero $a(p),\ b(p) \in F[p]$ there exist $q(p),\ r(p) \in F[p]$ such that

$$a(p) = q(p)b(p) + r(p), \quad d\big(r(p)\big) < d\big(b(p)\big). \tag{2}$$

Due to the existence of a division algorithm, $F[p]$ also satisfies the axioms of a principal ideal domain, and, in fact, it is an Euclidean domain.

$F[p]$ is also an integral domain so that it can be extended to the *field of fractions* denoted by $F(p)$. More general *rings of fractions* are discussed in Section 4.

A *polynomial IO-relation* S is defined by

$$S = \big\{(u, y) \in X \times X \mid a(p)y = b(p)u\big\}, \tag{3}$$

where (u, y) is an ordered input-output pair of the relation and $a(p), b(p) \in F[p]$. This formalism can be naturally extented to a multivariable case: here the IO-relation S from X^r to X^s is defined as

$$S = \big\{(u, y) \in X^r \times X^s \mid A(p)y = B(p)u\big\}, \tag{4}$$

where the matrix pair $(A(p), B(p))$, $A(p) \in F[p]^{s\times s}$ and $B(p) \in F[p]^{s\times r}$ is said to generate the IO-relation S. The defining equation in (4) can be written as

$$A(p)y - B(p)u = \left[\begin{array}{ccc} A(p) & \vdots & -B(p) \end{array}\right] \begin{bmatrix} y \\ u \end{bmatrix} = 0. \quad (5)$$

The partitioned matrix $[\,A(p) \vdots - B(p)\,]$ can be considered as a morphism $X^s \times X^r \cong X^{s+r} \to X^s$, whose kernel, as a set of ordered pairs, is the converse (relation) of S:

$$\begin{aligned} S^{-1} &= \{(y,u) \in X^s \times X^r \mid (u,y) \in S\} \\ &= \mathrm{Ker}\left[\begin{array}{ccc} A(p) & \vdots & -B(p) \end{array}\right]. \end{aligned} \quad (6)$$

Typically, an IO-relation may have an infinite number of different *generators* $[\,A(p) \vdots - B(p)\,]$, which are then said to be *input-output equivalent*. A generator $[\,A(p) \vdots - B(p)\,]$ is called *regular* if $\det A(p) \neq 0$ and a relation S is regular if it has a regular generator. The regularity is necessary from the realizability point of view, i.e. it is needed for constructing realizable input-output mappings.

In a typical system analysis or design problem we are, however, more interested in interconnected systems described by *compositions* of IO-relations, than merely in the analysis of one IO-relation. In order to have a practical theory for the composition of IO-relations, we have to assume further that the *domain* of each IO-relation (4) to be connected is ‘full’, i.e. $\mathcal{D}S = X^r$. This is guaranteed if the *range* of $\mathcal{R}A(p)$ in a regular generator $[\,A(p) \vdots - B(p)\,]$ (6) is the whole X^s, which, furthermore, is satisfied if the module X is *divisible*, i.e. if each non-zero $a(p) \in F[p]$ is surjective (i.e. an epimorphism) on X.

An arbitrary composition of a set of IO-relations describing the *subsystems* can be determined by the *interconnection constraints* (Blomberg and Ylinen, 1983). Every composition can be reduced to the general form depicted by Fig. 1.

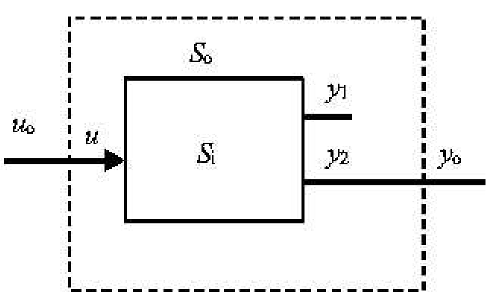

Fig. 1. General composition.

Here S_i is the *internal IO-relation* determined by the subsystems and interconnection constraints and S_0 is the *overall IO-relation* describing the system which we are interested in. Correspondingly, u_0 and y_0 are the chosen overall input and output, and y_1 constitutes the internal output consisting of the outputs of the interconnected subsystems. Each input of the subsystems is either an overall input or an internal output depending on the interconnection constraints.

Combining the IO-relations and interconnection constraints gives the following model for the internal IO-relation:

$$\begin{bmatrix} A_1(p) & A_2(p) & -B_1(p) \\ A_3(p) & A_4(p) & -B_2(p) \end{bmatrix} \begin{bmatrix} y_1 \\ y_0 \\ u_0 \end{bmatrix} = 0. \quad (7)$$

For constructing the model of the overall IO-relation

$$S_0 = \{(u_0, y_0) \mid \exists y_1 \left[(u_0, (y_1, y_0)) \in S_i\right]\}, \quad (8)$$

however, the internal output y_1 should be eliminated.

3. Order and Equivalence Relations for the IO-relations

In this section, we will show how a lattice structure can be introduced for our IO-relations defined over a polynomial ring $F[p]$. Note that the field F is arbitrary, and thus this theory also applies when the field F contains operators acting on a given signal space. The work follows closely the approaches adopted in (Ylinen, 1975; Ylinen and Blomberg, 1989).

Proposition 1. *Let S and S' be IO-relations generated by the pairs $(A(p), B(p)) \in F[p]^{s\times s} \times F[p]^{s\times r}$ and $(A'(p), B'(p)) \in F[p]^{s\times s} \times F[p]^{s\times r}$, respectively. Then $S \subset S'$ if and only if there exists an epimorphism of groups $\varphi : \mathcal{R}[\,A(p) \vdots - B(p)\,] \to \mathcal{R}[\,A'(p) \vdots - B'(p)\,]$, such that*

$$\left[\begin{array}{ccc} A'(p) & \vdots & -B'(p) \end{array}\right] = \varphi \circ \left[\begin{array}{ccc} A(p) & \vdots & -B(p) \end{array}\right]. \quad (9)$$

Furthermore, $S = S'$ if and only if φ is an isomorpism.

Proof. Assume first that (9) holds. Clearly, $\mathrm{Ker}[\,A(p) \vdots - B(p)\,] \subset \mathrm{Ker}(\varphi \circ [\,A(p) \vdots - B(p)\,])$ because φ is a morphism and, consequently, $S \subset S'$. In addition, $S = S'$ if and only if $\mathcal{R}[\,A(p) \vdots - B(p)\,] \cap \mathrm{Ker}\varphi = \{0\}$. Since $\mathrm{Ker}\varphi \subset \mathcal{R}[\,A(p) \vdots - B(p)\,]$, we have $S = S'$ if and only if $\mathrm{Ker}\varphi = \{0\}$, so that φ becomes an isomorphism between $\mathcal{R}[\,A(p) \vdots - B(p)\,]$ and $\mathcal{R}[\,A'(p) \vdots - B'(p)\,]$.

Assume now that $S \subset S'$, i.e. $\mathrm{Ker}[\,A(p)\ km\ - B(p)\,] \subset \mathrm{Ker}[\,A'(p) \vdots - B(p)\,]$. The group morphisms $[\,A(p) \vdots - B(p)\,]$ and $[\,A'(p) \vdots - B'(p)\,]$ can be decomposed into $[\,A(p) \vdots B(p)\,] = [\,A(p) \vdots - B(p)\,]^* \circ P$

and $[\,A'(p) \,\vdots\, - B'(p)\,] = [\,A'(p) \,\vdots\, - B'(p)\,]^* \circ P$, where P is the canonical surjection from $X^s \times X^r$ onto the factor group $X^s \times X^r/S^{-1}$. $[\,A(p) \,\vdots\, - B(p)\,]^* : X^s \times X^r/S^{-1} \to X^s$ and $[\,A'(p) \,\vdots\, - B'(p)\,]^* : X^s \times X^r/S^{-1} \to X^s$ are morphisms of groups and $[\,A(p) \,\vdots\, - B(p)\,]^*$ is a monomorpism of groups. Hence its left inverse exists and, consequently, $P = ([\,A(p) \,\vdots\, - B(p)\,]^*)^{-1} \circ [\,A(p) \,\vdots\, - B(p)]$, which gives

$$\begin{aligned}\begin{bmatrix} A'(p) & \vdots & -B'(p) \end{bmatrix} &= \begin{bmatrix} A'(p) & \vdots & -B'(p) \end{bmatrix}^* \\ &\quad \circ \left(\begin{bmatrix} A(p) & \vdots & -B(p) \end{bmatrix}^* \right)^{-1} \\ &\quad \circ \begin{bmatrix} A(p) & \vdots & -B(p) \end{bmatrix}. \end{aligned} \tag{10}$$

Define $\varphi \triangleq [\,A'(p) \,\vdots\, - B'(p)\,]^* \circ ([\,A(p) \,\vdots\, - B(p)\,]^*)^{-1}$. It is an epimorphism of groups from $\mathcal{R}[\,A(p) \,\vdots\, - B(p)\,]$ to $\mathcal{R}[\,A'(p) \,\vdots\, - B'(p)\,]$. If $S = S'$, then $[\,A'(p) \,\vdots\, - B'(p)\,]^*$ is a monomorphism of groups, which implies that φ is an isomorphism between $\mathcal{R}[\,A(p) \,\vdots\, - B(p)\,]$ and $\mathcal{R}[\,A'(p) \,\vdots\, - B'(p)\,]$. ■

This proposition is weak in the sense that it only implies the existence of a morphism φ but does not guarantee that this morphism would be also a matrix in $F[p]^{s\times s}$. The following proposition shows that this indeed is the case. The proof was given earlier only in (Ylinen and Blomberg, 1989). A shorter but a more module theoretic proof of the proposition can be found in (Hinrichsen and Prätzel-Wolters, 1980).

Proposition 2. *Let S and S' be two regular IO-relations generated by $[A(p) \,\vdots\, - B(p)]$ and $[\,A'(p) \,\vdots\, - B'(p)\,]$, respectively, with $A(p), A'(p) \in F[p]^{s\times s}$, $B(p), B'(p) \in F[p]^{s\times r}$. Suppose that $S \subset S'$. Assume that every nonzero element $c(p)$ of $F[p]$ is an epimorphism and $c(p) \in F[p]$ is an automorphism if and only if $c(p) \in F - \{0\}$. Then there exists a matrix $M(p) \in F[p]^{s\times s}$ such that*

$$\begin{bmatrix} A'(p) & \vdots & -B'(p) \end{bmatrix} = M(p) \begin{bmatrix} A(p) & \vdots & -B(p) \end{bmatrix}. \tag{11}$$

Furthermore, $S = S'$ if and only if $M(p)$ is unimodular.

Proof. $S \subset S'$ is equivalent to $S \cap S' = S$. This implies

$$\begin{aligned}&\operatorname{Ker}\begin{bmatrix} A(p) & \vdots & -B(p) \end{bmatrix} \\ &= \operatorname{Ker}\begin{bmatrix} A(p) & \vdots & -B(p) \end{bmatrix} \cap \operatorname{Ker}\begin{bmatrix} A'(p) & \vdots & -B'(p) \end{bmatrix} \\ &= \operatorname{Ker}\begin{bmatrix} A(p) & \vdots & -B(p) \\ A'(p) & \vdots & -B'(p) \end{bmatrix}. \end{aligned} \tag{12}$$

Because $F[p]$ is a principal ideal domain, we can find a unimodular matrix $N(p)$ such that

$$N(p)\begin{bmatrix} A(p) & \vdots & -B(p) \\ A'(p) & \vdots & -B'(p) \end{bmatrix} = \begin{bmatrix} \tilde{A}(p) & \vdots & -\tilde{B}(p) \\ 0 & \vdots & -\tilde{B}'(p) \end{bmatrix}, \tag{13}$$

which does not change the kernel. However, because the relation $S = \{(u, y) \mid A(p)y = B(p)u\}$ has as its domain the whole X^r, $\tilde{B}'(p)$ has to be zero. Thus

$$\begin{aligned}&\operatorname{Ker}\begin{bmatrix} A(p) & \vdots & -B(p) \end{bmatrix} \\ &= \operatorname{Ker} N(p)\begin{bmatrix} A(p) & \vdots & -B(p) \\ A'(p) & \vdots & -B'(p) \end{bmatrix} \\ &= \operatorname{Ker}\begin{bmatrix} \tilde{A}(p) & \vdots & -\tilde{B}(p) \\ 0 & \vdots & 0 \end{bmatrix} \\ &= \operatorname{Ker}\begin{bmatrix} \tilde{A}(p) & \vdots & -\tilde{B}(p) \end{bmatrix}. \end{aligned} \tag{14}$$

Conversely,

$$\begin{aligned}&\begin{bmatrix} A(p) & \vdots & -B(p) \\ A'(p) & \vdots & -B'(p) \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} Q_1(p) & \vdots & Q_2(p) \\ Q_3(p) & \vdots & Q_4(p) \end{bmatrix}}_{Q(p)} \begin{bmatrix} \tilde{A}(p) & \vdots & -\tilde{B}(p) \\ 0 & \vdots & 0 \end{bmatrix} \\ &= \begin{bmatrix} Q_1(p)\tilde{A}(p) & \vdots & -Q_1(p)\tilde{B}(p) \\ Q_3(p)\tilde{A}(p) & \vdots & -Q_3(p)\tilde{B}(p) \end{bmatrix}, \end{aligned} \tag{15}$$

where $Q(p) = N(p)^{-1}$. Because $[\,\tilde{A}(p) \,\vdots\, - \tilde{B}(p)\,]$ and $[\,Q_1(p)\tilde{A}(p) \,\vdots\, - Q_1(p)\tilde{B}(p)\,]$ have to generate the same IO-relation S, $Q_1(p)$ has to be unimodular. Thus $N(p)$ can be chosen so that $Q_1(p) = I$ and, consequently, $\tilde{A}(p) = A(p)$ and $\tilde{B}(p) = B(p)$. Thus $A'(p) = Q_3(p)A(p)$ and $B'(p) = Q_3(p)B(p)$ and $Q_3(p)$ qualifies as $M(p)$ in the proposition.

Finally, if $S = S'$, $M(p)$ must be an isomorphism, which implies that it is unimodular according to the assumptions. ■

The polynomial matrices satisfying (11) with unimodular $M(p)$ are said to be *row equivalent*. Thus, on the assumptions of Proposition 2, *two regular generators are input-output equivalent if and only if they are row equivalent*.

Let $(A(p), B(p)) \in F[p]^{s\times s} \times F[p]^{s\times r}$ generate a regular IO-relation. Let the rational matrix $G(p) \in$

$F(p)^{s\times r}$ be defined by $G(p) = A(p)^{-1}B(p)$. Then the IO-relation S generated by $(A(p), B(p))$ is said to be associated with $G(p)$, and $G(p)$ is called the *transfer matrix* determined by $(A(p), B(p))$. Two regular IO-relations associated with $G(p)$, as well as the corresponding generators, are called *transfer equivalent*. Obviously, row equivalent generators are transfer equivalent. The set of all transfer equivalent IO-relations associated with $G(p)$ can be ordered with a set inclusion, i.e. $S \le S' \iff S \subset S'$. The first element of this set, if it exists, is called the *minimal IO-relation* associated with $G(p)$.

Proposition 3. *Let S and S' be two regular IO-relations generated by $[\,A(p) \vdots -B(p)\,]$ and $[\,A'(p) \vdots -B'(p)\,]$, respectively, with $A(p), A'(p) \in F[p]^{s\times s}$, $B(p), B'(p) \in F[p]^{s\times r}$. Suppose that $S \subset S'$. Then S and S' are associated with the same transfer matrix $G(p) \in F(p)^{s\times r}$. Furthermore, a regular relation $\tilde{S}$ generated by $[\,\tilde{A}(p) \vdots -\tilde{B}(p)\,]$, $\tilde{A}(p) \in F[p]^{s\times s}$, $\tilde{B}(p) \in F[p]^{s\times r}$ is the minimal IO-relation associated with a transfer matrix $\tilde{G}(p) \in F(p)^{s\times r}$ if and only if $\tilde{A}(p)$ and $\tilde{B}(p)$ are left coprime.*

Proof. The first statement is a direct consequence of Proposition 2. Suppose now that $\tilde{A}(p)$ and $\tilde{B}(p)$ are left coprime. Assume that there exists a regular generator $[\,A(p) \vdots -B(p)\,]$ for S such that $S \subset \tilde{S}$. According to Proposition 2, there exists a non-unimodular $N(p) \in F[p]^{s\times s}$ such that $[\,\tilde{A}(p) \vdots -\tilde{B}(p)\,] = N(p)[\,A(p) \vdots -B(p)\,]$, which contradicts the assumption that $\tilde{A}(p)$ and $\tilde{B}(p)$ are left coprime. Suppose now that there exists a regular, minimal IO-relation $\tilde{S}$ associated with $G(p)$ and $\tilde{S}$ is generated by $[\,\tilde{A}(p) \vdots -\tilde{B}(p)\,]$. Assume further that $\tilde{A}(p)$ and $\tilde{B}(p)$ are not left coprime. Then $[\,\tilde{A}(p) \vdots -\tilde{B}(p)\,] = N(p)[\,A'(p) \vdots -B'(p)\,]$ for some $A'(p), N(p) \in F[p]^{s\times s}$, $B'(p) \in F[p]^{r\times s}$, and $N(p)$ is not unimodular. However, according to Proposition 2, this implies that $S' \subset \tilde{S}$, which leads to a contradiction. ∎

Finally, the existence of the minimal IO-relation for a transfer matrix $G(p) \in F(p)^{s\times r}$ can be shown with a constructive proof. For a reference, see, e.g. (Ylinen, 1975), where the construction is done by using the Smith-McMillan form of $G(p)$.

Note that the minimality of transfer equivalent relations can be studied in a more general, behavioural framework without dividing signals into inputs and outputs (Oberst, 1990). The minimality of the IO-relation associated with a transfer matrix $G(p)$ is an important concept in the sense that a non-minimal IO-relation can be decomposed into two subsystems, a minimal 'controllable' subsystem and an 'uncontrollable' subsystem, which is not affected by the input signals $u \in X^r$ (Blomberg and Ylinen, 1983; Willems, 1991).

A concrete (one-dimensional) example of a pair $(F[p], X)$ that satisfies the assumptions for the construction of the lattice structure is $(\mathbb{C}[\mathrm{d}/\mathrm{d}t], C^\infty(\mathbb{R}))$, where $C^\infty(\mathbb{R})$ is the space of infinitely differentiable complex-valued functions on $\mathbb{R}$ (Blomberg and Ylinen, 1983). However, if the subspace of compactly supported functions in $C^\infty(\mathbb{R})$, $C_c^\infty(\mathbb{R})$, is taken as X, the construction is not valid any more because in this space for arbitrary $a(\mathrm{d}/\mathrm{d}t) \in \mathbb{C}[\mathrm{d}/\mathrm{d}t]$, $x \in C_c^\infty(\mathbb{R})$, $a(\mathrm{d}/\mathrm{d}t)x = 0$ implies that either $a(\mathrm{d}/\mathrm{d}t) = 0$ or $x = 0$. Furthermore, it is quite easy to show that $a(\mathrm{d}/\mathrm{d}t) \in \mathbb{C}[\mathrm{d}/\mathrm{d}t]$ is an epimorphism on X if and only if $a(\mathrm{d}/\mathrm{d}t) \in \mathbb{C} - \{0\}$. Thus this structure contradicts the 'automorphism condition' in Proposition 2.

The propositions above pose a rigorous connection between IO-relations and their polynomial matrix descriptions. In particular, the effects of the manipulation of polynomial matrices on the corresponding IO-relations can be studied and 'safe' methods can be developed. On the other hand, the properties of the IO-relations can be tested using polynomial matrix manipulations.

4. Modules of Fractions

In this section, we analyse when it is possible to extend the scalar ring $F[p]$ of a module X to its field of fractions $F(p)$ or, more generally, to a *ring of fractions*. This material depends heavily on the results presented in (Northcott, 1968). Consider a subset D of a ring $F[p]$ (F is an arbitrary field) and assume that D is closed under multiplication and $0 \notin D$. To shorten the notation, we also assume that $1 \in D$. Then $F[p]$ can be extented to the ring $F[p]_D$ of fractions, where the elements are equivalence classes of the form $b(p)/a(p)$, $b(p) \in F[p]$, $a(p) \in D$. Two equivalent classes $b(p)/a(p), d(p)/c(p)$ are equal if and only if there exists $s(p) \in D$ such that $s(p)(c(p)a(p) - B(p)d(p)) = 0$. Addition and multiplication are defined as

$$b(p)/a(p) + d(p)/c(p) = \big(c(p)b(p) + a(p)d(p)\big)/\big(a(p)c(p)\big) \quad (16)$$

and

$$(b(p)/a(p))\,(d(p)/c(p)) = \big(b(p)d(p)\big)/\big(a(p)c(p)\big), \quad (17)$$

respectively. $F[p]$ can be embedded in $F[p]_D$ with the morphism $j_D : F[p] \to F[p]_D, a(p) \mapsto a(p)/1$.

Consider next the Abelian group X, which is a left module over $F[p]$. When $F[p]$ is extended to $F[p]_D$ then X has to be extented also to X_D, where an element of X_D is an equivalence class of the form $x/a(p)$,

$x \in X$ and $a(p) \in D$. Furthermore, equivalence classes $y/a(p) \in X_D$ and $x/b(p) \in X_D$ are equal if and only if there exists $c(p) \in D$, so that $c(p)(b(p)x - a(p)y) = 0$. The addition is defined in X_D as

$$x/a(p) + y/b(p) = \big(b(p)x + a(p)y\big)/\big(a(p)b(p)\big). \quad (18)$$

In order to make X_D a left module over $F[p]_D$, the scalar multiplication is defined as

$$(b(p)/a(p))\,(x/c(p)) = \big(b(p)x\big)/\big(a(p)c(p)\big). \quad (19)$$

X_D is also a $F[p]$ module and the mapping $j : X \to X_D, x \mapsto x/1$ is a morphism of $F[p]$-modules. Clearly, the embedding j is a monomorphism if and only if every $a(p) \in D$ is a monomorphism. In this case, X can be considered as a submodule of the $F[p]$-module X_D. Furthermore, j is an epimorphism if and only if every $a(p) \in D$ is an epimorphism. Thus X can be identified with X_D if and only if every $a(p) \in D$ is an isomorphism of $F[p]$-modules.

Note that if $D = F[p] - \{0\}$, then X_D is a vector space over the field F_D, and if each element in $F[p] - \{0\}$ is a monomorphism, X can be embedded in X_D. However, if we wish to consider X as a vector space over $F(p)$, we have to give for an arbitrary $x/a(p), x \in X$ and $a(p) \in F[p] - \{0\}$ the meaning $x/a(p) = a(p)^{-1}x$ (this vector space structure is quite commonly used in the control literature), where $a(p)^{-1}$ is the inverse of $a(p)$ as an endomorphism of X. Hence we have to further assume that each $a(p) \in F(p)$ is an epimorphism, otherwise for $x \notin \mathcal{R}(a(p))$, $a^{-1}x$ would not be well defined.

Consider now a SISO IO-relation $S \subset X \times X$ which is generated by $[\,a(p) \vdots -B(p)\,] \in F[p] \times F[p]$. If we cannot identify X with X_D, but we want to use the embedding $x \mapsto x/1$, an interesting problem would be to analyse whether there exists an IO-relation S_D in $X_D \times X_D$ such that the diagram of composite relations in Fig. 2 commutes, i.e. $j \circ S = S_D \circ j$, and how we can construct it. Note that the commutative diagrams are usually used for functions but the generalization to relations is straightforward. For the problem above we have the following result:

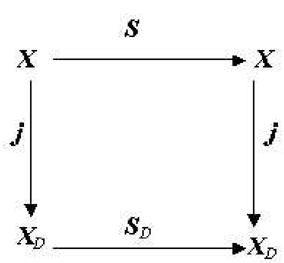

Fig. 2. Commutative relation diagram.

Proposition 4. *Consider the relation diagram of Fig. 2.*

(i) $j \circ S \subset S_D \circ j$ if and only if for every $(u, y) \in X \times X$ we have $(u, y) \in S \Rightarrow (u/1, y/1) \in S_D$.

(ii) The diagram is commutative, i.e. $j \circ S = S_D \circ j$, if for every $(u, y) \in X \times X$ we have $(u, y) \in S \iff (u/1, y/1) \in S_D$.

(iii) If the diagram is commutative and every $d(p) \in D$ is a monomorphism, then for every $(u, y) \in X \times X$ we have $(u, y) \in S \iff (u/1, y/1) \in S_D$.

Proof. (i) Suppose first that $j \circ S \subset S_D \circ j$, and take an arbitrary $(u, y) \in S$. Then $(u, y/1) \in j \circ S$, and further $(u, y/1) \in S_D \circ j$. Thus there exists $x/1$ such that $u/1 = x/1$ and $(x/1, y/1) \in S_D$, which implies that $(u/1, y/1) \in S_D$.

Conversely, suppose that $(u, y) \in S \Rightarrow (u/1, y/1) \in S_D$. Then an arbitrary $(u, y/1) \in j \circ S$ if and only if there exists a z such that $(u, z) \in S$ and $y/1 = z/1$. Consequently, there exists a z such that $(u/1, z/1) \in S_D$ and $y/1 = z/1$, which is equivalent to $(u/1, y/1) \in S_D$ and finally to $(u, y/1) \in S_D \circ j$.

(ii) Suppose next that $(u, y) \in S \iff (u/1, y/1) \in S_D$. Now an arbitrary $(u, y/1) \in j \circ S$ if and only if there exists a z such that $(u, z) \in S$ and $y/1 = z/1$ i.e. if and only if there exists a z such that $(u/1, z/1) \in S_D$ and $y/1 = z/1$. According to the previous paragraph, this is equivalent to $(u, y/1) \in S_D \circ j$.

(iii) Suppose that $j \circ S = S_D \circ j$. From (i) it follows that $(u, y) \in S \Rightarrow (u/1, y/1) \in S_D$. Take an arbitrary $(u/1, y/1) \in S_D$. We have $(u, y/1) \in S_D \circ j = j \circ S$, which implies that there exists a z such that $(u, z) \in S$ and $y/1 = z/1$. If every $d(p) \in D$ is a monomorphism then j is a monomorphism and, consequently, $y/1 = z/1 \Rightarrow y = z$. Thus $(u, y) \in S$. ■

Note that the condition $(u, y) \in S \Rightarrow (u/1, y/1) \in S_D$ is satisfied if S_D is chosen as

$$S_D = \big\{\big(u/c(p), y/d(p)\big) \in X_D \times X_D \mid a(p)y/d(p) = b(p)u/c(p)\big\}. \quad (20)$$

On the other hand, if $(u/1, y/1) \in S_D$, then also $(u/1, (y + z)/1) \in S_D$, where $d(p)z = 0$ for some $d(p) \in D$. This means that when the original IO-relation S is embedded into the structure of a module of fractions, signal pairs $(0, z)$ with $d(p)z = 0$ for some $d(p) \in D$ are added to the original pairs $(u, y) \in S$.

It is also easy to show that $[\,a(p) \vdots - B(p)\,]$ can be replaced by an arbitrary *transfer equivalent* generator $[\,a'(p) \vdots - b'(p)\,]$ satisfying

$$e(p)\begin{bmatrix} a(p) & \vdots & -B(p) \end{bmatrix} = e'(p)\begin{bmatrix} a'(p) & \vdots & -b'(p) \end{bmatrix},\ e(p), e'(p) \in D. \quad (21)$$

For example, if $p \triangleq \mathrm{d}/\mathrm{d}t$ and $X = C^\infty(\mathbb{R})$, then an arbitrary polynomial in $\mathbb{C}[p] - \{\mathbb{C}\}$ has a non-zero kernel, i.e. it is not a monomorphism. Thus the extension of the original IO-relation S by signal pairs $(0, z)$ with $d(p)z = 0$ for some $d(p) \in D$ is not a very consistent description of the original relation S, especially if D contains polynomials having zeros with positive real parts in $\mathbb{C}$. This can be avoided by making the denominators $d(p) \in D$ monomorphisms, e.g. by setting the initial conditions to zero. More formally, it can be done by restricting the original signal space X to the space

$$X_{0|0)} \triangleq \{x \mid x \in X,\, t < 0 \Rightarrow x(t) = 0\}\,. \qquad (22)$$

Because of this restriction, all non-zero polynomials are monomorphisms, so that the original IO-relation S becomes a mapping $X_{0|0)} \to X_{0|0)}$.

5. Modules of Fractions of Multidimensional Systems

In this section, we consider the possibility of using the algebraic machinery developed in previous sections in a multi-dimensional relation. So assume that we have a polynomial ring $F[p_1, p_2, \ldots, p_n] = F[\mathbf{p}]$, F is a field and $F[\mathbf{p}]$ can be considered as a subring of the ring of endomorphisms of a given Abelian group X. Our IO-relations are then of the form

$$S = \big\{(u, y) \in X^r \times X^s) \mid A(\mathbf{p})y = B(\mathbf{p})u\big\}, \qquad (23)$$

where $A(\mathbf{p}) \in F[\mathbf{p}]^{s\times s}, B(\mathbf{p}) \in F[\mathbf{p}]^{s\times r}$. Further suppose that the operators $\{p_1, p_2, \ldots, p_n\}$ are indeterminates, i.e. each polynomial in $F[\mathbf{p}]$ has a unique representation. This further implies that $F[\mathbf{p}]$ is an integral domain. However, it is not a principal ideal domain but a Noetherian domain.

Now we would like to pick up one of the operators, say p_n, and extend the original IO-relation 'against' this operator to the module of fractions over the principal ideal domain $F(p_1, p_2, \ldots, p_{n-1})[p_n]$ and to use the lattice structure constructed in Section 3 to analyse our system. According to the theory presented in previous sections, the extension keeps the IO-relation if each non-zero element in $F[p_1, p_2, \ldots, p_{n-1}]$ is an automorphism of X. Note, however, that if we do not want to keep the original IO-relation but accept its extension to the module of fractions X_D with $D = F[p_1, p_2, \ldots, p_{n-1}] - \{0\}$, the denominators do not need to be automorphisms.

Furthermore, in order to utilise our lattice structure for an arbitrary IO-relation defined as in (23), the following assumptions have to hold:

- The IO-relation has to be regular, i.e. $\det A(\mathbf{p}) \neq 0$.
- Each non-zero element of $F(p_1, p_2, \ldots, p_{n-1})[p_n]$ is an epimorphism.
- An element of $F(p_1, p_2, \ldots, p_{n-1})[p_n]$ is an automorphism if and only if it belongs to $F(p_1, p_2, \ldots, p_{n-1}) - \{0\}$.

One direct application of this algebraic system would be partial differential equations with constant coefficients (Hätönen and Ylinen, 2000). For example, suppose that X is the space $C^\infty(D)$ of infinitely differentiable complex-valued functions on an open set $D \subset \mathbb{R}^2$, $(x, t) \mapsto u(x, t)$, $p_1 = p_x \triangleq \partial/\partial x$, $p_2 = p_t \triangleq \partial/\partial t$ and $F = \mathbb{C}$. Now, the space X (over $\mathbb{C}$) can be extended to the vector space $X_{\mathbb{C}[p_x]^*}$ over $\mathbb{C}(p_x)$. The differentiation p_t can be extended to this space by

$$p_t\,(u/a(p_x)) = p_t u/a(p_x). \qquad (24)$$

Because, in general, the polynomials $a(p_x) \neq 0$ are not automorphisms of X, X itself cannot be considered as a subspace of $X_{\mathbb{C}[p_x]^*}$. However, if D is square, i.e. of the form $D_1 \times D_2$, and X is restricted, e.g. to the space

$$X_{0|x_0)} \triangleq \{u \mid u \in X, x < x_0 \Rightarrow u(x, t) = 0\} \qquad (25)$$

for some $x_0 \in D_1$, then for each (restricted) polynomial $a(p_x) \neq 0$ the equation

$$a(p_x)y = u, \quad y, u \in X_{0|x_0)} \qquad (26)$$

has a unique solution $y(\cdot, t)$ for each t and $u(\cdot, t)$. Furthermore, this solution is infinitely differentiable with respect to t. Hence each $a(p_x) \neq 0$ is an automorphism and $X_{0|x_0)}$ can be considered as a vector space over $\mathbb{C}(p_x)$.

Thus $X_{0|x_0)}$ can be considered as a module over $\mathbb{C}(p_x)[p_t]$ and the structure is analogous to the structure of time-invariant ordinary differential polynomials.

6. Analysis

Suppose again that X consists of (real or complex-valued) functions on an appropriate space-time domain and at least one of the dimensions of X is time or, more definitely, a suitable *time-interval* $T \subset \mathbb{R}$. Suppose further that p is the chosen basic operator and the other operators are included into the coefficient ring which, further, is extended to the field F of fractions in the way presented in previous sections. If the signal space X is such that the denominators are not automorphisms, it has to be extended to the module of fractions. However, we will not make any notational distinction between these two cases.

The *causality* of a system means that if there exists a cause-effect relationship between two variables, the future of the output is uniquely determined by the past of the system and the future of the input. This implies that if the past of the system is given, the output is uniquely determined by the input, i.e. for a given past the system is a mapping.

Thus the relation $S \subset X^r \times X^s$ generated by $[\,A(p) \vdots -B(p)\,]$ is *causal* if for each $(u,y) \in S$ and $t \in T$ the relation

$$\begin{aligned} S_{(u,y)|t)} \triangleq \{ & (u',y') \mid (u',y') \in S, \\ & (u',y') \mid (-\infty,t) \cap T \\ & = (u,y) \mid (-\infty,t) \cap T \} \end{aligned} \tag{27}$$

is the mapping $X^r_{u|t)} \to X^s_{y|t)}$, where

$$\begin{aligned} X^r_{u|t)} \triangleq \{ & u' \mid (u' \in X,\, u' \mid (-\infty,t) \cap T \\ & = u \mid (-\infty,t) \cap T \} \end{aligned} \tag{28}$$

and $X^s_{y|t)}$ is defined accordingly. Note that $S_{(u,y)|t)}$ can be written as

$$S_{(u,y)|t)} = (u,y) + S_{(0,0)|t)}. \tag{29}$$

Thus the IO-relation S is causal if the polynomial matrices are such that $A(p) \mid X^s_{0|t)}$ is a monomorphism and $\mathcal{R}B(p) \mid X^r_{0|t)} \subset \mathcal{R}A(p) \mid X^s_{0|t)}$ is not a linear mapping except if $(u,y) \mid t) = (0,0)|t)$.

Consider the example for the nD case of Section 5, i.e. the polynomials $\mathbb{C}[p_x][p_t]$. Suppose that S is a IO-relation generated by

$$a(p_x)y(\cdot,t) = u(\cdot,t), \quad t \in T, \tag{30}$$

where the input-output dependence is 'pointwise' with respect to time t. Suppose further that the boundary conditions are given by boundary values for output y and they are 'fixed', i.e. they do not depend on input u. It is natural to require that this kind of system should be causal. This implies that the IO-relation S should be a mapping, i.e. the boundary conditions are such that the morphism $a(p_x)$ is invertible. One way to reach this is the restriction of X to $X_{0|x)}$(25), i.e. the use of zero boundary values. This is not necessary because the extension to the module of fractions X_D with $D = \mathbb{C}[p_x] - \{0\}$ is possible, too. The restricted 'rational signals' are defined by $(x/1) \mid t) = x \mid t)/1$. Note, however, that in this case we accept the fact that the IO-relations describing the systems are unique only up to the transfer equivalence and their outputs can contain arbitrary additional terms z such that $d(p_x)z = 0$ for some $d(p_x) \neq 0$.

The relation generated by $[\,A(p) \vdots -B(p)\,]$ is *stable* if the solutions y of the equation

$$A(p)y = 0 \tag{31}$$

get asymptotically close to zero as time $t \to \infty$.

The relation *generated* by $[\,A(p) \vdots -B(p)\,]$ is *controllable* if $A(p)$ and $B(p)$ are *left coprime*, i.e. their common left divisors are all unimodular.

Using elementary row operations and the division algorithm, the model of an arbitrary composition described in Section 2 can be brought to an upper-triangular form (Blomberg and Ylinen, 1983; Ylinen, 1975):

$$\begin{bmatrix} A_1(p) & A_2(p) & -B_1(p) \\ 0 & A_4(p) & -B_2(p) \end{bmatrix} \begin{bmatrix} y_1 \\ y_0 \\ u_0 \end{bmatrix} = 0, \tag{32}$$

where u_0 and y_0 are the overall input and output and y_1 the internal output of the composition, respectively. If for each $(u_0,\, y_0)$ satisfying the equation

$$A_4(p)y_0 = B_2(p)u_0 \tag{33}$$

there exists a y_1 such that

$$A_1(p)y_1 = -A_2(p)y_0 + B_1(p)u_0, \tag{34}$$

the overall IO-relation S_0 determined by the composition is generated by $[\,A_4(p) \vdots -B_2(p)\,]$. Furthermore, if $A_1(p)$ is unimodular, the composition is said to be $(y_1(u_0,y_0)-)$*observable*.

7. Observer Design

Consider the composition above. The observer design problem is to construct a system $\widehat{S}$, an *observer* with two inputs y_0 and u_0 so that its output $\widehat{y}_1$ estimates y_1, i.e. the error $\widetilde{y}_1 = y_1 - \widehat{y}_1$ is as small as possible and stable regardless of the input u_0. The problem is depicted in Fig. 3. There are many different solutions to the estimation problem. The *observer* type estimators are based on the system model so that the observer model and the system model belong to the same class of systems.

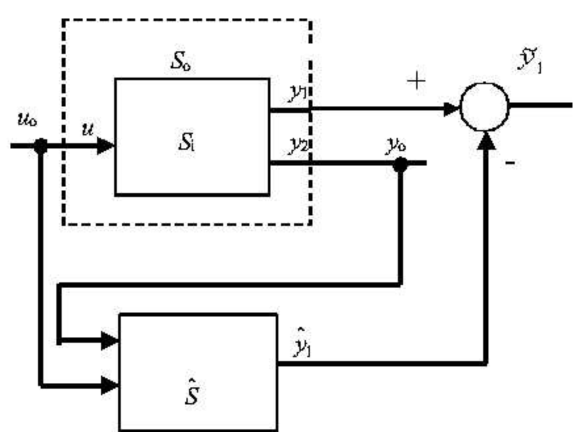

Fig. 3. Observer design problem.

It is natural to require that the correct y_1 be a possible output of the observer, i.e. if $(u,(y_1,y_2)) \in S_i$, then $((y_2,u),y_1) \in \widehat{S}$. Then Proposition 2 gives that each generator of the observer $\widehat{S}\ [\,C(p) \vdots -D_1(p)\ -D_2(p)\,]$ has to satisfy (Blomberg and Ylinen, 1983; Ylinen, 1975):

$$\begin{bmatrix} C(p) & -D_1(p) & -D_2(p) \\ 0 & A_4(p) & -B_2(p) \end{bmatrix} = \underbrace{\begin{bmatrix} T_1(p) & T_2(p) \\ 0 & I \end{bmatrix}}_{T(p)} \begin{bmatrix} A_1(p) & A_2(p) & -B_1(p) \\ 0 & A_4(p) & -B_2(p) \end{bmatrix} \tag{35}$$

for some $[\,T_1(p) \vdots T_2(p)\,]$. Furthermore, the error $\widetilde{y}_1$ satisfies

$$T_1(p)A_1(p)\widetilde{y}_1 = 0. \tag{36}$$

The behaviour of the observer should be robust with respect to parameter variations, which means that it has to be proper.

Left multiplication of $T(p)$ by another matrix of the same type results again in a matrix of the same type. Therefore the condition (35) can be used repeatedly for constructing a suitable $T(p)$. In particular, $T(p)$ can be factored to

$$\begin{bmatrix} T_1(p) & T_2(p) \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & T_2(p) \\ 0 & I \end{bmatrix} \begin{bmatrix} T_1(p) & 0 \\ 0 & I \end{bmatrix}. \tag{37}$$

Thus a candidate for the matrix $T_1(p)$ determining the error dynamics can be chosen first and then elementary row operations are used to realize the properness provided that the order of $T_1(p)A_1(p)$ is high enough. If the properness cannot be achieved, the generator is multiplied by a new $T_1(p)$ and the use of elementary row operations is continued, and so on, until a satisfactory result is obtained.

8. Feedback Controller Design

Consider next the feedback controller design for the relation S generated by $[\,A(p) \vdots -B(p)\,]$ with input u and output y. The problem is to construct a relation S_2, a *feedback controller*, with input y and output u such that the overall system behaves satisfactorily, is stable, robust, etc. The feedback composition is depicted by Fig. 4. The feedback controller is assumed to belong to the same class of relations as the controlled relation S.

It can be shown (Blomberg and Ylinen, 1983; Ylinen, 1975) that the generator $[\,C(p) \vdots -D(p)\,]$ of the feedback controller satisfies

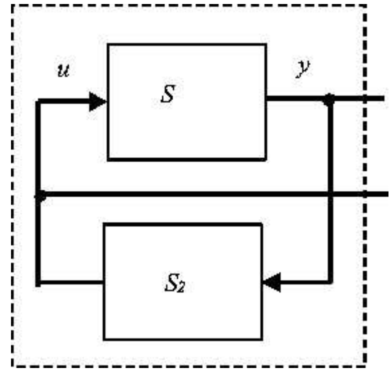

Fig. 4. Feedback composition.

$$\underbrace{\begin{bmatrix} A_1(p) & -B_1(p) \\ -D(p) & C(p) \end{bmatrix}}_{A_i(p)} = \underbrace{\begin{bmatrix} I & 0 \\ T_3(p) & T_4(p) \end{bmatrix}}_{T(p)} \underbrace{\begin{bmatrix} A_1(p) & -B_1(p) \\ X(p) & Y(p) \end{bmatrix}}_{P(p)} \tag{38}$$

for some $[\,T_3(p) \vdots T_4(p)\,]$ and a unimodular $P(p)$.

Here $[\,A_1(p) \vdots -B_1(p)\,]$ represents the controllable part of the controlled system, i.e.

$$\left[\ A(p)\ \vdots\ -B(p)\ \right] = L(p) \left[\ A_1(p)\ \vdots\ -B_1(p)\ \right], \tag{39}$$

where $L(p)$ is the greatest common left factor of $A(p)$ and $B(p)$. The so-called *first candidate* $P(p)$ satisfies

$$\left[\ A_1(p)\ \vdots\ B_1(p)\ \right] P(p)^{-1} = \left[\ I\ \vdots\ 0\ \right] \tag{40}$$

and can be constructed by applying *elementary column operations* to $[\,A_1(p) \vdots -B_1(p)\,]$.

The closed-loop behaviour of the overall system is determined by $T_4(p)$ and the uncontrollable part $L(p)$. Analogously to the condition (35), also the condition (38) can be used repeatedly. $T(p)$ can be factored to

$$\begin{bmatrix} I & 0 \\ T_3(p) & T_4(p) \end{bmatrix} = \begin{bmatrix} I & 0 \\ T_3(p) & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & T_4(p) \end{bmatrix}. \tag{41}$$

An appropriate matrix $T_4(p)$ is chosen first and then elementary row operations are applied in order to obtain a proper feedback controller. If this fails, the resulting generator is multiplied by a new $T_4(p)$, and so on.

9. Illustrative Example

Consider a cooling system where a metal wire is pulled out from a heating treatment with constant velocity, and the control problem is to manipulate the temperature distribution of the pulled wire to a desired temperature profile by utilising a cooling equipment wrapped around the pulled metal wire.

This cooling system can be roughly described by the following partial differential equation:

$$k_1 \frac{\partial T(x,t)}{\partial t} + V_x k_2 \frac{\partial T(x,t)}{\partial x} = k_3 \frac{\partial^2 T(x,t)}{\partial x^2} + k_4\big(T_{\mathrm{in}}(x,t) - T(x,t)\big), \quad (42)$$

where k_1,k_2,k_3 and k_4 are constants describing the heat transfer properties of the different materials found in the system, V_x is the constant pulling speed, $T(x,t)$ is the temperature distribution of the wire and $T_{\mathrm{in}}(x,t)$ is the temperature distribution of the cooling equipment. The initial and boundary conditions for the system are

$$T(0,t) = 1 + f(t), \;\; T(x,0) = 1, \;\; \frac{\partial T(L,t)}{\partial x} = 0, \quad (43)$$

where f is an unknown disturbance and L is the length of the cooling equipment.

In order to utilize the polynomial approach, all signals are assumed to be infinitely differentiable with respect to space x and time t, i.e. they are considered as elements of $C^\infty(\mathbb{R} \times \mathbb{R})$ even though $C^\infty((0,L) \times \mathbb{R})$ could also be possible. The two-dimensional polynomial ring $\mathbb{C}[p_x, p_t]$ is chosen as the scalar ring. Thus the equivalent representation of the system in polynomial form is

$$(a_1 p_t + a_0)\, T(x,t) = b_0 T_{\mathrm{in}}(x,t), \quad (44)$$

where

$$a_1 = k_1, \quad a_0 = -k_3 p_x^2 + V_x k_2 p_x + k_4, \quad b_0 = k_4. \quad (45)$$

9.1. Observer Design

In practical applications it is quite unrealistic to assume that the continuous temperature distribution of the pulled wire would be directly available. To this direction, in our case it is assumed that only N evenly placed temperature point measurements can be used and the continuous temperature is estimated by interpolating the temperature profile between two points. In addition there is white noise superimposed on the original measurement signals. Thus the first problem is to estimate the continuous temperature distribution based on the dynamical model in (42) before any controller design, i.e. one should calculate an estimate $T_{\mathrm{e}}(x,t)$ of the true temperature distribution $T(x,t)$ based on the measured temperature distribution $T_{\mathrm{m}}(x,t)$ and $T_{\mathrm{in}}(x,t)$, where the accuracy of $T_{\mathrm{e}}(x,t)$ should be improved using the accuracy of $T_{\mathrm{m}}(x,t)$. To take into account the effect of the 'measurement error' in the observer design, the following equation has to be added into the system described by (42):

$$T_{\mathrm{m}}(x,t) = T(x,t) + v(x,t), \quad (46)$$

where $T_{\mathrm{m}}(x,t)$ is the measured temperature profile and $v(x,t)$ describes the effect of interpolation and white noise in the measured temperature $T_{\mathrm{m}}(x,t)$. According to the theory presented in Section 7, the first candidate for the observer is the uppermost row in

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & a_1 p_t + a_0 & -b_0 \end{bmatrix} \begin{bmatrix} T_{\mathrm{e}} \\ T_{\mathrm{m}} \\ T_{\mathrm{in}} \end{bmatrix} = 0, \quad (47)$$

i.e. the estimate of the distribution would be directly the measured distribution $T_{\mathrm{m}}(x,t)$ and the initial conditions for $T_{\mathrm{e}}(x,t)$ are

$$T_{\mathrm{e}}(x,t) = T_{\mathrm{m}}(0,t), \quad \frac{\partial T_{\mathrm{e}}(x,L)}{\partial x} = \frac{\partial T_{\mathrm{m}}(0,L)}{\partial x}, \quad T_{\mathrm{e}}(x,0) = T_{\mathrm{m}}(x,0). \quad (48)$$

However, due to the measurement error, this candidate is a poor choice, and new 'poles' have to be added into the system. A natural choice for the 'pole polynomial' seems to be $a_1 p_t + a_0 + \alpha$, where α is a positive constant such that the error behaviour is stable. This gives

$$\begin{bmatrix} a_1 p_t + a_0 + \alpha & -(a_1 p_t + a_0 + \alpha) & 0 \\ 0 & a_1 p_t + a_0 & -b_0 \end{bmatrix}$$

and by adding the second row to the first row (i.e. using elementary row operations) one gets

$$\begin{bmatrix} a_1 p_t + a_0 + \alpha & -\alpha & -b_0 \\ 0 & a_1 p_t + a_0 & -b_0 \end{bmatrix}. \quad (49)$$

Note that only elementary operations of $\mathbb{C}[p_x, p_t]$ were used so that the extension to the module of fractions was not needed. From (49) the estimate can be written in the form

$$k_1 p_t T_{\mathrm{e}}(x,t) = \alpha\big(T_{\mathrm{m}}(x,t) - T_{\mathrm{e}}(x,t)\big) - V_x k_2 \frac{\partial T_{\mathrm{e}}(x,t)}{\partial x} + k_3 \frac{\partial^2 T_{\mathrm{e}}(x,t)}{\partial x^2} + k_4\left(T_{\mathrm{in}}(x,t) - T_{\mathrm{e}}(x,t)\right), \quad (50)$$

which is more or less a Kalman filter type of structure.

To evaluate how the observer performs in 'practice', the original system was simulated together with the observer. The simulations were done in the Matlab environment by discretising the partial differential equations using standard methods. The observer input was the noisy interpolated signal, as was explained earlier. On the other hand, $T(0,t)$ was selected to be a constant function that had low frequency oscillations superimposed on it. In Fig. 5 the original signal T, the noisy interpolated signal $T_{\rm m}$, and the estimated temperature profile $T_{\rm e}$ are shown versus space at a selected time point. The observer seems to work just as the theory suggested.

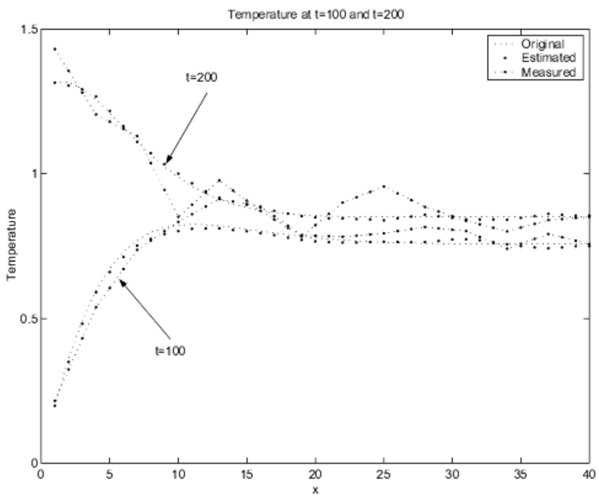

Fig. 5. Estimation of the temperature profile.

9.2. Controller Design

To demonstrate the controller design methodology, it was decided that for the controller problem in the cooling system the essential requirement is to have a temperature profile that would match as accurately as possible a reference temperature profile $T_{\rm ref}(x)$ even when there are disturbances in the incoming temperature $T(0,t)$. In the controller design it is assumed that the temperature profile of $T_{\rm in}(x,t)$ can be manipulated directly. The controller design methodology described in Section 8, however, can be used only for stabilising a system so that if the system has a non-zero output at $t=0$ and at the same time the feedback is switched on, the output of the system will settle back to the zero position asymptotically due to feedback as $t\to\infty$. A simple way to overcome this problem is to transform the original signals into difference signals from reference signals.

In Fig. 6 the resulting signal flowchart is shown, where S_1 refers to the original system, S_2 is the observer, S_3 is the controller, and M denotes the fact that only the interpolated noisy signal is available from

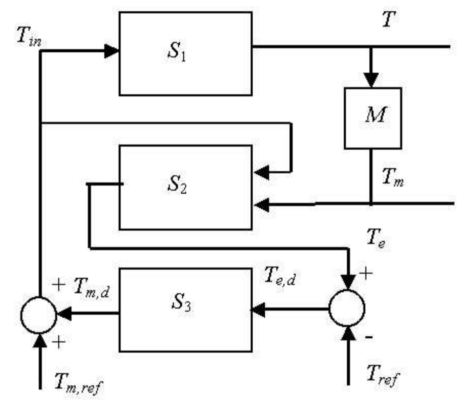

Fig. 6. Signal flowchart of the closed loop system.

the original system. The difference signals are $T_{\rm e,d} = T_{\rm e} - T_{\rm e,ref}$ and $T_{\rm in,d} = T_{\rm in} - T_{\rm in,ref}$. In order to work with the error signal, the model in (42) has to be rewritten for

$$T_d(x,t) = T(x,t) - T_{\rm ref}(x,t). \qquad (51)$$

By inserting (51) into (42) one gets

$$\begin{aligned}&\big(k_1p_t - k_3p_x^2 + k_2V_xp_x + k_4\big)\,T_{\rm d}\\ &\quad = k_4T_{\rm in,d} + k_4T_{\rm in,ref} + (k_3p_x^2 - k_2V_xp_x - k_4)T_{\rm ref}.\end{aligned}$$

The resulting system is affine (i.e. with a zero input there will be a non-zero output from the system) because of the 'exogenous' terms related to $T_{\rm ref}$ and $T_{\rm in,ref}$. In order to utilise the design procedure shown in Section 8, this constant term should be manipulated to zero. This is easily achieved by solving

$$T_{\rm in,ref} = \frac{1}{k_4}(-k_3p_x^2 + k_2V_xp_x + k_4)T_{\rm ref} \qquad (52)$$

for $T_{\rm in,ref}$. This equation can be further simplified by selecting $T_{\rm ref}(x)$ as a function that decreases (increases) linearly from $T_{\rm ref}(0)$ to $T_{\rm ref}(L)$ but satisfies the boundary conditions for T. In this case one can reduce (52) to

$$T_{\rm in,ref} = \frac{1}{k_4}(-k_2V_xc_1 - k_4T_{\rm ref}),$$

where c_1 is the slope of $T_{\rm ref}$.

Now both $T_{\rm in,d}$ and T_d can be considered as signals $u\in C^\infty(\mathbb{R}\times\mathbb{R})$ such that $u(0,t)=0$ and $\partial u(L,t)/\partial x = 0$ for all t. However, in order to get boundary conditions for making an originally unknown non-zero $a(p_x)$ a monomorphism, the signal space is restricted to the space

$$\begin{aligned}X \;=\; \Big\{ & u\in C^\infty(\mathbb{R}\times\mathbb{R}) \mid u(0,t)=0,\\ & n>0 \Rightarrow \frac{\partial^n u(L,t)}{\partial x^n} = 0\Big\}. \qquad (53)\end{aligned}$$

After transforming the original system into a suitable form for the controller design, the next step is to find a first candidate for the controller. Simple calculations show that the lowest row of

$$A(p_t)\begin{bmatrix} T_d \\ T_{\text{in,d}} \end{bmatrix} = \begin{bmatrix} a_1p_t + a_0 & -b_0 \\ -(a_1/b_0)p_t & 1 \end{bmatrix}\begin{bmatrix} T_d \\ T_{\text{in,d}} \end{bmatrix} = 0 \quad (54)$$

is a suitable candidate (because $\det A(p_t) = \text{constant}$). In order to ensure that $T_{\text{d}}(x,t)$ will asymptotically approach zero as $t \to \infty$, a new 'pole factor' $p_t + \lambda$ has to be introduced. After adding this 'pole' and two elementary row operations, the system can be written as

$$\begin{bmatrix} a_1p_t + a_0 + \lambda & -b_0 \\ -a_0 + \lambda a_1 & b_0 \end{bmatrix}. \quad (55)$$

In these operations the elementary operations of $\mathbb{C}(p_x)[p_t]$ are also needed, which means that the signal space should also be extended to the corresponding module of fractions. However, due to the scaling of variables, their boundary values are zero so that the extension is not needed.

The controller is only proper with respect to time and to increase the robustness of the system one more pole should be added. However, in order to keep the calculations simple, we settle for this proper controller. From (55) it is directly seen that the resulting controller is the solution of the following partial differential equation:

$$T_{\text{in,d}}(x,t) = \frac{1}{k_4}\Big(- k_3\frac{\partial^2 T_{\text{d}}(x,t)}{\partial x^2} + V_x k_2 \frac{\partial T_{\text{d}}(x,t)}{\partial x} - \alpha k_1 T_{\text{d}}(x,t)\Big), \quad (56)$$

where T_{d} can be replaced by the estimate $T_{\text{d,e}}$.

In Fig. 7 it is shown how the temperature distribution of the pulled wire evolves as a function of time in the simulation model. The figure displays nicely how the temperature distribution approaches asymptotically the reference distribution as a function of time.

10. Conclusions

Distributed-parameter systems with parameters varying with respect to time and space are very difficult to analyze and design. In this paper a methodology based on the generalization of the polynomial systems theory of ordinary time-invariant linear systems has been presented. The basic structure was the ring of polynomials with polynomial coefficients.

The analysis and design of a distributed-parameter control system proved the applicability of the methodology.

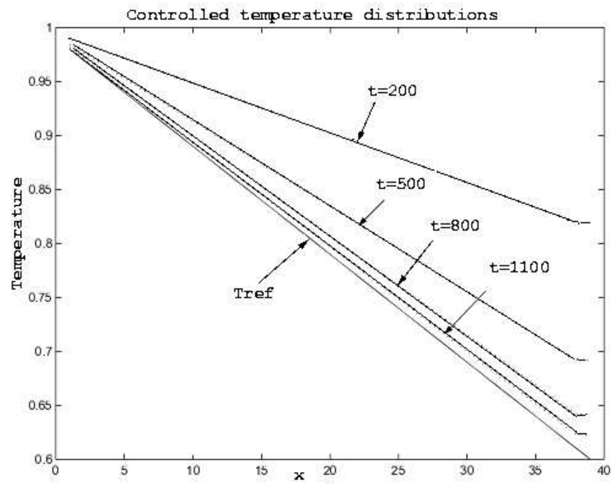

Fig. 7. Control of the temperature profile.

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