STURM-LIOUVILLE SYSTEMS ARE RIESZ-SPECTRAL SYSTEMS

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The class of Sturm-Liouville systems is defined. It appears to be a subclass of Riesz-spectral systems, since it is shown that the negative of a Sturm-Liouville operator is a Riesz-spectral operator on $L^2(a,b)$ and the infinitesimal generator of a C_0-semigroup of bounded linear operators.

Keywords: Sturm-Liouville system, Riesz-spectral system, infinite-dimensional state-space system, C_0-semigroup

1. Introduction

A fundamental concept in the analysis of distributed parameter systems are C_0-semigroups of bounded linear operators (see, e.g., (Curtain and Zwart, 1995) or (Pazy, 1983)). A typical framework is the following class of systems:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{1}$$

$$y(t) = Cx(t) + Du(t), \tag{2}$$

where x, u and y are the system state, input and output, respectively, A is a densely defined differential linear operator on an (infinite-dimensional) Hilbert space (e.g., $L^2(a,b)$, $a,\ b \in \mathbb{R}$), which generates a C_0-semigroup, and B, C and D are bounded linear operators. Moreover, if A is a Riesz-spectral operator, it possesses several interesting properties, regarding in particular observability and controllability.

In many physical systems (e.g., vibration problems in mechanics, diffusion problems), A, or $-A$, is a *Sturm-Liouville operator* (see, e.g., Renardy and Rogers, 1993, Naylor and Sell, 1982, Ray, 1981, p. 157). This is also the case for chemical reactor models with axial dispersion (see, e.g., Winkin *et al.*, 2000, Laabissi *et al.*, 2001). In order to encompass all these applications in one single unifying framework, it is natural to define the class of *Sturm-Liouville systems*. This is accomplished in Section 2.

Many theoretical results regarding Sturm-Liouville (S-L) operators or S-L problems are available in the scientific literature (see, e.g., Sagan, 1961; Birkhoff, 1962; Young, 1972; Renardy and Rogers, 1993). In Section 3 we deduce from these properties that any S-L system is a Riesz-spectral system on $L^2(a,b)$. To the authors' knowledge, the concept of the Sturm-Liouville *system* is new and so is the result concerning its connection with Riesz spectral systems, under this form. The authors would like to stress the fact that this result is obtained by gathering a number of properties that are dispersed in the literature, and expressed in a form that can be useful for system theory and control, by emphasizing the concept of Sturm-Liouville systems. Such an application to systems analysis is given in Section 4.

2. Sturm-Liouville Systems

First let us recall the definition of Sturm-Liouville operators (see, e.g., Naylor and Sell, 1982, Def. 7.5.1).

Definition 1. Consider the operator $\mathcal{A}$ defined on the domain

$$D(\mathcal{A}) = \Big\{ f \in L^2(a,b) :\ f,\ \frac{\mathrm{d}f}{\mathrm{d}z} \text{ absolutely continuous,} \quad \frac{\mathrm{d}^2 f}{\mathrm{d}z^2} \in L^2(a,b), \text{ and } \alpha_a \frac{\mathrm{d}f}{\mathrm{d}z}(a) + \beta_a f(a) = 0, \quad \alpha_b \frac{\mathrm{d}f}{\mathrm{d}z}(b) + \beta_b f(b) = 0 \Big\}, \tag{3}$$

where a and b are real numbers, $(\alpha_a, \beta_a) \neq (0, 0)$ and $(\alpha_b, \beta_b) \neq (0, 0)$. $\mathcal{A}$ is said to be a *Sturm-Liouville operator* if

$$\forall\, f \in D(\mathcal{A}),$$
$$\mathcal{A}f = \frac{1}{\rho(z)}\left(\frac{\mathrm{d}}{\mathrm{d}z}\left(-p(z)\frac{\mathrm{d}f}{\mathrm{d}z}(z)\right) + q(z)f(z)\right), \quad (4)$$

where p, $\mathrm{d}p/\mathrm{d}z$, q and ρ are real-valued and continuous functions, such that $\rho > 0$ and $p > 0$.

Note that this definition only corresponds to regular S-L problems (since a and b are assumed to be finite).

Based on the concept of the S-L operator, the class of Sturm-Liouville systems is defined as follows:

Definition 2. Consider the linear state-space system Σ defined by (1) and (2), where A is a linear operator on the Hilbert space $L^2(a, b)$ ($a,\ b \in \mathbb{R}$), B is a bounded linear operator from the Hilbert space U to $L^2(a, b)$, C is a bounded linear operator from $L^2(a, b)$ to the Hilbert space Y, and D is a bounded linear operator from U to Y. Σ is called a *Sturm-Liouville system* if $-A$ is a Sturm-Liouville operator.

Remarks:

1. One can ask why to consider $-A$ instead of A in Definition 2. Actually, if we took A to be an S-L operator, some results of the next section would not apply.

2. Many processes involving a diffusion phenomenon may be modelled by S-L systems, and convection-dispersion reactors in particular (Ray, 1981, Example 4.2.3; Laabissi *et al.*, 2001).

3. Main Result

The following theoretical result is reported:

Theorem 1. *Any Sturm-Liouville system Σ is a Riesz-spectral system.*

Proof. In view of the definition of Riesz-spectral systems (Curtain and Zwart, 1995, Def. 4.1.1), it is sufficient to prove the following lemma:

Lemma 1. *Let A be the negative of a Sturm-Liouville operator (4) defined on its domain $D(A)$ given by (3).*

Then

(i) A is a Riesz-spectral operator, and

(ii) A is the infinitesimal generator of a C_0-semigroup of bounded linear operators on $L^2(a, b)$.

Proof of Lemma 1. (i) *A is a Riesz spectral operator.* By the definition of a Riesz-spectral operator (Curtain and Zwart, 1995, definition 2.3.4), it should be shown that:

(a) A is *closed*,

(b) its eigenvalues λ_n are *simple*,

(c) $\overline{\{\lambda_n, n \in \mathbb{N}\}}$ is *totally disconnected*, i.e. $\forall a, b \in \overline{\{\lambda_n, n \in \mathbb{N}\}}, [a, b] \not\subseteq \overline{\{\lambda_n, n \in \mathbb{N}\}}$

(d) the set of the corresponding eigenvectors $\{\phi_n, n \geq 1\}$ is a *Riesz basis* of $L^2(a, b)$, i.e., it is an orthonormal basis with respect to an equivalent inner product (see Young, 1980, Theorem 9.2).

One can use the properties of an S-L operator spectrum since the eigenvalues of the S-L operator $-A$ are $-\lambda_n$ with ϕ_n as their corresponding eigenvectors.

Therefore the eigenvalues of A are real (Naylor and Sell, 1982, Theorem 7.5.6), countable and simple (Sagan, 1961, Theorem V.8), and the set of the corresponding normalized eigenvectors $\{\phi_n, n \geq 1\}$ is an orthonormal basis with respect to the equivalent inner product $\langle \cdot, \cdot \rangle_\rho$ (Sagan, 1961, Chapter 2.4):

$$\langle \phi_m, \phi_n \rangle_\rho = \int_0^1 \rho(z)\phi_m(z)\phi_n(z)\,\mathrm{d}z = 0$$

for $m \neq n$. (More recently, all these properties of S-L operators were also reported in (Renardy and Rogers, 1993, Thms. 7.96 and 7.97; Pryce, 1993, Thms. 2.3 and 2.4).)

Moreover, A is closed. Indeed, $-A$ is closed since any S-L operator is an invertible linear operator with a bounded linear inverse (Curtain and Zwart, 1995, pp. 82–83 and Thm. A.3.46).

It remains to prove that $\overline{\{\lambda_n, n \in \mathbb{N}\}}$ is totally disconnected. To this end, consider λ such that $\lambda \in \rho(A)$. Then Naylor and Sell (1982, Thm. 7.5.5) show that $(\lambda I - A)^{-1}$ is compact for any S-L operator A. Thus the spectrum of $\lambda I - A$ is discrete, i.e., it consists only of isolated eigenvalues (Curtain and Zwart, 1995, Lemma A.4.19), and so is the spectrum of A.

(ii) *A generates a C_0-semigroup.* As A is a Riesz spectral operator with simple eigenvalues λ_n, it generates a C_0-semigroup if, and only if, $\sup_{n \geq 1} \lambda_n < \infty$ (Curtain and Zwart, 1995, Thm. 2.3.5(c)). From (Sagan, 1961, Thm. V.7; Renardy and Rogers, 1993, Thm. 7.96.2), it is known that the spectrum of any S-L operator is bounded from below. Hence there exists λ_0 such that for all n, $-\lambda_n > \lambda_0$, i.e., $\lambda_n < -\lambda_0 < +\infty$. ■

Remarks:

1. Note that the second result of Lemma 1 would not hold if A was an S-L operator. Indeed, following, e.g., (Renardy and Rogers, 1993, Thm. 7.97.2),

the eigenvalues of the S-L operator A tend to $+\infty$. Then (Curtain and Zwart, 1995, Thm. 2.3.5(c)) the Riesz-spectral operator A does not generate a C_0-semigroup.

2. The link between S-L and Riesz-spectral operators was already touched upon in the literature. For example, in (Belinskiy and Dauer, 1997; Zhidkov, 2000) it is shown that the eigenfunctions of a regular Sturm-Liouville problem form a Riesz basis for particular boundary conditions.

3. The converse of Theorem 1 does not hold. Indeed, there are Riesz-spectral operators whose negatives are not Sturm-Liouville ones. For example, the system operator associated with the undamped wave equation with Dirichlet boudary conditions is a Riesz-spectral operator, see, e.g., (Curtain and Zwart, 1995, Ex. 2.3.8). This operator is defined as follows:

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5}$$

on its domain

$$D(A) = D(A_0) \oplus D(A_0^{1/2}),$$

where A_0 is the linear operator given by $A_0 x := -\mathrm{d}^2x/\mathrm{d}z^2$ on the domain

$$D(A_0) = \Big\{ x \in L^2(0,1) : f,\ \frac{\mathrm{d}f}{\mathrm{d}z} \text{ are absolutely continuous}, \frac{\mathrm{d}^2x}{\mathrm{d}z^2} \in L^2(0,1), \text{ and } x(0) = 0 = x(1) \Big\}. \tag{6}$$

However, the operator $-A$ is obviously not a Sturm-Liouville operator. This is confirmed by the fact that all the eigenvalues of this operator are located on the imaginary axis, see (Curtain and Zwart, 1995, Ex. 2.3.8).

More generally, any Riesz-spectral system whose C_0-semigroup generator spectrum does not consist of real eigenvalues is not an S-L system.

4. Case Study

Consider the linear operator $A : D(A) \to L^2(0,1)$ given by

$$Ax = D\frac{\mathrm{d}^2x}{\mathrm{d}z^2} - v\frac{\mathrm{d}x}{\mathrm{d}z} - kx, \tag{7}$$

where D, v and k are positive constants, on its domain

$$D(A) = \Big\{ x \in L^2(0,1) : x,\ \frac{\mathrm{d}x}{\mathrm{d}z} \text{ are absolutely continuous}, \frac{\mathrm{d}^2x}{\mathrm{d}z^2} \in L^2(0,1), \text{ and } D\frac{\mathrm{d}x}{\mathrm{d}z}(0) - vx(0) = 0 = \frac{\mathrm{d}x}{\mathrm{d}z}(1) \Big\}. \tag{8}$$

In (Winkin *et al.*, 2000, pp. 355–356) it is shown that $-A$ is an S-L operator of the form (4), where the functions ρ, p and q are given by

$$\rho(z) = \exp\Big(-\frac{v}{D}z\Big), \quad p(z) = D\rho(z), \quad q(z) = -k.$$

Hence any system of the form (1)–(2), where the operator A is given by (7)–(8), is an S-L system.

Such a system plays an important role in the linear dynamical description of isothermal or non-isothermal tubular reactor models with axial dispersion, see, e.g., (Winkin *et al.*, 2000; Laabissi *et al.*, 2001). In such models the state variable x typically denotes a reactant or product concentration, whereas the parameters D, v and k denote the axial dispersion coefficient, the fluid superficial velocity and the kinetic constant, respectively.

Moreover, such a system Σ is also an *intereseting* example of a Riesz-spectral system having a non-trivial (i.e., non-orthonormal) Riesz basis of eigenvectors, see (Winkin *et al.*, 2000, Lem. 5.1).

It is known (Winkin *et al.*, 2000) that the eigenvalues of the operator A defined by (7) and (8) are given by the following expression:

$$\lambda_n = -\frac{s_n^2 v^2}{4D} - k < -\Big(\frac{v^2}{4D} + k\Big) < 0, \tag{9}$$

where $\{s_n,\ n \geq 1\}$ is the set of all the solutions to the resolvent equation

$$\tan\Big(\frac{s}{2D}\Big) = \frac{2vs}{s^2 - v^2}, \quad s > 0,$$

such that $0 < s_n < s_{n+1}$ for all $n \geq 1$.

As any system Σ of the form (1)–(2), where the operator A is given by (7) and (8), is an S-L system, it can be deduced from Theorem 1 that it is a Riesz-spectral system. Hence, since $\sup_{n\geq 1} \lambda_n < 0$ (in view of (9)), it is easily deduced from (Curtain and Zwart, 1995, Thm. 2.3.5.d) that the operator A is the generator of an exponentially stable C_0-semigroup.

Therefore, owing to Theorem 1 and the property (Curtain and Zwart, 1995, Thm. 2.3.5.d) of Riesz-spectral operators, we are able to verify the exponential stability of the S-L system Σ more rapidly and in an easier way than it is done in (Winkin *et al.*, 2000).

5. Conclusion

In this paper the class of Sturm-Liouville systems was defined. Then it was shown that they are a subclass of Riesz-spectral systems. By performing this proof, on the basis of the properties of S-L operators it was shown that these are Riesz spectral operators and infinitesimal generators of C_0-semigroups. Although some results exist for some particular systems (e.g., Winkin *et al.*, 2000), the last theoretical result was not available in the scientific literature, to the authors' knowledge.

The immediate consequence of this result is that the properties of Riesz-spectral systems and operators (Curtain and Zwart, 1995; Kuiper and Zwart, 1993) can be used in the analysis or control of S-L systems, in particular for convection-diffusion-reaction systems.

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