ARGUMENT INCREMENT STABILITY CRITERION FOR LINEAR DELTA MODELS

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Currently used stability criteria for linear sampled-data systems refer to the standard linear difference equation form of the system model. This paper presents a stability criterion based on the argument increment rule modified for the delta operator form of the sampled-data model. For the asymptotic stability of this system form it is necessary and sufficient that the roots of the appropriate characteristic equation lie inside a circle in the left half of the complex plane, the radius of which is inversely proportional to the sampling period. Therefore the argument increment of the system characteristic polynomial of an asymptotically stable delta model has to increase by $2\pi n$ if this circle has been run around in the counter-clockwise direction. The criterion developed based on this principle permits not only the proof of the system stability itself, but also the approximation of the dominant roots of its characteristic equation.

Keywords: delta model, stability criterion, conformal mapping, delta operator, argument increment rule

1. Introduction

For the purposes of the analysis and synthesis of process control, continuous or sampled-data linear models are mostly used. The Laplace transform or the Z-transform are usually preferred when theoretical results are to be obtained (Ogata, 1995). Assuming zero initial conditions, the Laplace transform variable formally corresponds to the derivative operator with respect to time. In a similar manner, with zero initial conditions, the Z-transform variable formally corresponds to the shifting operator of the sampling period T. The completely different principles of the two operators imply that only a weak analogy between the two transforms can be expected. Specifically, it is impossible to consider a limit conversion from the sampled-data to the continuous representation of dynamic systems on the basis of Z-transform domain, since the differences between the samples approach zero as $T \to 0$. For this reason, the delta transform (Bobál *et al.*, 1999; Feuer and Goodwin, 1996; Middleton and Gooddwin, 1989) allowing convenient conversion into a continuous representation as $T \to 0$ has been developed.

Similarly to a sampled-data model described in terms of the shifting operator q, the δ operator (forward difference) can be applied to obtain a representation of the sampled-data model. For a sequence of samples $\{f_k\}$ in discrete time k, the δ operator is defined (Feuer and Goodwin, 1996; Middleton and Gooddwin, 1989) as follows:

$$\delta f_k = \frac{f_{k+1} - f_k}{T}. \tag{1}$$

Applying the shifting operator q to the sequence (1) yields

$$\delta f_k = \frac{q-1}{T} f_k \tag{2}$$

or $\delta = (q-1)/T$, and therefore

$$q = 1 + T\delta. \tag{3}$$

Remark 1. The stability of a linear system does not depend on the initial conditions. If the zero initial conditions are considered, the models described in terms of the Z-transform and in terms of the shifting operator q are of the same formal form. Similarly, the models described in terms of the delta transform or in terms of the δ operator (forward difference) are formally the same.

2. Linear Delta Model

A sampled-data system given by a linear difference equation of the n-th order

$$\tilde{a}_n\, y_{k+n} + \tilde{a}_{n-1}\, y_{k+n-1} + \cdots + \tilde{a}_0\, y_k = \tilde{b}_0\, u_k + \tilde{b}_1\, u_{k+1} + \cdots + \tilde{b}_{n-1}\, u_{k+n-1} \tag{4}$$

containing constant real coefficients $\tilde{a}_i, \ \ i = 0, 1, \ldots, n$ and $\tilde{b}_j$, $j = 0, \ldots, n-1$ with its input data sequence $\{u_k\}$ and an output data sequence $\{y_k\}$ can be written down with the use of the q operator as

$$\tilde{a}_n\, q^n y_k + \tilde{a}_{n-1}\, q^{n-1} y_k + \cdots + \tilde{a}_0\, y_k = \tilde{b}_0\, u_k + \tilde{b}_1\, q\, u_k + \cdots + \tilde{b}_{n-1}\, q^{n-1} u_k. \tag{5}$$

After the application of the formula (3), the following delta model is obtained:

$$a_n\,\delta^n y_k + a_{n-1}\,\delta^{n-1} y_k + \cdots + a_0\,y_k = b_0\,u_k + b_1\,\delta u_k + \cdots + b_{n-1}\,\delta^{n-1} u_k, \quad (6)$$

where the coefficients $a_i,\ i = 0, 1, \ldots, n$ and $b_j,\ j = 0, \ldots, n-1$ result from (3) and (5), and the left-hand side coefficients are

$$a_i = \sum_{j=i}^{n} \binom{i+j}{i} \tilde{a}_j\,T^i. \quad (7)$$

3. Stability of the Linear Delta Model

The stability of a sampled-data system in the conventional form (4) can be proved by well-known criteria. However, the delta form (6) results in a different characteristic equation and therefore in a different form of the stability condition as well.

Definition 1. The delta model (6) is *asymptotically stable* if the solution y_k of the homogeneous equation

$$a_n\,\delta^n y_k + a_{n-1}\,\delta^{n-1} y_k + \cdots + a_0\,y_k = 0, \quad (8)$$

n being the order of the delta model, is a convergent sequence with the property

$$\lim_{k\to\infty} y_k = 0. \quad (9)$$

Any particular solution of the homogeneous equation (8) is of the general form

$$y_k = Ck^i(1+T\lambda)^k, \quad i = 0, 1, \ldots, \nu - 1, \quad (10)$$

where C is a coefficient (complex-valued in general), k is discrete time, T is the sampling period and λ is a root of the characteristic equation

$$H(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0, \quad (11)$$

complex-valued in general, and ν is the multiplicity of this root.

Assuming for simplicity that there are n distinct and single roots $\lambda_i,\ i = 1, 2, \ldots, n$ of the characteristic equation, the general solution of the homogeneous equation (8) can be expressed in the form

$$y_k = \sum_{i=1}^{n} C_i\left(1 + T\,\lambda_i\right)^k, \quad (12)$$

where $C_i,\ i = 1, 2, \ldots, n$ are complex or real coefficients. With reference to (Feuer and Goodwin, 1996; Middleton and Gooddwin, 1989), the following stability condition of a linear delta model results from (9) and (12):

Lemma 1. *Let $H(\lambda)$ be the characteristic polynomial of the delta model (6). If $\lambda_i,\ i = 1, 2, \ldots$ are the zeros of $H(\lambda)$, either single or multiple, then the system (6) is asymptotically stable if and only if the following inequality holds:*

$$|1 + T\,\lambda_i| < 1 \quad (13)$$

for each of the roots λ_i. In other words, the condition (13) means that all the roots $\lambda_i,\ i = 1, 2, \ldots$ of the characteristic equation (11) of a stable system lie inside the circle with centre at $S \equiv [-T^{-1}, 0]$ and radius T^{-1}, see Fig. 1.

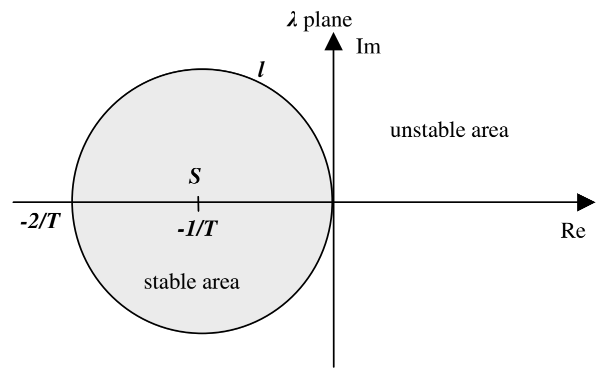

Fig. 1. Stability domain.

4. Argument Increment Stability Criterion

In this section, a stability criterion for delta models based on the argument increment will be introduced. This criterion for the stability proof of linear sampled-data systems even with transport delay was published in (Zítek and Petrova, 2001). Further in this paper, the criterion is used not only for the stability assessment, but also for estimating the roots of the characteristic equation (11). In fact, the introduced stability criterion is a modification of the contour stability criterion (Chemodanov, 1977; Zítek, 1990; 2001) utilized for the stability proof of a sampled-data dynamic system represented by the linear difference equation of the form (5). If a linear sampled-data dynamic system in the delta model form (6) is represented by its characteristic polynomial $H(\lambda)$ as in (11), the $H(\lambda)$ argument increment evaluation can be applied in its stability check.

As the n-th order polynomial $H(\lambda)$ is a holomorphic function with n zeros $\lambda_i,\ i = 1, 2, \ldots, n$, given as the roots of the characteristic equation (11), and can be expressed in the factorized form

$$H(\lambda) = a_n\,(\lambda - \lambda_1)\,(\lambda - \lambda_2)\,\ldots\,(\lambda - \lambda_n). \quad (14)$$

Following the rule of complex multiplication, the $H(\lambda)$ argument is given as the sum of partial argument increments appropriate to the particular root factors $\lambda - \lambda_i,\ i = 1, 2, \ldots$, i.e.,

$$\Delta \arg H(\lambda) = \sum_{i=1}^{n} \Delta \arg(\lambda - \lambda_i). \tag{15}$$

Let a circle l be given by the following formula:

$$l = \left\{ \lambda \mid \lambda = \frac{1}{T}\big(\exp(j\,\varphi) - 1\big),\, T > 0, \varphi \in [0, 2\pi] \right\}, \tag{16}$$

representing the δ model stability boundary in the λ complex plane. If the complex variable λ has circumscribed the circle (16) in the counter-clockwise direction, see Fig. 2, then for λ_i located *inside* this circle we have

$$\Delta_l \arg(\lambda - \lambda_i) = 2\pi, \tag{17}$$

and for λ_j located *outside* we get

$$\Delta_l \arg(\lambda - \lambda_j) = 0. \tag{18}$$

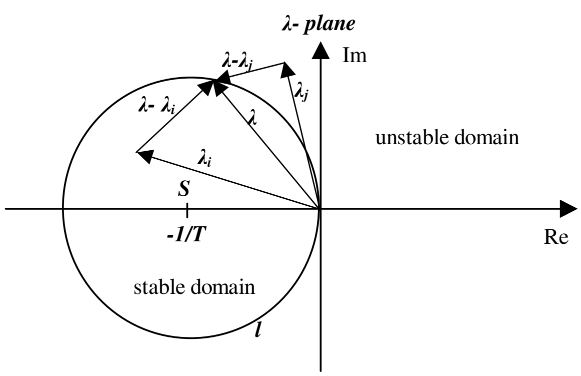

Fig. 2. One possible placement of the zeros of $H(\lambda)$ as in (11).

If the linear delta model (6) is stable, all the roots of the characteristic equation (11) must be located inside the circle l and, therefore, the following stability criterion holds:

Theorem 1. *A linear system described by the delta model (6) is asymptotically stable if and only if the $H(\lambda)$ argument increment attains the angle*

$$\Delta_l \arg H(\lambda) = 2\pi\, n \tag{19}$$

when λ given by (16) runs around the circle l in the counter-clockwise direction.

Proof. The required argument increment $\Delta_l \arg H(\lambda)$ results directly from the property (17) and from the necessary and sufficient condition that all the roots of (11) lie inside the circle l, given by $\lambda = (\exp(j\varphi) - 1)/T$. If any of the roots lies outside this circle, the argument is less than $n2\pi$ (necessity). On the other hand, if n of $H(\lambda)$ zeros, distinct or identical (i.e., multiple) lie inside l, then the argument increment is $n2\pi$ (sufficiency).

The evaluation of the $H(\lambda)$ argument results in

$$\begin{aligned} H(\lambda)|_{\lambda = \frac{1}{T}(\exp(j\varphi)-1)} &= H\left(\frac{1}{T}\ (\exp(j\,\varphi) - 1)\right) \\ &= \sum_{i=0}^{n} a_i \left(\frac{1}{T}\ (\exp(j\,\varphi) - 1)\right)^i, \end{aligned} \tag{20}$$

i.e., apparently in a periodic function $H^*(\varphi)$ with period 2π, which corresponds to a closed contour. This contour corresponding to the interval $\varphi \in [0, 2\pi]$ is the so-called hodograph and the argument increment can be determined from the number of its loops. The hodograph evaluation can be simplified in the following manner. Since the complex conjugates λ and $\bar{\lambda}$ result in complex conjugates $H(\lambda)$ and $H(\bar{\lambda})$ again, the second half of the hodograph is symmetrical with respect to the real axis with its first half obtained for $\varphi \in [0, \pi]$. For this reason, for deciding on the system stability it is sufficient to calculate only the first half of the hodograph, which is appropriate to the interval $\varphi \in [0, \pi]$, and to reduce the argument increment condition to the following requirement:

$$\underset{\varphi \in [0,\pi]}{\Delta} \arg H\left(\frac{1}{T}\ (\exp(j\,\varphi) - 1)\right) = n\,\pi. \tag{21}$$

Example 1. Using the stability criterion (21), we wish to check the stability of the linear delta model given by the homogeneous equation

$$\delta^4 y_k + 1.8\delta^3 y_k + 2.42\delta^2 y_k + 1.674\delta y_k + 0.6649 y_k = 0, \tag{22}$$

where the sampling period is $T = 0.5\,s$.

The characteristic polynomial of (22) is as follows:

$$H(\lambda) = \lambda^4 + 1.8\lambda^3 + 2.42\lambda^2 + 1.674\lambda + 0.6649. \tag{23}$$

By substituting $(\exp(j\,\varphi) - 1)/T$ for λ and by sketching the hodograph of the function $H((\exp(j\,\varphi) - 1)/T)$ over the interval $\varphi \in [0, \pi]$, the contour in Fig. 3 is obtained. The details of this contour close to the origin are given in Fig. 4. The argument increment of the function $H((\exp(j\,\varphi) - 1)/T)$ appropriate to the interval $\varphi \in [0, \pi]$ is of 4π and, therefore, it follows that

$$\underset{\varphi \in [0,\pi]}{\Delta} \arg H\left(\frac{1}{T}\ (\exp(j\,\varphi) - 1)\right) = 4\pi. \tag{24}$$

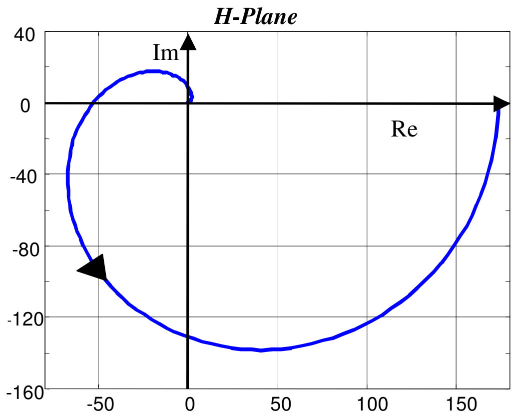

Fig. 3. Hodograph for stability assessment.

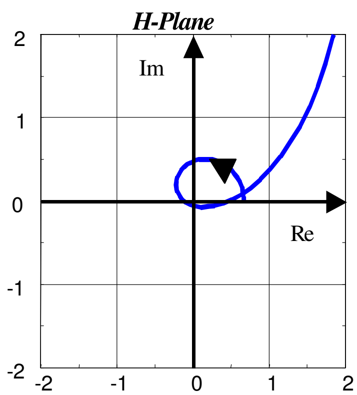

Fig. 4. Hodograph close to its origin in detail.

Because of the fourth order of the characteristic polynomial (23), i.e., $n = 4$, the condition (21) is satisfied and this linear delta model is stable.

In this case, it is easy to check the obtained solution by the evaluation of the roots of the characteristic polynomial (23), the zeros of which have the following values:

$$\lambda_{1,2} = -0.3 \pm j, \quad \lambda_{3,4} = -0.6 \pm 0.5j \qquad (25)$$

and all the values lie inside the stable domain enclosed by the circle of diameter $T^{-1} = 2$ centred at the point $S \equiv [-T^{-1}, 0] = [-2, 0]$. Thus, the stability condition (13) is satisfied. ♦

5. Estimation of the $H(\lambda)$ Zero Position

Since the polynomial $H(\lambda)$ is a holomorphic function, $H(\lambda)$ can be used not only for assessing the stability of the delta model, but also for evaluating the rate of the stability. Moreover, it even allow us to estimate the positions of some of the roots of the characteristic equation (11). These conclusions can be ascertained using the fact that a holomorphic function provides a *conformal mapping* from λ to the H complex plane. This means that two elementary lines of the same length crossing at point N within an angle β are mapped again as two lines of the same length crossing at the mapped point N within the same angle β (Zítek, 2001).

The conformal mapping of the λ-plane onto the H-plane according to the function $H = H(\lambda)$, where $H(\lambda)$ is determined by (11), is shown in Fig. 5. Using this mapping, all the roots of the characteristic equation (11) are mapped into the origin of the H-plane $O \equiv [0, 0]$, i.e., $H : \lambda_i \to O$ for $i = 1, 2, \ldots, n$ (only one root $\lambda_i$is shown in Fig. 5). The circle l which encloses the stable domain is mapped into a curve l', i.e., $H : l \to l'$. The line r_i depicted in Fig. 5 passes through the root λ_i and through the centre point $S \equiv [-T^{-1}, 0]$ of the circle l, and intersects the circle at point M_i.

The beam r_i is then mapped onto the H-plane as a contour r_i', i.e., $H : r_i \to r_i'$, passing through the mapped points S' and M_i'. In addition, since the mapping is conformal, curves r_i' and l' intersect each other with angle $\pi/2$ at point M_i'. Moreover, in the neighbourhood of this point, the proportion is kept, i.e., the ratio of the segments measured on l and r_i in the vicinity of M_i is equal to the ratio of the corresponding segments on l' and r_i' in the vicinity of the point M_i'.

The above features of the conformal mapping can be utilized for the approximate estimation of some zeros of the characteristic polynomial $H(\lambda)$. Having obtained the contour l' given by $H((\exp(j\,\varphi) - 1)/T)$ it is possible to drop a perpendicular line from the origin $O \equiv [0, 0]$ towards the nearest part of l'. At the bottom of this perpendicular line, cf. point N_i' in Figs. 6 and 7, the value of the angle φ determines approximately the angle with which the corresponding beam r_i intersects the real axis in the λ-plane at point S. Both the length d_i' of the drawn perpendicular line corresponding to the distance between the origin O and the bottom of the perpendicular line (i.e., the distance measured from M_i' to point O) and also the length of the segment u_i' corresponding to the angle increment $\Delta\varphi_i$ on l' in the neighbourhood of the bottom N_i' in Figs. 6 and 7 can then serve as an approximation of the distance d_i between M_i and the corresponding root λ_i of the characteristic equation (11). As is depicted in Fig. 6, the following approximate relationships hold for the estimated root:

$$\lambda_i = \left(\frac{1}{T} - d_i\right) \exp(j\,\varphi_i) - \frac{1}{T}, \qquad (26)$$

$$\frac{d_i}{d_i'} \approx \frac{u_i}{u_i'} = \frac{\Delta\varphi_i/T}{u_i'}. \qquad (27)$$

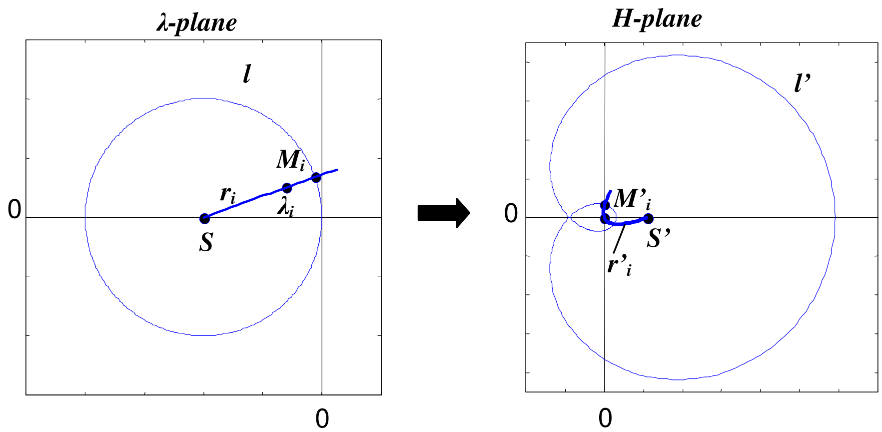

Fig. 5. Conformal mapping of the λ-plane onto the H-plane.

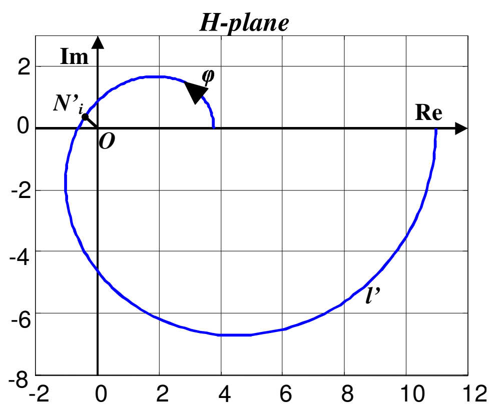

Fig. 6. Hodograph of Example 2.

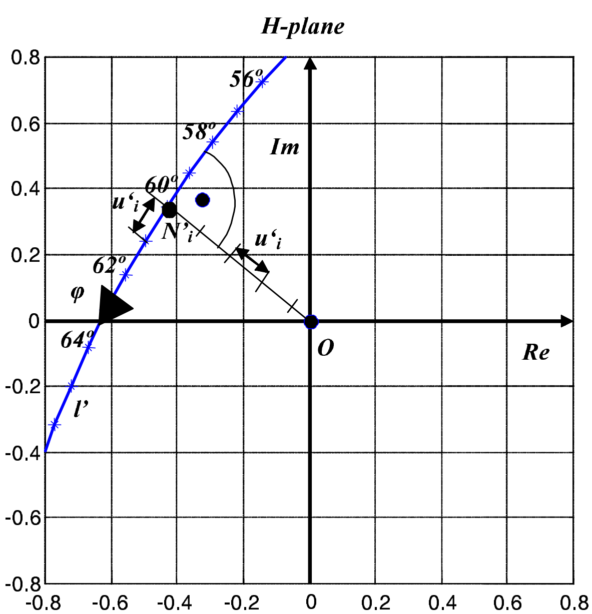

Fig. 7. Details of the hodograph in the neighbourhood of the origin.

Inserting (27) into (26), the following formula results for approximated λ_i:

$$\lambda_i = \frac{1}{T}\left[\left(1 - \frac{d_i'}{u_i'}\,\Delta\varphi\right)\exp\left(j\varphi\right) - 1\right]. \qquad (28)$$

The above approximation is based on replacing segments with straight lines. Therefore, the closer the point N_i' to the origin, the more accurate the estimation. On the contrary, this estimation method may become unsuitable for estimating the roots lying close to the point S. However, the estimation can be modified in the following way: Instead of the hodograph $H((\exp(j\,\varphi) - 1)/T)$, a mapping of a circle of a smaller radius $\rho < 1/T$ can be used, resulting in the modified hodograph $H\left(\rho\,(\exp(j\,\varphi) - 1)\right)$, and the estimation can be performed in a similar manner.

Example 2. Using the conformal mapping of the stability boundary, we wish to estimate the zeros of the characteristic polynomial (roots of the characteristic equation) of the delta model with the homogeneous equation of the form

$$\delta^2 y_k + 2.2\delta\, y_k + 3.77 y_k = 0 \qquad (29)$$

with sampling period $T = 0.5$.

The characteristic polynomial $H\left(\lambda\right)$ is given by

$$H\left(\lambda\right) = \lambda^2 + 2.2\lambda + 3.77. \qquad (30)$$

Figure 6 is obtained by substituting $(\exp(j\,\varphi) - 1)/T$ for λ into the polynomial (30) and by drawing the hodograph of the function $H((\exp(j\,\varphi) - 1)/T)$ for $\varphi \in [0, \pi]$. As indicated by the shape of the hodograph in Fig. 6, the examined linear delta model is stable because the argument increment is of 2π and $n = 2$. Thus, the condition (21) is satisfied. The stability of the delta model implies that all the roots are placed inside the area enclosed by the circle l, see Fig. 1. The roots of the characteristic equation can be estimated from Fig. 7, where the details of the hodograph curve in the neighbourhood of the origin O are depicted. The point N_i' is located by dropping the perpendicular line to l' from the origin O. The angle $\varphi = 60^\circ = \pi/3$ at the bootom of this perpendicular line approximately determines the angle φ_i with which the corresponding beam r_i and the real axis intersect. The point N_i' is an estimate of the M_i' location. The distance between the origin O and N_i' and the length of the segment u_i' on l' corresponding to the increment $\Delta\varphi_i$ in the vicinity of N_i' yields estimates of λ_i and M_i. For the chosen argument increment of $\Delta\varphi_i = 1^\circ = \pi/180$ in the introduced example, the corresponding segment of l' has the length u_i' and the distance between N_i' and O is d_i'. From Fig. 7 it can be seen that $d_i' \approx 4.5\, u_i'$. Therefore, after substitution into (28), the root estimate is

$$\lambda_i = \frac{1}{0.5}\left[\left(1 - 4.5\frac{\pi}{180}\right)\exp\left(j\frac{\pi}{3}\right) - 1\right] = -1.0785 + 1.596j. \tag{31}$$

Hence the resulting complex conjugate pair of roots is estimated as $\lambda_{1,2} = -1.0785 \pm 1.596j$. By comparing this estimate with the actual roots of the characteristic equation $\lambda_{1,2} = -1.1 \pm 1.6j$, it is evident that the estimation is rather accurate due to a relatively small distance between the estimated roots and the circle l. ♦

6. Damping and the Decay Ratio

To assess the oscillatory behaviour of delta models, the formula (12) can be used, from which it is apparent that the i-th particular solution

$$^{i}y_k = C_i(1 + T\,\lambda_i)^k \tag{32}$$

of the homogeneous equation (9) is monotonous for a real and single λ_i, and for a complex conjugate pair of λ_i it results in an oscillating sequence. In order to obtain the final homogeneous solution (12) as not oscillating, all the particular solutions have to be monotonous.

In the case of the oscillating i-th particular solution, when $\arg(1 + T\,\lambda_i) \neq 0$, the oscillation period k_{p_i}, expressed as the number of sampling periods T, is given as a real number:

$$k_{p_i} = \frac{2\,\pi}{\arg(1 + T\,\lambda_i)}. \tag{33}$$

The oscillation decay rate, η_i, as the ratio of two successive amplitudes, is determined by the formula

$$\begin{aligned}\eta_i &= \frac{^{i}y_{k+k_{p_i}}}{^{i}y_k} \\ &= \frac{C_i\ |1 + T\,\lambda_i|^{k+k_{p_i}}\ \exp(j\arg(1 + T\,\lambda_i)\,(k + k_{p_i}))}{C_i\ |1 + T\,\lambda_i|^{k}\ \exp(jk\arg(1 + T\,\lambda_i))} \\ &= |1 + T\,\lambda_i|^{k_{p_i}}.\end{aligned} \tag{34}$$

From (34) it is clear that the closer the root λ_i of the characteristic equation (11) to the circle l, the lower the decay results.

7. Conclusions

Due to the similarity in spectral characteristics, the delta models are advantageous in applications where a sampled-data model serves as an approximation to an originally continuous model. The presented criterion does not serve as a stability check only, but it provides an effective method of investigating the dynamic properties of sampled-data systems described in the delta form. Properties such as the frequency of natural oscillations or their damping rate can be assessed approximately by this method from the estimates of the dominant roots of the characteristic equation. An important advantage of the presented criterion is its straightforward computational implementation. The evaluation of the function $H((\exp(j\,\varphi) - 1)/T)$ can always be performed over the same set of φ values, evenly dividing the interval $\varphi \in [0, 180^\circ]$ regardless of the system character.

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