EXTENSION OF THE CAYLEY-HAMILTON THEOREM TO CONTINUOUS–TIME LINEAR SYSTEMS WITH DELAYS

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The classical Cayley-Hamilton theorem is extended to continuous-time linear systems with delays. The matrices $A_0, A_1, \dots, A_h \in \mathbb{R}^{n\times n}$ of the system with h delays $\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{h} A_i x(t-hi) + Bu(t)$ satisfy $nh+1$ algebraic matrix equations with coefficients of the characteristic polynomial $p(s,w) = \det\left[I_n s - A_0 - A_1 w - \dots - A_h w^h\right]$, $w = e^{-hs}$.

Keywords: Cayley-Hamilton theorem, continuous-time, linear system, delay, extension.

1. Introduction

The classical Cayley-Hamilton theorem (Gantmacher, 1974; Lancaster, 1969) says that every square matrix satisfies its own characteristic equation. The Cayley-Hamilton theorem was extended to rectangular matrices (Kaczorek, 1995c), block matrices (Kaczorek, 1995b; Victoria, 1982), pairs of commuting matrices (Chang and Chan, 1992; Lewis, 1982; 1986; Mertizios and Christodoulous, 1986), pairs of block matrices (Kaczorek, 1998), and standard and singular two-dimensional linear (2-D) systems (Kaczorek, 1992/93, 1994; 1995a; Smart and Barnett, 1989; Theodoru, 1989).

The Cayley-Hamilton theorem and its generalizations have been used in control systems, electrical circuits, systems with delays, singular systems, 2-D linear systems, etc., (Gałkowski, 1996; Kaczorek, 1992/93; 1995c; Lancaster, 1969).

In (Kaczorek, 2005), the Cayley-Hamilton theorem was been extended to n-dimensional (n-D) real polynomial matrices. An extension of the Cayley-Hamilton theorem to discrete-time linear systems with delay was given in (Busłowicz and Kaczorek, 2004).

In this note the classical Cayley-Hamilton theorem is extended to continuous-time linear systems with delays. It will be shown that matrices of the n-th order system with h delays satisfy $(nh+1)$ algebraic equations.

2. Main Result

Let $\mathbb{R}^{n\times m}$ be the set of $n\times m$ real matrices and $\mathbb{R}^n := \mathbb{R}^{n\times 1}$. Consider the continuous-time linear system with h delays described by the equation

$$\begin{aligned}\dot{x}(t) = {} & A_0 x(t) + A_1 x(t-d) + \dots \\ & + A_h x(t-hd) + Bu(t),\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ are respectively the state and input vectors, $A_k \in \mathbb{R}^{n\times n}$, $k = 0,1,\dots,h$, $B \in \mathbb{R}^{n\times m}$, and d is the delay.

The characteristic polynomial of (1) has the form

$$\begin{aligned}p(s,w) &= \det\left[I_n s - A_0 - A_1 w - \dots - A_h w^h\right] \\ &= s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0,\end{aligned} \tag{2}$$

where $w = e^{-\mathrm{d}s}$ and

$$\begin{aligned}a_{n-1} = a_{n-1}(w) &= a_{n-1,h}w^h + \dots + a_{n-1,1}w \\ &\quad + a_{n-1,0} \\ a_{n-2} = a_{n-2}(w) &= a_{n-2,2h}w^{2h} + \dots + a_{n-2,1}w \\ &\quad + a_{n-2,0} \\ &\ \vdots \\ a_0 = a_0(w) &= a_{0,nh}w^{nh} + \dots + a_{0,1}w + a_{0,0}.\end{aligned} \tag{3}$$

The coefficients a_{kj}, $k = 0, 1, \ldots, n-1$ and $j = 0, 1, \ldots, nh$, depend on the entries of matrices $A_0, A_1, \ldots, A_h$.

Let

$$\left[I_n s - \left(A_0 + A_1 w + \cdots + A_h w^h\right)\right]^{-1} = I_n s^{-1} + \Phi_1 s^{-2} + \Phi_2 s^{-3} + \ldots, \tag{4}$$

where

$$\Phi_i = \Phi_i(w) = (A(w))^i \quad \text{for} \quad i = 1, 2, \ldots$$

and

$$A(w) = A_0 + A_1 w + \cdots + A_h w^h. \tag{5}$$

Using the well-known relation $\operatorname{Adj} M = M^{-1} \det M$ between the adjoint matrix $\operatorname{Adj} M$, the inverse matrix M^{-1} and its determinant $\det M$, taken in conjuction with (2) and (4), we can write

$$\begin{aligned}\operatorname{Adj} A(w) &= \left[I_n s^{-1} + \Phi_1 s^{-2} + \Phi_2 s^{-3} + \ldots\right] \\ &\quad \times \left(s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0\right).\end{aligned} \tag{6}$$

Note the adjoint matrix $\operatorname{Adj} A(w)$ is a polynomial matrix in non-negative powers of s. Thus equating the coefficients at the same powers of s^{-1} of (6) yields

$$\Phi_{n+k-1} + a_{n-1}\Phi_{n+k-2} + \cdots + a_1 \Phi_k + a_0 \Phi_{k-1} = 0 \quad \text{for } k = 1, 2, \ldots \quad (\Phi_0 = I_n). \tag{7}$$

From (7) for $k = 1$ we have (cf. the Cayley-Hamilton theorem):

$$\Phi_n + a_{n-1}\Phi_{n-1} + \cdots + a_1 \Phi_1 + a_0 I_n = 0 \tag{8}$$

with coefficients a_k depending on w.

From (5) we have

$$\begin{aligned}\Phi_i &= \left(A_0 + A_1 w + \cdots + A_h w^h\right)^i \\ &= A_0^i + \big(A_0 A_1 A_0^{i-2} + A_1 A_0^{i-1} \\ &\quad + \cdots + A_0^{i-2} A_1 A_0\big) w \\ &\quad + \big(A_0 A_2 A_0^{i-2} + A_1^2 A_0^{i-2} + A_2 A_0^{i-1} \\ &\quad + A_1 A_0 A_1 A_0^{i-3} + A_0 A_1 A_0 A_1^{i-3} \\ &\quad + A_1 A_0^2 A_1^{i-3} + \cdots A_0^2 A_1^{i-2} A_0^2\big) w^2 \\ &\quad + \cdots + A_h^i w^{hi} \quad \text{for } i = 1, 2, \ldots.\end{aligned} \tag{9}$$

The substitution of (9) and (3) into (8) yields

$$\begin{aligned}&A_0^n + \left(A_0 A_1 A_0^{n-2} + A_1 A_0^{n-1} + \cdots + A_0^{n-2} A_1 A_0\right) w \\ &+ \left(A_0 A_2 A_0^{n-2} + A_1^2 A_0^{n-2} + \cdots + A_0^2 A_1^{n-2} A_0^2\right) w^2 \\ &+ \cdots + A_h^n w^{nh} \\ &+ \left(a_{n-1,h} w^h + \cdots + a_{n-1,1} w + a_{n-1,0}\right) \\ &\times \Big[A_0^{n-1} + \big(A_0 A_1 A_0^{n-3} + A_1 A_0^{n-2} \\ &\quad + \cdots + A_0^{n-3} A_1 A_0\big) w \\ &\quad + \left(A_0 A_2 A_0^{n-3} + A_1^2 A_0^{n-3} + \cdots + A_0^2 A_1^{n-3} A_1^2\right) w^2 \\ &\quad + \cdots + A_h^{n-1} w^{(n-1)h} + \cdots + \\ &\quad + \left(a_{1,n(h-1)} w^{n(h-1)} + \cdots + a_{11} w + a_{10}\right)\Big] \\ &\times \left[A_0 + A_1 w + \cdots + A_h w^h\right] \\ &+ \left(a_{0,nh} w^{nh} + \cdots + a_{01} w + a_{00}\right) I_n = 0.\end{aligned} \tag{10}$$

From (10) we have the following $nh + 1$ equations:

$$\begin{aligned}&A_0^n + a_{n-1,0} A_0^{n-1} + \cdots + a_{10} A_0 + a_{00} I_n = 0, \\ &A_0 A_1 A_0^{n-2} + A_1 A_0^{n-1} \quad + \cdots + A_0^{n-2} A_1 A_0 \\ &\quad + a_{n-1,0}\big(A_0 A_1 A_0^{n-3} + A_1 A_0^{n-2} \\ &\quad + \cdots + A_0^{n-3} A_1 A_0\big) + a_{n-1,1} A_0^{n-1} \\ &\quad + \cdots + a_{10} A_1 + a_{11} A_0 + a_{01} I_n = 0, \\ &\qquad \vdots \\ &A_h^n + a_{0,nh} I_n = 0.\end{aligned} \tag{11}$$

Therefore, the following theorem has been proved:

Theorem 1. *Matrices $A_k \in \mathbb{R}^{n \times n}$, $k = 0, 1, \ldots, h$ of the continuous-time linear system with h delays (1) satisfy the $nh + 1$ algebraic matrix equations (11).*

Note that the first equation of (11) expresses the Cayley-Hamilton theorem for the system (1) without delay ($h = 0$).

Example 1. Consider the system with

$$A_0 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{12}$$

In this case the characteristic polynomial (2) has the form

$$\begin{aligned} p(s,w) &= \det\left[I_2 s - A_0 - A_1 w - A_2 w^2\right] \\ &= \begin{vmatrix} s-1-w & -2-w^2 \\ -1-w^2 & s-w \end{vmatrix} \\ &= s^2 - (2w+1)s - \left(w^4 + 2w^2 - w + 2\right) \end{aligned}$$

$$\begin{aligned} a_1 &= a_1(w) = a_{11}w + a_{10} = -2w - 1, \\ a_0 &= a_0(w) = a_{04}w^4 + a_{03}w^3 + a_{02}w^2 + a_{01}w \\ &\qquad + a_{00} = -w^4 - 2w^2 + w - 2. \end{aligned}$$

Taking into account the fact that $n = h = 2$, from (11) we obtain the following equations:

$$\begin{aligned} &A_0^2 + a_{10}A_0 + a_{00}I_2 \\ &= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^2 - \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} &A_0A_1 + A_1A_0 + a_{11}A_0 + a_{10}A_1 + a_{01}I_2 \\ &= 2\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &\quad + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} &A_1^2 + A_0A_2 + A_2A_0 + a_{10}A_2 + a_{11}A_1 + a_{02}I_2 \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \\ &\quad - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} &A_1A_2 + A_2A_1 + a_{11}A_2 + a_{03}I_2 \\ &= 2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$$A_2^2 + a_{04}I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

3. Concluding Remarks

The classical Cayley-Hamilton theorem was extended to continuous-time linear systems with delays. It was shown that the matrices $A_k \in \mathbb{R}^{n\times n}$, $k = 0,1,\dots,h$ of the system (1) satisfy the $nh + 1$ algebraic equations (11) with coefficients a_{kj}, $k = 0,1,\dots,n-1$ and $j = 0,1,\dots,nh$, of the characteristic polynomial (2). The proposed extension can be generalized to rectangular matrices and block matrices (Kaczorek, 1995b; Kaczorek, 1995c; Victoria, 1982). An open problem is the extension of the theorem to singular continuous-time linear systems with delays.

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Received: 30 March 2005
Revised: 4 May 2005