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STABILITY AND STABILIZABILITY OF A CLASS OF UNCERTAIN DYNAMICAL SYSTEMS WITH DELAYS

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This paper deals with a class of uncertain systems with time-varying delays and norm-bounded uncertainty. The stability and stabilizability of this class of systems are considered. Linear Matrix Inequalities (LMI) delay-dependent sufficient conditions for both stability and stabilizability and their robustness are established.

Keywords: dynamical systems, time-varying delays, time-varying uncertainty, norm-bounded robust stability, robust stabilizability

1. Introduction

Time delays can be a cause of instability and performance degradation. Dynamical systems with time delays have attracted a lot of researchers mainly from the control community and many results on this class of systems have been reported in the literature. We refer the reader to (Boukas and Liu, 2002; Mahmoud, 2000) and the references therein for more information.

The study of stability and/or stabilizability is, in general, based on one of the following techniques: the Lyapunov-Razumikhin method yielding delay-independent conditions (Hale, 1977; Hmamed, 1997; Li and Souza, 1996; 1997a; Mahmoud, 2000; Niculescu *et al.*, 1994; Su, 1994; Su and Huang, 1992; Sun *et al.*, 1997; Wang *et al.*, 1987; Xu, 1995; Xu and Liu, 1994) and the Lyapunov-Krasovskii approach, yielding delay-dependent conditions (Boukas and Liu, 2002; Mahmoud, 2000).

From the practical point of view, one is interested in conditions that constrain the upper bound of the delay and the lower and upper bounds of the first derivative of the time-varying delay. Since, in general, the delay is time varying, it can be usually represented by a function $h(t)$, and bounded by a constant $\bar{h}$. It is therefore desirable to have conditions that depend on the upper bound of the time-varying delay and on the lower and upper bounds of the first derivative of the time-varying delay.

The goal of this paper is to investigate the class of dynamical uncertain linear systems with multiple time-varying delays and to develop sufficient conditions for stability, stabilizability and their robustness, which depend on the upper bounds of the delays and on the lower and upper bounds of first derivative of time-varying delays. The Lyapunov-Krasovskii approach will be used in this paper.

In addition to that, the result is based on parameter-dependent Lyapunov functions and the obtained sufficient conditions are dilated Linear Matrix Inequalities (LMI). However, we will restrict our presentation to systems with a single delay in order to make it clearer and avoid a complicated notation. The paper is organized as follows: In Section 2, the problem is formulated and the required assumptions are given. Section 3 deals with stability and robust stability. Section 4 covers the stabilizability and robust stabilizability of the class of systems under study. Section 5 presents some numerical examples to show the usefulness of the proposed results.

2. Problem Statement

Consider the following class of systems with multiple time-varying delays:

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + G(t)x\big(t-h(t)\big) + B(t)u(t),\\ y(t) &= C(t)x(t),\end{aligned} \tag{1}$$

where $x(t)$ is the state vector, $u(t)$ is the control input, $h(t)$ is the time-varying delay of the system, and

$$\begin{aligned} A(t) &= A + D_a F_a(t) E_a, \\ G(t) &= G + D_g F_g(t) E_g, \\ B(t) &= B + D_b F_b(t) E_b. \end{aligned} \tag{2}$$

Here A, G, B, D_a, E_a, D_g, E_g, D_b and E_b are given matrices with appropriate dimensions, and $F_a(t)$, $F_g(t)$ and $F_b(t)$ represent system uncertainties satisfying the following assumption.

Assumption 1. *Assume that the uncertainties $F_a(t)$, $F_g(t)$ and $F_b(t)$ are Lebesgue measurable functions which are bounded according to*

$$F_a^\top(t) R_a F_a(t) \le R_a, \tag{3}$$

$$F_g^\top(t) R_g F_g(t) \le R_g, \tag{4}$$

$$F_b^\top(t) R_b F_b(t) \le R_b, \tag{5}$$

and R_a, R_b and R_g are given matrices with appropriate dimensions.

Remark 1. The uncertainties that satisfy (3)–(5) will be referred to as admissible uncertainties. Notice that the uncertainties $F_a(t)$, $F_g(t)$ and $F_b(t)$ can depend on the system state and the developed results will remain valid. However, in the present paper we will consider only the case of time-varying uncertainties.

Assumption 2. *The time-varying delay $h(t)$ is assumed to satisfy the following conditions:*

$$0 \le h(t) \le \bar{h} < \infty, \tag{6}$$

$$0 \le \dot{h}(t) \le \mu < 1, \tag{7}$$

where $\bar{h}$ and μ are given positive constants.

Remark 2. The case of multiple time-varying delays in the model

$$\dot{x}(t) = A(t)x(t) + \sum_{i=1}^{p} G_i(t) x\big(t - h_i(t)\big) + B(t)u(t) \tag{8}$$

can be dealt with by taking

$$\mathbb{G}(t) = \begin{bmatrix} G_1(t) & \dots & G_p(t) \end{bmatrix},$$

$$x_h(t) = \begin{bmatrix} x^\top\big(t - h_1(t)\big) & \dots & x^\top\big(t - h_p(t)\big) \end{bmatrix}^\top,$$

which allows us to rewrite (8) as

$$\dot{x}(t) = A(t)x(t) + \mathbb{G}(t)x_h(t) + B(t)u(t),$$

and all the subsequent developments will be carried out analogously.

In the remainder of this paper the notation is standard unless specified otherwise. $L > 0$ (respectively, $L < 0$) means that the matrix L is symmetric and positive definite (resp. symmetric and negative definite). The Kronecker product of two matrices Z and W is a block matrix H with generic block entries $H_{ij} = W_{ij} Z$, i.e.,

$$W \otimes Z = \left[W_{ij} Z \right]_{ij}.$$

The symbol ‘Sym’ means

$$\mathrm{Sym}(W) = W + W^\top.$$

3. Stability Problem

In order to investigate system stability, we assume that the control satisfies $u(t) = 0$ for every time instant t and thus our system becomes

$$\dot{x}(t) = A(t)x(t) + G(t)x\big(t - h(t)\big). \tag{9}$$

The following result provides a sufficient condition for robust stability:

Theorem 1. *If there exist positive-definite matrices X, P_1 and P_2 such that the LMI condition (10) is feasible, with*

$$\begin{aligned} \alpha_{11} &= (A+G)\,X + X\,(A+G)^\top + \lambda X E_a^\top R_a E_a X \\ &\quad + \lambda X E_g^\top R_g E_g X, \\ \alpha_{44} &= P_1 + \lambda P_1 E_g^\top R_g E_g P_1, \\ \alpha_{55} &= P_2 + \lambda P_2 E_g^\top R_g E_g P_2, \\ \alpha_{77} &= R_g + \lambda D_g^\top E_g^\top R_g E_g D_g, \\ \alpha_{16} &= \bar{h} X E_g^\top R_g, \\ \alpha_{66} &= (1-\mu) R_g, , \\ \alpha_{33} &= (1-\mu) P_2, \end{aligned}$$

then the system (9) is robustly stable.

Remark 3. The procedure followed to derive (10) reveals some similarities as the one in (Lee and Lee, 1999; 2000). However, we note that there is only one LMI condition to handle the implicit relation between P_1, P_2 and X.

Remark 4. Theorem 1 is intended for checking the stability of the time-varying delay system (9). In the case of a constant time delay, other alternatives exist. We quote, e.g., approaches based on a bivariate characteristic equation. However, the characteristic equation is difficult to deal with and more efficient approaches are proposed in the litterature, see, e.g., (Sen, 2002) and the references therein.

$$\begin{bmatrix} \alpha_{11} & XA^\top & XG^\top & \bar{h}GP_1 & \bar{h}GP_2 & \alpha_{16} & GD_g & D_a & \bar{h}D_g & \bar{h}D_g & D_g \\ AX & P_1 & 0 & 0 & 0 & 0 & 0 & D_a & 0 & 0 & 0 \\ GX & 0 & \alpha_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{h}P_1G^\top & 0 & 0 & \alpha_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{h}P_2G^\top & 0 & 0 & 0 & \alpha_{55} & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_{16}^\top & 0 & 0 & 0 & 0 & \alpha_{66} & 0 & 0 & 0 & 0 & 0 \\ D_g^\top G^\top & 0 & 0 & 0 & 0 & 0 & \alpha_{77} & 0 & 0 & 0 & 0 \\ D_a^\top & D_a^\top & 0 & 0 & 0 & 0 & 0 & -\lambda R_a & 0 & 0 & 0 \\ \bar{h}D_g^\top & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda R_g & 0 & 0 \\ \bar{h}D_g^\top & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda R_g & 0 \\ D_g^\top & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda R_g \end{bmatrix} < 0 \tag{10}$$

$$\begin{bmatrix} \beta_{11} & XA^\top & XG^\top & \beta_{14} & \beta_{15} & \beta_{16} & GD_g & D_a & D_b & \bar{h}D_g & \bar{h}D_g & D_g \\ AX & P_1 & 0 & 0 & 0 & 0 & 0 & D_a & D_b & 0 & 0 & 0 \\ GX & 0 & \beta_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_{14}^\top & 0 & 0 & \beta_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_{15}^\top & 0 & 0 & 0 & \beta_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_{16}^\top & 0 & 0 & 0 & 0 & \beta_{66} & 0 & 0 & 0 & 0 & 0 & 0 \\ D_g^\top G^\top & 0 & 0 & 0 & 0 & 0 & \beta_{77} & 0 & 0 & 0 & 0 & 0 \\ D_a^\top & D_a^\top & 0 & 0 & 0 & 0 & 0 & -\lambda R_a & 0 & 0 & 0 & 0 \\ D_b^\top & D_b^\top & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda R_b & 0 & 0 & 0 \\ \bar{h}D_g^\top & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda R_g & 0 & 0 \\ \bar{h}D_g^\top & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda R_g & 0 \\ D_g^\top & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda R_g \end{bmatrix} < 0 \tag{14}$$

4. Robust State Feedback Stabilization Problem

In this section we will consider the class of systems given by

$$\dot{x}(t) = A(t)x(t) + G(t)x\big(t-h(t)\big) + Bu(t), \tag{11}$$

where the matrices $A(t)$, $B(t)$ and $G(t)$ are given by (2).

In this section we restrict ourselves to the state feedback case given by

$$u(t) = Kx(t). \tag{12}$$

Then the closed-loop system is described by

$$\begin{aligned} \dot{x}(t) &= \left(A+BK+\begin{bmatrix} D_a & D_b \end{bmatrix}\begin{bmatrix} F_a(t) & 0 \\ 0 & F_b(t) \end{bmatrix}\begin{bmatrix} E_a \\ E_bK \end{bmatrix}\right) \\ &\quad \times x(t) + \big(G + D_gF_g(t)E_g\big)x\big(t-h(t)\big). \end{aligned} \tag{13}$$

The closed-loop robust stability conditions can be obtained by a direct application of Theorem 1 which we formulate in Theorem 2.

Theorem 2. *If there exist a non-zero matrix S and positive-definite matrices X, P_1 and P_2 such that the condition (14) is feasible, with*

$$\begin{aligned} \beta_{11} &= (A+G)\,X + X\,(A+G)^\top + BS + S^\top B^\top \\ &\quad + \lambda \begin{bmatrix} XE_a^\top & S^\top E_b^\top \end{bmatrix}\begin{bmatrix} R_a & 0 \\ 0 & R_b \end{bmatrix}\begin{bmatrix} E_aX \\ E_bS \end{bmatrix}, \end{aligned}$$

$$\beta_{44} = P_1 + \lambda P_1 E_g^\top R_g E_g P_1,$$

$$\beta_{55} = P_2 + \lambda P_2 E_g^\top R_g E_g P_2,$$

$$\beta_{77} = R_g + \lambda D_g^\top E_g^\top R_g E_g D_g,$$

$$\beta_{16} = \bar{h}XE_g^\top R_g,$$

$$\beta_{66} = (1-\mu)R_g,$$

$$\beta_{33} = (1-\mu)P_2,$$

$$\beta_{14} = \bar{h}GP_1,$$

$$\beta_{15} = \bar{h}GP_2,$$

then the system (11) is robustly stabilizable with the state feedback gain given by

$$K = SX^{-1}.$$

Proof. Following similar arguments as in the proof of Theorem 1, we get the desired result. ■

Remark 5. It is worth noticing that the condition (14) is not an LMI in the present form, but a simple Schur complement leads to an LMI that can be easily solved using any LMI solver.

Remark 6. The control law (12) takes account only of the current state because we assume that the delay is varying and eventually unknown. One can use a linear time delay controller (Marchenko *et al.*, 1996, Eqn. (5)) in the case of a well-known delay, which, obviously, will lead to less conservatism compared with the present controller (12).

5. Illustrative Example

Example 1. In this example we deal with the stability problem

$$\dot{x}(t) = A(t)x(t) + G(t)x\big(t-h(t)\big) \tag{15}$$

with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad G = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \tag{16}$$

$$D = D_d = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \; E_a = E_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{17}$$

In the case of a constant delay, that is, $\dot{h}(t) = 0$, Table 1 shows a comparison of our result with some previous conditions from the literature guaranteeing the stability of the uncertain time-delay system. ♦

Example 2. In this example (Lee and Lee, 1999) we deal with the stabilization problem. For this purpose, we consider the open-loop system

$$\dot{x}(t) = A(t)x(t) + G(t)x\big(t-h(t)\big) + B(t)u(t), \tag{18}$$

Table 1. Comparison of the proposed method with previous works.

	τ
Li and Souza (1997b)	0.2013
Kim (2001)	0.2412
Lee and Lee (1999)	0.4708
Our results	0.525

where the controller has the form

$$u(t) = Kx(t). \tag{19}$$

The system matrices are given as follows:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{20}$$

$$D = D_d = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad E_a = E_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{21}$$

Applying Theorem 2, we get the stabilizing state feedback

$$K = \begin{bmatrix} 0.1908 & -3.5593 \end{bmatrix} \tag{22}$$

for any constant time delay $\bar{h} \leq 0.299s$.

Figures 1 and 2 give the behavior of the system states for a time delay $\bar{h} = 0.299$ s with two different uncertainty matrices. ♦

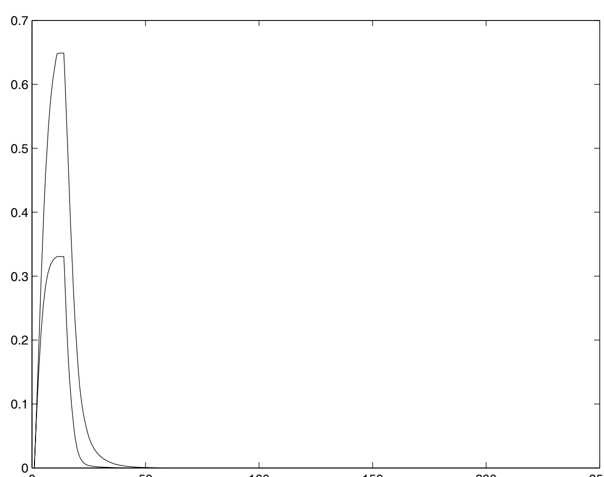

Fig. 1. Behavior of state components for the uncertainty matrices $F_a = F_d = 0.2 \times I$.

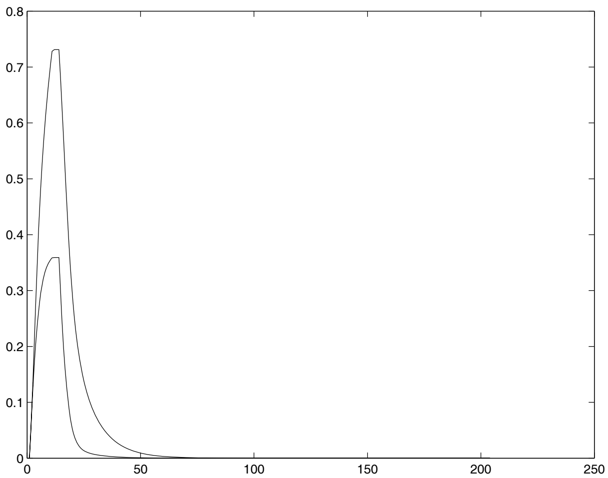

Fig. 2. Behavior of state components for the uncertainty matrices $F_a = F_d = 0.9 \times I$.

6. Conclusion

This paper deals with a class of dynamical linear uncertain systems with time-varying delays in the state. The uncertainty is assumed to be norm bounded. Delay-dependent sufficient conditions have been developed to check the robust stability. A state feedback controller is considered for the stabilizability problem, and delay-dependent conditions are developed to check the stabilizability problem. The obtained conditions are formulated by means of LMI conditions. The extension to multiple time varying delays can be dealt with according to Remark 2.

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References

Boukas E.-K. and Liu Z.-K. (2002): *Deterministic and Stochastic Time-Delay Systems*. — Boston: Birkhäuser, Marcel Dekker.

de la Sen M. (2002): *Stability test for two common classes of linear time-delay systems and hybrid systems.* — Lutianian Math. J., Vol. 42, No. 2, pp. 153–168.

Hale J.-K. (1977): *Theory of Functional Differential Equations*. — New York: Springer.

Hmamed A. (1997): *Further results on the robust stability of uncertain linear systems including delayed perturbations*. — Automatica, Vol. 33, No. 9, pp. 1763–1765.

Kim J.-H. (2001): *Delay and its time derivative dependent robust stability of time-delayed linear systems with uncertainty*. — IEEE Trans. Automat. Contr., Vol. 46, No. 5, pp. 789–792.

Lee B. and Lee J.-G. (1999): *Robust stability and stabilization of linear delayed systems with structured uncertainty*. — Automatica, Vol. 35, No. 6, pp. 1149–1154.

Lee B. and Lee J.-G. (2000): *Robust control of uncertain systems with input delay and input sector nonlinearity*. — Proc. 39th IEEE Conf. Decision and Control, North Sydney, Australia, Vol. 5, pp. 4430–4435.

Li X. and de Souza C.-E. (1996): *Criteria for robust stability of uncertain linear systems with time-varying state delays*. — Proc. 13th IFAC World Congress, San Francisco, CA, pp. 137–142.

Li X. and de Souza C.-E. (1997a): *Criteria for robust stability of uncertain linear systems with state delays*. — Automatica, Vol. 33, No. 9, pp. 1657–1662.

Li X. and de Souza C.-E. (1997b): *Delay dependent robust stability and stabilization of uncertain linear delay systems: A linear matrix inequality approach*. — IEEE Trans. Automat. Contr., Vol. 42, No. 8, pp. 1144–1148.

Li X., Fu M., and de Souza C.-E. (1992): *H_∞ control and quadratic stabilization of systems with parameter uncertainty via output feedback*. — IEEE Trans. Automat. Contr., Vol. 37, No. 8, pp. 1253–1256.

Mahmoud M.-S. (2000): *Robust Control and Filtering for Time-Delay Systems*. — New York: Marcel-Dekker.

Marchenko V.-M., Borkovskaja I.-M. and Jakimenko A.-A. (1996): *Linear state-feedback for after-effect systems: stabilization and modal control*. — Proc. 13th IFAC World Congress, San-Francisco, USA, pp. 441–446.

Niculescu S.-I., de Souza C.-E., Dion J.-M. and Dugard L. (1994): *Robust stability and stabilization of uncertain linear systems with state delay: Single dealy case*. — Proc. IFAC Symp. Robust Control Design, Rio de Janero, Brazil, pp. 469–474.

Su J.-H. (1994): *Further results on the robust stability of linear systems with a single time-delay*. — Syst. Contr. Lett., Vol. 23, pp. 375–379.

Su T.J. and Huang C.-G. (1992): *Robust stability of delay dependence for linear systems*. — IEEE Trans. Automat. Contr., Vol. 37, No. 10, pp. 1656–1659.

Sun Y.J., Hsieh J.-G. and Yang H.-C. (1997): *On the stability of uncertain systems with multiple time-varying delays*. — IEEE Trans. Automat. Contr., Vol. 42, No. 1, pp. 101–105.

Wang S.-S., Chen B.-S. and Lin T.-P. (1987): *Robust stability of uncertain time-delay systems*. — Int. J. Contr., Vol. 46, No. 4, pp. 963–976.

Xu B. (1995): *On delay-independent stability of large scale systems with time-delays*. — IEEE Trans. Automat. Contr., Vol. 40, No. 5, pp. 930–933.

Xu B. and Liu Y. (1994): *An improved Razimukhin-type theorem and its applications*. — IEEE Trans. Automat. Contr., Vol. 39, No. 4, pp. 839–841.

Appendices

Appendix A

Consider the function

$$V(t)=\int_{-\bar h}^{0}\int_{a(t+\theta)}^{t} f(\tau)\,\mathrm{d}\tau\,\mathrm{d}\theta,$$

where the function $a(t)$ is differentiable. Assume that $f(t)$ is the first derivative of $H(t)$, that is,

$$\dot H(t) = f(t).$$

Then we have

$$\int_{t+\theta}^{t} f(\tau)\,\mathrm{d}\tau=\int_{a(t+\theta)}^{t}\dot H(\tau)\,\mathrm{d}\tau=H(t)-H\big(a(t+\theta)\big)$$

and

$$\begin{aligned}\int_{-\bar h}^{0}\big(H(t)-H\big(a(t+\theta)\big)\big)\,\mathrm{d}\theta &= H(t)\int_{-\bar h}^{0}\mathrm{d}\theta-\int_{-\bar h}^{0}H\big(a(t+\theta)\big)\,\mathrm{d}\theta\\ &= \bar hH(t)-\int_{-\bar h}^{0}H\big(a(t+\theta)\big)\,\mathrm{d}\theta.\end{aligned}$$

Hence $V(t)$ is given by

$$V(t)=\bar hH(t)-\int_{-\bar h}^{0}H\big(a(t+\theta)\big)\,\mathrm{d}\theta,$$

and its derivative can be written as

$$\begin{aligned}\dot V(t) &= \bar h\dot H(t)-\int_{-\bar h}^{0}\dot a(t+\theta)\dot H\big(a(t+\theta)\big)\,\mathrm{d}\theta\\ &= \bar hf(t)-\int_{-\bar h}^{0}\dot a(t+\theta)f\big(a(t+\theta)\big)\,\mathrm{d}\theta,\end{aligned}$$

i.e., the first derivative of $V(t)$ is given by

$$\dot V(t)=\bar hf(t)-\int_{-\bar h}^{0}\dot a(t+\theta)f\big(a(t+\theta)\big)\,\mathrm{d}\theta.$$

Appendix B: Proof of Theorem 1

Let us consider the Lyapunov candidate function

$$\mathcal V=\mathcal V_0+\mathcal V_1+\mathcal V_2+\mathcal V_3$$

with

$$\mathcal V_0 = x^\top(t)Px(t), \tag{23}$$

$$\mathcal V_1 = \int_{-\bar h}^{0}\int_{t+\theta}^{t} x^\top(\tau)A^\top\left(\bar hP_1\right)^{-1}Ax(\tau)\,\mathrm{d}\tau\,\mathrm{d}\theta, \tag{24}$$

$$\begin{aligned}\mathcal V_2 = \frac{1}{1-\mu}\int_{-\bar h}^{0}\int_{t+\theta-h(t+\theta)}^{t} & x^\top(\tau)G^\top\left(\bar hP_2\right)^{-1}\\ &\times Gx(\tau)\,\mathrm{d}\tau\,\mathrm{d}\theta,\end{aligned} \tag{25}$$

$$\begin{aligned}\mathcal V_3 = \frac{1}{1-\mu}\int_{-\bar h}^{0}\int_{t+\theta-h(t+\theta)}^{t} & x^\top(\tau)\\ &\times E_g^\top\left(\bar hR_g\right)E_gx(\tau)\,\mathrm{d}\tau\,\mathrm{d}\theta.\end{aligned} \tag{26}$$

To derive a sufficient robust stability condition we have to compute the first derivative with respect to time t of the Lyapunov candidate function $\mathcal V$ along the system trajectory. The first derivatives of $\mathcal V_1$, $\mathcal V_2$ and $\mathcal V_3$ are respectively given by

$$\begin{aligned}\dot{\mathcal V}_1 = {} & x^\top(t)A(t)^\top P_1^{-1}A(t)x(t)\\ & -\int_{-\bar h}^{0}x^\top(t+\theta)A(t+\theta)^\top\left(\bar hP_1\right)^{-1}\\ & \times A(t+\theta)x(t+\theta)\,\mathrm{d}\theta,\end{aligned}$$

$$\begin{aligned}\dot{\mathcal V}_2 = {} & \frac{1}{1-\mu}\Big[x^\top(t)G^\top P_2^{-1}Gx(t)\\ & -\int_{-\bar h}^{0}\big(1-\dot h(t+\theta)\big)x^\top\big(t+\theta-h(t+\theta)\big)\\ & \times G^\top\left(\bar hP_2\right)^{-1}Gx\big(t+\theta-h(t+\theta)\big)\,\mathrm{d}\theta\Big]\\ \le {} & \frac{1}{1-\mu}x^\top(t)G^\top P_2^{-1}Gx(t)\\ & -\int_{-\bar h}^{0}x^\top\big(t+\theta-h(t+\theta)\big)\\ & \times G^\top\left(\bar hP_2\right)^{-1}Gx\big(t+\theta-h(t+\theta)\big)\,\mathrm{d}\theta,\end{aligned}$$

$$\begin{aligned}\dot{\mathcal V}_3 = {} & \frac{1}{1-\mu}\Big[\bar h^2x^\top(t)E_g^\top R_gE_gx(t)\\ & -\int_{-\bar h}^{0}\big(1-\dot h(t+\theta)\big)x^\top\big(t+\theta-h(t+\theta)\big)\\ & \times E_g^\top\left(\bar hR_g\right)E_gx\big(t+\theta-h(t+\theta)\big)\,\mathrm{d}\theta\Big]\\ \le {} & \frac{1}{1-\mu}\bar h^2x^\top(t)E_g^\top R_gE_gx(t)\\ = {} & -\int_{-\bar h}^{0}x^\top\big(t+\theta-h(t+\theta)\big)\\ & \times E_g^\top\left(\bar hR_g\right)E_gx\big(t+\theta-h(t+\theta)\big)\,\mathrm{d}\theta.\end{aligned}$$

The first derivative of $\mathcal{V}_0$ with respect to time is

$$\begin{aligned}\dot{\mathcal{V}}_0 &= 2\dot{x}^\top(t)Px(t)\\ &= 2x^\top(t)P\left(A(t)x(t)+G(t)x\big(t-h(t)\big)\right)\\ &= 2x^\top(t)PA(t)Px(t)\\ &\quad + 2x^\top(t)PG(t)x\big(t-h(t)\big). \end{aligned} \tag{27}$$

The standard result

$$x\big(t-h(t)\big)-x(t) = -\int_{-h(t)}^{0}\dot{x}(t+\theta)\,\mathrm{d}\theta,$$

expressing the delayed state as a function of the current state, will be used in what follows.

The term containing the matrix $G(t)$ in (27) is expanded as follows:

$$\begin{aligned}&2x^\top(t)PG(t)x\big(t-h(t)\big)\\ &\quad= 2x^\top(t)PG(t)x(t)\\ &\qquad - 2x^\top(t)PG(t)\int_{-h(t)}^{0}\dot{x}(t+\theta)\,\mathrm{d}\theta\\ &\quad= 2x^\top(t)PG(t)x(t)\\ &\qquad - 2x^\top(t)PG(t)\int_{-h(t)}^{0}A(t+\theta)x(t+\theta)\,\mathrm{d}\theta\\ &\qquad - 2x^\top(t)PG(t)\int_{-h(t)}^{0}G(t+\theta)\\ &\qquad\quad \times x\big(t+\theta-h(t+\theta)\big)\,\mathrm{d}\theta. \end{aligned} \tag{28}$$

Note that the term

$$-2x^\top(t)PG(t)\int_{-h(t)}^{0}A(t+\theta)x(t+\theta)\,\mathrm{d}\theta$$

can be bounded as

$$\begin{aligned}&-2x^\top(t)PG(t)\int_{-h(t)}^{0}A(t+\theta)x(t+\theta)\,\mathrm{d}\theta\\ &\quad\le \int_{-\bar{h}}^{0}x^\top(t)PG(t)\left(\bar{h}P_1\right)G(t)^\top Px(t)\,\mathrm{d}\theta\\ &\qquad+\int_{-\bar{h}}^{0}x^\top(t+\theta)A^\top(t+\theta)\left(\bar{h}P_1\right)^{-1}\\ &\qquad\quad\times A(t+\theta)x(t+\theta)\,\mathrm{d}\theta. \end{aligned} \tag{29}$$

Also, note that the term

$$-2x^\top(t)PG(t)\int_{-h(t)}^{0}G(t+\theta)x\big(t+\theta-h(t+\theta)\big)\,\mathrm{d}\theta$$

in (28) can be bounded as follows:

$$\begin{aligned}&-2x^\top(t)PG(t)\int_{-h(t)}^{0}G(t+\theta)x\big(t+\theta-h(t+\theta)\big)\,\mathrm{d}\theta\\ &= -2x^\top(t)PG(t)\int_{-h(t)}^{0}Gx\big(t+\theta-h(t+\theta)\big)\,\mathrm{d}\theta\\ &\quad -2x^\top(t)PG(t)D_g\\ &\qquad\times\int_{-h(t)}^{0}F_g(t+\theta)E_gx\big(t+\theta-h(t+\theta)\big)\,\mathrm{d}\theta\\ &\le \int_{-\bar{h}}^{0}x^\top(t)PG(t)\left(\bar{h}P_2\right)G(t)^\top Px(t)\,\mathrm{d}\theta\\ &\quad+\bar{h}x^\top(t)PG(t)D_g\left(\bar{h}R_g\right)^{-1}D_g^\top G^\top(t)Px(t)\\ &\quad+\int_{-\bar{h}}^{0}x^\top\big(t+\theta-h(t+\theta)\big)G^\top\left(\bar{h}P_2\right)^{-1}\\ &\qquad\times Gx\big(t+\theta-h(t+\theta)\big)\,\mathrm{d}\theta\\ &\quad+\int_{-h(t)}^{0}x^\top\big(t+\theta-h(t+\theta)\big)E_g^\top F_g^\top(t+\theta)\left(\bar{h}R_g\right)\\ &\qquad\times F_g(t+\theta)E_gx\big(t+\theta-h(t+\theta)\big)\,\mathrm{d}\theta\\ &\le \bar{h}^2x^\top(t)PG(t)P_2G(t)^\top Px(t)\\ &\quad+x^\top(t)PG(t)D_gR_g^{-1}D_g^\top G^\top(t)Px(t)\\ &\quad+\int_{-\bar{h}}^{0}x^\top\big(t+\theta-h(t+\theta)\big)G^\top\left(\bar{h}P_2\right)^{-1}\\ &\qquad\times Gx\big(t+\theta-h(t+\theta)\big)\,\mathrm{d}\theta\\ &\quad+\int_{-h(t)}^{0}x^\top\big(t+\theta-h(t+\theta)\big)\\ &\qquad\times E_g^\top\left(\bar{h}R_g\right)E_gx\big(t+\theta-h(t+\theta)\big)\,\mathrm{d}\theta. \end{aligned}$$

Hence, collecting all the expressions above, we have for the first derivative of $\mathcal{V}_0$

$$\begin{aligned}\dot{\mathcal{V}}_0 &\le x^\top(t)\Big(P\left(A(t)+G(t)\right)+\left(A(t)+G(t)\right)^\top P\Big)x(t)\\ &\qquad\times x^\top(t)\Big(\bar{h}^2PG(t)P_1G^\top(t)P\\ &\qquad+\bar{h}^2PG(t)P_2G^\top(t)P\\ &\qquad+PG(t)D_gR_g^{-1}D_g^\top G^\top(t)P\Big)x(t)\\ &\quad+\int_{-h(t)}^{0}x^\top(t+\theta)A^\top(t+\theta)\left(\bar{h}P_1\right)^{-1}\\ &\qquad\times A(t+\theta)x(t+\theta)\,\mathrm{d}\theta \end{aligned}$$

$$\mathcal{M}(A(t),G(t),X)$$
$$=\begin{bmatrix} \alpha_{11} & XA(t)^\top & XG^\top & \bar{h}G(t)P_1 & \bar{h}G(t)P_2 & \bar{h}XE_g^\top R_g & G(t)D_g \\ A(t)X & -P_1 & 0 & 0 & 0 & 0 & 0 \\ GX & 0 & -(1-\mu)P_2 & 0 & 0 & 0 & 0 \\ \bar{h}P_1G^\top(t) & 0 & 0 & -P_1 & 0 & 0 & 0 \\ \bar{h}P_2G^\top(t) & 0 & 0 & 0 & -P_2 & 0 & 0 \\ \bar{h}R_gE_gX & 0 & 0 & 0 & 0 & -(1-\mu)R_g & 0 \\ D_g^\top G^\top(t) & 0 & 0 & 0 & 0 & 0 & -R_g \end{bmatrix} < 0 \tag{30}$$

$$+\int_{-h(t)}^{0} x^\top(t+\theta-h(t+\theta))G^\top\left(\bar{h}P_2\right)^{-1}$$
$$\times Gx(t+\theta-h(t+\theta))\,\mathrm{d}\theta$$
$$+\int_{-h(t)}^{0} x^\top\left(t+\theta-h(t+\theta)\right)E_g^\top\left(\bar{h}R_g\right)$$
$$\times E_gx(t+\theta-h(t+\theta))\,\mathrm{d}\theta.$$

Recall that we have already obtained that

$$\int_{-\bar{h}}^{0} x^\top(t+\theta)A(t+\theta)^\top\left(\bar{h}P_1\right)^{-1}A(t+\theta)x(t+\theta)\,\mathrm{d}\theta$$
$$= -\dot{\mathcal{V}}_1 + x^\top(t)A(t)^\top P_1^{-1}A(t)x(t),$$

$$\int_{-\bar{h}}^{0} x^\top(t+\theta-h(t+\theta))G^\top\left(\bar{h}P_2\right)^{-1}$$
$$\times Gx(t+\theta-h(t+\theta))\,\mathrm{d}\theta$$
$$\le -\dot{\mathcal{V}}_2 + \frac{1}{1-\mu}x^\top(t)G^\top P_2^{-1}Gx(t),$$

$$\int_{-\bar{h}}^{0} x^\top(t+\theta-h(t+\theta))E_g^\top\left(\bar{h}R_g\right)$$
$$\times E_gx(t+\theta-h(t+\theta))\,\mathrm{d}\theta$$
$$\le -\dot{\mathcal{V}}_3 + \frac{\bar{h}^2}{1-\mu}x^\top(t)E_g^\top R_gE_gx(t),$$

from which we get

$$\dot{\mathcal{V}}_0+\dot{\mathcal{V}}_1+\dot{\mathcal{V}}_2+\dot{\mathcal{V}}_3$$
$$\le 2x^\top(t)P\left(A(t)+G(t)\right)x(t)$$
$$+\bar{h}^2x^\top(t)PG(t)P_1G^\top(t)Px(t)$$
$$+x^\top(t)A(t)^\top P_1^{-1}A(t)x(t)$$
$$+\bar{h}^2x^\top(t)PG(t)P_2G^\top(t)Px(t)$$
$$+\frac{1}{1-\mu}x^\top(t)G^\top P_2^{-1}Gx(t)x^\top(t)PG(t)D_gR_g^{-1}$$
$$\times D_g^\top G^\top(t)Px(t)$$
$$+\frac{\bar{h}^2}{1-\mu}x^\top(t)E_g^\top R_gE_gx(t).$$

Hence our condition becomes

$$\left(A(t)+G(t)\right)X + X\left(A(t)+G(t)\right)^\top + \bar{h}^2G(t)P_1G^\top(t)$$
$$+XA(t)^\top P_1^{-1}A(t)X + \bar{h}^2G(t)P_2G^\top(t)$$
$$+\frac{1}{1-\mu}XG^\top P_2^{-1}GX + G(t)D_gR_g^{-1}D_g^\top G^\top(t)$$
$$+\frac{\bar{h}^2}{1-\mu}XE_g^\top R_gE_gX < 0,$$

from which we easily deduce the condition (30), with

$$\alpha_{11} = \left(A(t)+G(t)\right)X + X\left(A(t)+G(t)\right)^\top.$$

At this step we replace $A(t)$ and $G(t)$ by their expressions (2) and get Eqn. (31), with

$$\mathcal{F}(t) = \begin{bmatrix} F_a(t) & \\ & I_3\otimes F_g(t) \end{bmatrix}.$$

The application of Lemma 1 (see Appendix C) allows us to get

$$\mathcal{M}(A,G,X) + \mathcal{D}^\top\left(\lambda R\right)^{-1}\mathcal{D} + \mathcal{E}\left(\lambda R\right)\mathcal{E}^\top$$

$$
\begin{aligned}
&\mathcal{M}\big(A(t),G(t),X\big) = \mathcal{M}(A,G,X)\\
&+\operatorname{Sym}\left\{\begin{bmatrix} D_a & D_g & D_g & D_g & D_g\\ D_a & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\mathcal{F}(t)\begin{bmatrix} E_aX & 0 & 0 & 0 & 0 & 0\\ E_gX & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \bar{h}E_gP_1 & 0 & 0 & 0\\ 0 & 0 & 0 & \bar{h}E_gP_2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \bar{h}E_gD_g \end{bmatrix}\right\} < 0, \qquad (31)
\end{aligned}
$$

with

$$
\mathcal{D} = \begin{bmatrix} D_a^\top & D_a^\top & 0 & 0 & 0 & 0 & 0\\ D_g^\top & 0 & 0 & 0 & 0 & 0 & 0\\ D_g^\top & 0 & 0 & 0 & 0 & 0 & 0\\ D_g^\top & 0 & 0 & 0 & 0 & 0 & 0\\ D_g^\top & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
\mathcal{E} = \begin{bmatrix} XE_g^\top & XE_a^\top & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \bar{h}P_1E_g^\top & 0 & 0\\ 0 & 0 & 0 & \bar{h}P_2E_g^\top & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \bar{h}D_g^\top E_g^\top \end{bmatrix},
$$

and it becomes obvious that the condition (10) can be easily deduced.

Appendix C

Lemma 1. *(Li et al., 1992) Let Z, E, F, R and Δ be matrices of appropriate dimensions. Assume that Z is symmetric, R is symmetric and positive definite, and $\Delta^\top R\Delta \le R$. Then*

$$Z + E\Delta F + F^\top\Delta^\top E^\top < 0$$

if and only if there exists a scalar $\lambda > 0$ satisfying

$$Z + E\big(\lambda R\big)E^\top + F^\top\big(\lambda R\big)^{-1}F < 0.$$

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