ON THE COMPUTATION OF THE MINIMAL POLYNOMIAL OF A POLYNOMIAL MATRIX

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The main contribution of this work is to provide two algorithms for the computation of the minimal polynomial of univariate polynomial matrices. The first algorithm is based on the solution of linear matrix equations while the second one employs DFT techniques. The whole theory is illustrated with examples.

Keywords: minimal polynomial, discrete Fourier transform, polynomial matrix, linear matrix equations

1. Introduction

It is well known from the Cayley Hamilton theorem that every matrix $A \in \mathbb{R}^{r\times r}$ satisfies its characteristic equation (Gantmacher, 1959), i.e., if $p(s) := \det(sI_r - A) = s^r + p_1 s^{r-1} + \cdots + p_r$, then $p(A) = 0$. The Cayley Hamilton theorem is still valid for all cases of matrices over a commutative ring (Atiyah and McDonald, 1964), and thus for multivariable polynomial matrices. Another form of the Cayley-Hamilton theorem, also known as the relative Caley-Hamilton theorem, is given in terms of the fundamental matrix sequence of the resolvent of the matrix, i.e., if $(sI_r - A)^{-1} = \sum_{i=0}^{\infty} \Phi_i s^{-i}$ then $\Phi_k + p_1\Phi_{k-1} + \cdots + p_r\Phi_{k-r} = 0$. The Cayley-Hamilton theorem was investigated for the matrix pencil case $A(s) = A_0 + A_1 s$ in (Mertzios and Christodoulou, 1986), and the respective relative Cayley-Hamilton theorem in (Lewis, 1986). The Cayley-Hamilton theorem was extended to matrix polynomials (Fragulis, 1995; Kitamoto, 1999; Yu and Kitamoto, 2000), to standard and singular bivariate matrix pencils (Givone and Roesser, 1973; Ciftcibaci and Yuksel, 1982; Kaczorek, 1995a; 1989; Vilfan, 1973), M-D matrix pencils in (Gałkowski, 1996; Theodorou, 1989) and n-d polynomial matrices (Kaczorek, 2005). The Cayley-Hamilton theorem was also extended to non-square matrices, non-square block matrices and singular 2D linear systems with non-square matrices (Kaczorek, 1995b; 1995c; 1995d). The reason behind the interest in the Caley-Hamilton theorem is its applications in control systems, i.e., the calculation of controllability and observability grammians and the state-transition matrix, electrical circuits, systems with delays, singular systems, 2-D linear systems, the calculation of the powers of matrices and inverses, etc.

Of particular importance for the determination of the characteristic polynomial of a polynomial matrix $A(s) = A_0 + A_1 s + \cdots + A_q s^q \in \mathbb{R}^{r\times r}[s]$ are: (a) the *Faddeev-Leverrier* algorithm (Faddeev and Faddeeva, 1963; Helmberg *et al.*, 1993) which is fraction free and needs $r^3(r-1)$ polynomial multiplications, (b) the *CHTB* method presented in (Kitamoto, 1999), which needs $r^3(q+1)$ polynomial multiplications (its shortcomings are that it cannot be used for a polynomial matrix $A(s)$ when A_0 has multiple eigenvalues, and it needs to compute first the eigenvalues and eigenvectors of A_0), and (c) the *CHACM* method presented in (Yu and Kitamoto, 2000), which needs $\frac{7}{12}r^4 + O(r^3)$ polynomial multiplications (a *CHTB* method given with an artifical constant matrix in order to release the restrictions of the *CHTB* method, which needs no condition on the given matrix, does not have to solve any eigenvalue problem and is fraction free). Except for the characteristic polynomial of a constant matrix, say $p(s)$, with the nice property $p(A) = 0$, there is also another polynomial, known as the minimal polynomial, say $m(s)$, which is the least degree monic polynomial that satisfies the equation $m(A) = 0$ (Gantmacher, 1959). Since the minimal polynomial has a lower degree than the characteristic polynomial, it helps us to solve faster problems such as the computation of the inverse or power of a matrix.

A number of algorithms have been proposed for the computation of the minimal polynomial of a constant matrix (Augot and Camion, 1997), but there is not much interest in polynomial matrices of one or more variables. Therefore, the aim of this work is to propose two algorithms for the computation of the minimal polynomial of univariate polynomial matrices. The first one is presented in Section 2 and is based on the solution of linear matrix equations, while the second one is based on Discrete Fourier Transform (DFT) techniques and is presented in Section 3. The proposed algorithms are illustrated via examples.

2. Computation of the Minimal Polynomial of Univariate Polynomial Matrices

Consider the polynomial matrix

$$A(s) = \sum_{i=0}^{q} A_i s^i \in \mathbb{R}^{r\times r}[s], \tag{1}$$

where q is the greatest power of s in $A(s)$.

Definition 1. Every polynomial

$$p(z,s) = z^p + p_1(s) z^{p-1} + \cdots + p_p(s)$$

for which

$$p\big(A(s), s\big) = A(s)^p + p_1(s) A(s)^{p-1} + \cdots + p_p(s) I_r = 0 \tag{2}$$

is called the *annihilating polynomial* for the polynomial matrix $A(s) \in \mathbb{R}^{r\times r}[s]$. The monic annihilating polynomial with a lower degree in z is called the *minimal polynomial*.

It is well known that the characteristic polynomial $p(z,s) = \det(zI_r - A(s))$ is an annihilating polynomial, but not necessarily a minimal polynomial.

Example 1. Let

$$A(s) = \begin{bmatrix} s & 1 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix}.$$

Then

$$\begin{aligned} p(z,s) &= \det(zI_3 - A(s)) \\ &= \det \begin{bmatrix} z-s & -1 & 0 \\ 0 & z-s & 0 \\ 0 & 0 & z-s \end{bmatrix} \\ &= (z-s)^3 = z^3 - 3sz^2 + 3s^2 z - s^3 \end{aligned}$$

and

$$\begin{aligned} p\big(A(s), s\big) &= (A(s) - sI_3)^3 \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^3 = 0_{3,3}. \end{aligned}$$

♦

The coefficients of the characteristic polynomial can be computed in a recursive way by an algorithm presented in (Fragulis *et al.*, 1991; Kitamoto, 1999; Yu and Kitamoto, 2000). As we shall see below, the characteristic polynomial of the above example is not the only polynomial of the third order that satisfies (2), and does not coincide with the minimal polynomial. Let now

$$B(s) = \sum_{i=0}^{p} B_i s^i \in \mathbb{R}^{r\times r}[s],$$

where p is the greatest power of s in $B(s)$. Then the product of $B(s)A(s)$ is given by

$$B(s)A(s) = \sum_{l=0}^{p+q} \left(\sum_{i=0}^{l} B_i A_{l-i} \right) s^l.$$

Note that the coefficient matrices with indices greater than p (resp. q) for $B(s)$ (resp. for $A(s)$) are taken to be zero. If $B(s) = \Phi_{0,0} := I_r$, then

$$A(s) =: \sum_{l=0}^{q} \Phi_{1,l} s^l \equiv \Phi_{0,0} A(s) = \sum_{l=0}^{q} \left(\sum_{i=0}^{l} \Phi_{0,i} A_{l-i} \right) s^l,$$

where $\Phi_{0,l} = 0$, $\forall l \neq 0$, and thus

$$\Phi_{1,l} = \sum_{i=0}^{l} \Phi_{0,i} A_{l-i} = A_l.$$

Similarly, if we set $B(s) = \sum_{i=0}^{q} \Phi_{1,i} s^i = A(s)$, where $\Phi_{1,i} = A_i$, $i = 0, 1, \ldots, q$, then

$$\begin{aligned} A^2(s) &=: \sum_{l=0}^{2q} \Phi_{2,l} s^l \equiv A(s)A(s) \\ &= \sum_{l=0}^{2q} \left(\sum_{i=0}^{l} \Phi_{1,i} A_{l-i} \right) s^l, \end{aligned}$$

and thus

$$\Phi_{2,l} = \sum_{i=0}^{l} \Phi_{1,i} A_{l-i}.$$

In the general case, where $\Phi_{k,i}$ is the matrix coefficient of s^i in the matrix $A(s)^k$, we have

$$A^k(s) = \begin{cases} I_r & \text{if } k = 0, \\ \sum_{l=0}^{kq} \Phi_{k,l} s^l & \text{if } k \geq 1, \end{cases} \tag{3}$$

where

$$\Phi_{k,l} = \sum_{i=0}^{l} \Phi_{k-1,i} A_{l-i} \quad \text{with } l = 0, 1, \ldots, kq \tag{4}$$

and $\Phi_{1,l} = A_l$, $\Phi_{0,0} = I_r$.

Let now the minimal polynomial of $A(s)$ be of the form

$$p(z,s) = z^m + p_{m-1}(s)z^{m-1} + \cdots + p_1(s)z + p_0(s),$$

where $m \le r$, with

$$p_i(s) = \sum_{k=0}^{(m-i)q} p_{i,k}s^k, \quad p_{i,k} \in \mathbb{R}. \tag{5}$$

Then (2) can be rewritten as

$$p(A(s),s) = A(s)^m + p_{m-1}(s)A(s)^{m-1} + \cdots + p_1(s)A(s) + p_0(s)I_r = 0_{r,r}, \tag{6}$$

or, equivalently,

$$p_{m-1}(s)A(s)^{m-1} + \cdots + p_1(s)A(s) + p_0(s)I_r = -A(s)^m. \tag{7}$$

Equation (7) can be rewritten as

$$\sum_{i=0}^{mq} f_i s^i = -\sum_{i=0}^{mq} \Phi_{m,i}s^i. \tag{8}$$

Using (3), (4) and (7) in (8), we get the formula

$$f_k = \sum_{i=0}^{m-1} \sum_{j=\min(k,(m-i)q)}^{\max(0,k-iq)} \Phi_{m-i,j}p_{m-i,k-j} \tag{9}$$

for $k = 0,1,\ldots,mq$. Define now the matrices

$$F_m = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{mq} \end{bmatrix}, \quad \overline{\Phi}_m = \begin{bmatrix} -\Phi_{m,0} \\ -\Phi_{m,1} \\ \vdots \\ -\Phi_{m,mq} \end{bmatrix},$$

$$P_m = \begin{bmatrix} p_{m-1,0}I_r \\ p_{m-1,1}I_r \\ \vdots \\ p_{m-1,q}I_r \\ \hline p_{m-2,0}I_r \\ \vdots \\ p_{m-2,2q}I_r \\ \hline \vdots \\ \hline p_{0,0}I_r \\ \vdots \\ p_{0,mq}I_r \end{bmatrix},$$

and Φ_m, cf. Eqn. (10), where

$$n_1 = r(mq+1),$$

$$m_1 = r\sum_{i=1}^{m}(iq+1) = \left(\frac{1}{2}qm(m+1)+m\right)r,$$

and $\Phi_{0,0} = I_r, \Phi_{1,i} = A_i$. From (9) and (10) we have

$$F_m = \Phi_m P_m = \overline{\Phi}_m. \tag{11}$$

Let Φ_i^m be the matrix that contains $i \bmod r$ columns of the matrix Φ_m and K_i^m be the matrix that contains i columns of the matrix $\overline{\Phi}_m$. Then (11) can be rewritten as

$$\underbrace{\begin{bmatrix} \Phi_1^m \\ \Phi_2^m \\ \vdots \\ \Phi_r^m \end{bmatrix}}_{\mathcal{F}_m} \underbrace{\begin{bmatrix} p_{m-1,0} \\ p_{m-1,1} \\ \vdots \\ p_{m-1,q} \\ \hline p_{m-2,0} \\ \vdots \\ p_{m-2,2q} \\ \hline \vdots \\ \hline p_{0,0} \\ \vdots \\ p_{0,mq} \end{bmatrix}}_{\mathcal{P}_m} = \underbrace{\begin{bmatrix} K_1^m \\ K_2^m \\ \vdots \\ K_r^m \end{bmatrix}}_{\mathcal{K}_m}, \tag{12}$$

where $\mathcal{F}_m \in \mathbb{R}^{n_2\times m_2}, n_2 = r^2(qm+1), m_2 = \frac{1}{2}qm(m+1)+m,$ with $m \le r$. Note that

$$\begin{aligned}\lambda &= \frac{n_2}{m_2} = \frac{r^2(qm+1)}{\frac{1}{2}qm(m+1)+m} \ge \frac{r(qm+1)}{\frac{1}{2}q(m+1)+1} \\ &= 2\frac{r(qm+1)}{(mq+1)+(q+1)} \overset{m\ge 1}{\ge} 2m\frac{r(qm+1)}{2(mq+1)} \\ &= rm \ge 1,\end{aligned}$$

and thus, in general, the number of rows of $\mathcal{F}_m$ is greater than or equal to the number of its columns, with equality in the case where $r = m = 1$. Note that the relation (9) and the matrices presented in (10), and therefore in (12), are used in (Fragulis, 1995) for the computation of the characteristic polynomial of a polynomial matrix, but in a wrong form. By using known numerical procedures, such as the Gauss elimination method, the QR factorization or the Cholesky factorization, we can easily determine the values of $p_{i,j}$ and therefore the polynomials $p_i(s)$, $i = 0,1,\ldots,m-1$. It is easily checked that the upper bound for m is r, i.e., the degree of the characteristic polynomial. An algorithm for the computation of the

$$
\Phi_m = \left[\underbrace{\begin{array}{ccccc}
\Phi_{m-1,0} & 0 & \cdots & \cdots & 0 \\
\Phi_{m-1,1} & \Phi_{m-1,0} & \cdots & \cdots & 0 \\
\Phi_{m-1,2} & \Phi_{m-1,1} & \Phi_{m-1,0} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\Phi_{m-1,q} & \Phi_{m-1,q-1} & \Phi_{m-1,q-2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\Phi_{m-1,(m-1)q} & \Phi_{m-1,(m-1)q-1} & \Phi_{m-1,(m-1)q-2} & \cdots & \Phi_{m-1,(m-2)q} \\
0 & \Phi_{m-1,(m-1)q} & \Phi_{m-1,(m-1)q-1} & \cdots & \Phi_{m-1,(m-2)q+1} \\
0 & 0 & \Phi_{m-1,(m-1)q} & \cdots & \Phi_{m-1,(m-2)q+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \Phi_{m-1,(m-1)q}
\end{array}}_{q+1} \right.
$$

$$
\cdots \underbrace{\begin{array}{ccccc}
\Phi_{2,0} & 0 & \cdots & \cdots & 0 \\
\Phi_{2,1} & \Phi_{2,0} & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \cdots & 0 \\
\Phi_{2,2q} & \Phi_{2,2q-1} & \cdots & \cdots & 0 \\
0 & \Phi_{2,2q} & \cdots & \cdots & 0 \\
0 & 0 & \ddots & \cdots & 0 \\
\vdots & \vdots & \cdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \Phi_{2,0} \\
\vdots & \vdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \cdots & \cdots & \Phi_{2,2q-1} \\
0 & 0 & \cdots & \cdots & \Phi_{2,2q}
\end{array}}_{(m-2)q+1}
\underbrace{\begin{array}{ccccc}
\Phi_{1,0} & 0 & \cdots & \cdots & 0 \\
\Phi_{1,1} & \Phi_{1,0} & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \cdots & 0 \\
\Phi_{1,q} & \Phi_{1,q-1} & \cdots & \cdots & 0 \\
0 & \Phi_{1,q} & \cdots & \cdots & 0 \\
0 & 0 & \ddots & \cdots & 0 \\
\vdots & \vdots & \cdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \Phi_{1,0} \\
\vdots & \vdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \cdots & \cdots & \Phi_{1,q-1} \\
0 & 0 & \cdots & \cdots & \Phi_{1,q}
\end{array}}_{(m-1)q+1}
$$

$$
\left. \underbrace{\begin{array}{cccccc}
\Phi_{0,0} & 0 & 0 & \cdots & 0 & 0 \\
0 & \Phi_{0,0} & 0 & \cdots & 0 & 0 \\
0 & 0 & \Phi_{0,0} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \Phi_{0,0} & 0 \\
0 & 0 & 0 & \cdots & 0 & \Phi_{0,0}
\end{array}}_{mq+1} \right] \in \mathbb{R}^{n_1 \times m_1}, \tag{10}
$$

minimal polynomial or otherwise for the coefficients $p_{i,j}$ for $i = 0, 1, \ldots, m-1$ and $j = 0, 1, \ldots, mq$ is given below.

Algorithm 1. *Computation of the minimal polynomial of a polynomial matrix*

Step 1. $x = 0$.

Step 2. *Do*
$x = x + 1$
Define the matrix $\Phi_x \in \mathbb{R}^{n_1 \times m_1}$ *(see (10)), with* $n_1 = r(xq+1)$ *and* $m_1 = r\sum_{i=1}^{x}(iq+1)$.
Rewrite the equations $\Phi_x P_x = \overline{\Phi}_x$ *as* $\mathcal{F}_x \mathcal{P}_x = \mathcal{K}_x$ *(see (12)).*
While NOT $\mathcal{F}_x \mathcal{P}_x = \mathcal{K}_x$ *has a solution.*

Step 3. *The coefficients of the minimal polynomial are given by the solution of the system* $\mathcal{F}_x \mathcal{P}_x = \mathcal{K}_x$.

The above algorithm can be easily modified in order to find an annihilating polynomial of the same degree with the characteristic polynomial of the polynomial matrix $A(s)$ as follows:

Algorithm 2. *Computation of an annihilating polynomial of the same degree with the characteristic polynomial of a polynomial matrix*

Step 1. *Construct the matrices*
$\Phi_{i,0}, \quad \Phi_{i,1}, \quad \ldots, \quad \Phi_{i,iq}, \quad i = 0, 1, \ldots, r-1.$

Step 2. *Define the matrix* Φ_r *(see (10)).*

Step 3. *Construct the matrices* $\mathcal{F}_r$ *and* K_r *that contain* $i \bmod r$ *columns of* Φ_r *and* $\overline{\Phi}$*, respectively.*

Step 4. *The coefficients of the annihilating polynomial are given by the solution of the system (12) with* $m = r$*, i.e.,* $\mathcal{F}_r \mathcal{P}_r = \mathcal{K}_r$.

The main advantage of the above method of determining of the coefficients $p_{i,j}$ is the use of numerically stable procedures for the solution of the linear system (12), while the main disadvantage is the use of large-scale matrices. The upper bound for the complexity of the algorithm for the computation of the minimal polynomial is $O\left(\frac{1}{10}q^3r^9\right)$ (q is the greatest power of s in $A(s)$, r stands for the dimension of the matrix $A(s)$) and this is the case where the minimal polynomial coincides with the characteristic polynomial. The lower bound for the complexity of the above algorithm is $O\left(\frac{1}{2}q^3r^4\right)$. The complexity can also be reduced by using fast matrix multiplication techniques (Coppersmith and Winograd, 1990) with the complexity $O\left(n^{2.376}\right)$ and fast linear matrix solvers that exploit the sparsity of the matrices, i.e., a conjugate-gradient linear system solver, with the complexity $O\left(n^2\right)$ instead of $O\left(n^3\right)$, which we have used for the computation of the complexity of Algorithm 2. Now, if we take into account the fact that the upper bound for the complexity of polynomial multiplication of polynomials with degree at most rq is $O(rq\log(rq))$ (by using FFT transforms), then the CHACM method for the computation of the characteristic polynomial (Yu and Kitamoto, 2000) needs $r^3(q+1)$ polynomial multiplications or otherwise the upper bound for its complexity is $O\left(r^4q(q+1)\log(rq)\right)$, which is better than those in Algorithm 2. However, Algorithm 2 may gives better results than those in (Yu and Kitamoto, 2000) in the case where the minimal polynomial has a much smaller degree in z than the characteristic polynomial.

Example 2. Let

$$A(s) = \begin{bmatrix} s & 1 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_0} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_1} s \in \mathbb{R}[s]^{3\times 3}.$$

Step 1. $x = 0$.

Step 2. $x = 1$.
Define the compound matrices

$$\Phi_1 = \begin{bmatrix} I_3 & 0 \\ 0 & I_3 \end{bmatrix} \in \mathbb{R}^{6\times 6},$$

$$P_1 = \begin{bmatrix} p_{0,0}I_3 \\ p_{0,1}I_3 \end{bmatrix}, \quad \overline{\Phi}_1 = \begin{bmatrix} -\Phi_{1,0} \\ -\Phi_{1,1} \end{bmatrix},$$

where

$$\Phi_{1,0} = A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Phi_{1,1} = A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can easily check that the matrix equation $\Phi_1 P_1 = \overline{\Phi}_1$ *or equivalently* $P_1 = \overline{\Phi}_1$ *has no solution and thus we proceed to the next step i.e.* $x = x + 1$. *Define the compound matrices*

$$\Phi_2 = \begin{bmatrix} \Phi_{1,0} & 0 & I_3 & 0 & 0 \\ \Phi_{1,1} & \Phi_{1,0} & 0 & I_3 & 0 \\ 0 & \Phi_{1,1} & 0 & 0 & I_3 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|ccc|ccc|ccc|ccc}
0&1&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
1&0&0&0&1&0&0&0&0&1&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&1
\end{array}\right]$$

$$\in \mathbb{R}^{9\times 15},$$

$$P_2 = \begin{bmatrix} p_{1,0}I_3 \\ p_{1,1}I_3 \\ p_{0,0}I_3 \\ p_{0,1}I_3 \\ p_{0,2}I_3 \end{bmatrix}, \quad \overline{\Phi}_2 = \begin{bmatrix} -\Phi_{2,0} \\ -\Phi_{2,1} \\ -\Phi_{2,2} \end{bmatrix},$$

where

$$\Phi_{2,0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Phi_{2,1} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Phi_{2,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Rewrite the equations $\Phi_2 P_2 = \overline{\Phi}_2$ *as* $\mathcal{F}_2\mathcal{P}_2 = \mathcal{K}_2$ *(see (12)) with*

$$\mathcal{F}_2 = \begin{bmatrix} 0&0&1&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 1&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&1&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 1&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&0 \\ 0&1&0&0&0 \\ 1&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&1&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 1&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&1&0&0&1 \end{bmatrix}, \quad \mathcal{P}_2 = \begin{bmatrix} p_{1,0} \\ p_{1,1} \\ \hline p_{0,0} \\ p_{0,1} \\ p_{0,2} \end{bmatrix}, \quad \mathcal{K}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$

and get the unique solution

$$\begin{bmatrix} p_{1,0} \\ p_{1,1} \\ \hline p_{0,0} \\ p_{0,1} \\ p_{0,2} \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ \hline 0 \\ 0 \\ 1 \end{bmatrix}.$$

Step 3. *The coefficients of the minimal polynomial are given by the solution of the system* $\mathcal{F}_2\mathcal{P}_2 = \mathcal{K}_2$

$$p(z,s) = (z-s)^2 = z^2 - 2sz + s^2.$$

In case we would like to apply the Algorithm 2, then we need to proceed one step more i.e. $x = r = 3$. *Define the compound matrices*

$$\Phi_3 = \begin{bmatrix} \Phi_{2,0} & 0 & \Phi_{1,0} & 0 & 0 & I_3 & 0 & 0 & 0 \\ \Phi_{2,1} & \Phi_{2,0} & \Phi_{1,1} & \Phi_{1,0} & 0 & 0 & I_3 & 0 & 0 \\ \Phi_{2,2} & \Phi_{2,1} & 0 & \Phi_{1,1} & \Phi_{1,0} & 0 & 0 & I_3 & 0 \\ 0 & \Phi_{2,2} & 0 & 0 & \Phi_{1,1} & 0 & 0 & 0 & I_3 \end{bmatrix}$$

$$= \left[\begin{array}{ccccccccccccccccccc}
0&0&0& & & &0&1&0& & & & & & & & & & \\
0&0&0& &0& &0&0&0& &0& & &0& &I&0&0&0 \\
0&0&0& & & &0&0&0& & & & & & & & & & \\
0&2&0&0&0&0&1&0&0&0&1&0& & & & & & & \\
0&0&0&0&0&0&0&1&0&0&0&0& &0& &0&I&0&0 \\
0&0&0&0&0&0&0&0&1&0&0&0& & & & & & & \\
1&0&0&0&2&0& & & &1&0&0&0&1&0& & & & \\
0&1&0&0&0&0& &0& &0&1&0&0&0&0&0&0&I&0 \\
0&0&1&0&0&0& & & &0&0&1&0&0&0& & & & \\
 & & &1&0&0& & & & & & &1&0&0& & & & \\
 &0& &0&1&0& &0& & &0& &0&1&0&0&0&0&I \\
 & & &0&0&1& & & & & & &0&0&1& & & &
\end{array}\right].$$

$$P_3 = \begin{bmatrix} p_{2,0}I_3 \\ p_{2,1}I_3 \\ \hline p_{1,0}I_3 \\ p_{1,1}I_3 \\ p_{1,2}I_3 \\ \hline p_{0,0}I_3 \\ p_{0,1}I_3 \\ p_{0,2}I_3 \\ p_{0,3}I_3 \end{bmatrix}, \quad \overline{\Phi}_3 = \begin{bmatrix} -\Phi_{3,0} \\ -\Phi_{3,1} \\ -\Phi_{3,2} \\ -\Phi_{3,3} \end{bmatrix}$$

where

$$\Phi_{3,3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Phi_{3,2} = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Phi_{3,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Phi_{3,0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then we construct the matrices $\mathcal{F}_3, \mathcal{K}_3$ and $\mathcal{P}_3$ as defined above:

$$\mathcal{F}_3 = \begin{bmatrix} 0\,0\,0 & 0\,0\,1 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,1 & 0\,0\,0 & 1\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 1\,0\,0 & 1\,0\,0 & 0\,1\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,1\,0 & 0\,1\,0 & 0\,0\,1 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,1 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,1 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 2\,0\,0 & 1\,0\,0 & 0\,0\,0 \\ 0\,0\,1 & 0\,0\,0 & 1\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,2\,0 & 0\,1\,0 & 0\,0\,0 \\ 1\,0\,0 & 1\,0\,0 & 0\,1\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,1\,0 & 0\,1\,0 & 0\,0\,1 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,1 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,1 & 0\,0\,0 & 1\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 1\,0\,0 & 1\,0\,0 & 0\,1\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,1\,0 & 0\,1\,0 & 0\,0\,1 \end{bmatrix}, \quad \mathcal{K}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\mathcal{P}_3 = \left[\begin{array}{c} p_{2,0} \\ p_{2,1} \\ \hline p_{1,0} \\ p_{1,1} \\ p_{1,2} \\ \hline p_{0,0} \\ p_{0,1} \\ p_{0,2} \\ p_{0,3} \end{array}\right],$$

and solve the equations $\mathcal{F}_3\mathcal{P}_3 = \mathcal{K}_3$. The coefficients of the annihilating polynomial of degree 3 in z are given by the solution of the above system of equations:

$$\mathcal{P}_3 = \left[\begin{array}{c} p_{2,0} \\ p_{2,1} \\ \hline p_{1,0} \\ p_{1,1} \\ p_{1,2} \\ \hline p_{0,0} \\ p_{0,1} \\ p_{0,2} \\ p_{0,3} \end{array}\right] = \left[\begin{array}{c} -\frac{1}{6}z_1 + \frac{1}{3}z_4 - \frac{1}{6}z_8 \\ -\frac{4}{3} - \frac{1}{6}z_2 + \frac{1}{3}z_5 - \frac{1}{6}z_9 \\ \hline 0 \\ \frac{1}{3}z_1 - \frac{2}{3}z_4 + \frac{1}{3}z_8 \\ -\frac{1}{3} + \frac{1}{3}z_2 - \frac{2}{3}z_5 + \frac{1}{3}z_9 \\ \hline 0 \\ 0 \\ -\frac{1}{6}z_1 + \frac{1}{3}z_4 - \frac{1}{6}z_8 \\ \frac{2}{3} - \frac{1}{6}z_2 + \frac{1}{3}z_5 - \frac{1}{6}z_9 \end{array}\right].$$

Thus the annihilating polynomial of order 3 in z will have the following form:

$$\begin{aligned} p(z,s) = z^3 &+ \Big[\Big(-\frac{4}{3} - \frac{1}{6}z_2 + \frac{1}{3}z_5 - \frac{1}{6}z_9\Big)s \\ &+ \Big(-\frac{1}{6}z_1 + \frac{1}{3}z_4 - \frac{1}{6}z_8\Big)\Big]z^2 \\ &+ \Big[\Big(-\frac{1}{3} + \frac{1}{3}z_2 - \frac{2}{3}z_5 + \frac{1}{3}z_9\Big)s^2 \\ &+ \Big(\frac{1}{3}z_1 - \frac{2}{3}z_4 + \frac{1}{3}z_8\Big)s\Big]z \\ &+ \Big[\Big(\frac{2}{3} - \frac{1}{6}z_2 + \frac{1}{3}z_5 - \frac{1}{6}z_9\Big)s^3 \\ &+ \Big(-\frac{1}{6}z_1 + \frac{1}{3}z_4 - \frac{1}{6}z_8\Big)s^2\Big], \end{aligned}$$

or, equivalently, if we set $x = z_2 - 2z_5 + z_9 - 10$ and $y = z_1 - 2z_4 + z_8$,

$$\begin{aligned} p(z,s) = {} & z^3 + \Big(\Big(-\frac{9}{3} - \frac{1}{6}x\Big)s + \Big(-\frac{1}{6}y\Big)\Big)z^2 \\ & + \Big(\Big(\frac{9}{3} + \frac{1}{3}x\Big)s^2 + \Big(\frac{1}{3}y\Big)s\Big)z \\ & + \Big(\Big(-1 - \frac{1}{6}x\Big)s^3 + \Big(-\frac{1}{6}y\Big)s^2\Big) \\ = {} & (z-s)^2\Big(z - \frac{(6+x)s + y}{6}\Big). \end{aligned}$$

Note that (a) $p(A(s), s) = 0$ and (b) the characteristic polynomial is a special case of $p(z,s)$ for $x = 0$, $y = 0$. Any other polynomial of the form $p'(z,s) = (z-s)^2(z - a(x_1, x_2, \dots, x_n, s))$ is also an annihilating polynomial of degree 3 in z since, as we have shown above, the minimal polynomial of the matrix $A(s)$ is $(s-z)^2$. ♦

3. DFT Calculation of a Minimal Polynomial

The main disadvantage of the algorithm presented in the previous section is its complexity. In order to overcome this difficulty, we can use other techniques such as interpolation methods. Schuster and Hippe (1992) use interpolation techniques to find the inverse of a polynomial matrix. The speed of interpolation algorithms can be increased by using Discrete Fourier Transforms (DFT) techniques or better Fast Fourier Transforms (FFT). Some of the advantages of DFT-based algorithms are that there are very efficient algorithms available both in software and hardware, and that they are parallel in nature (through symmetric multiprocessing or other techniques). Paccagnella and Pierobon (1976) use FFT methods for the computation of the determinant of a polynomial matrix. In this section we present an algorithm based on the Discrete Fourier Transform (DFT) which is by an order of magnitude faster than the algorithm presented in the previous section.

Multidimensional Fourier Transforms arise very frequently in many scientific fields such as image processing, statistics, etc. Let us now present the strict definition of a DFT pair. Consider the finite sequences $X(k_1,k_2)$ and $\tilde{X}(r_1,r_2)$, $k_i, r_i = 0,1,\dots,M_i$. In order for the sequence $X(k_1,k_2)$ and $\tilde{X}(r_1,r_2)$ to constitute a DFT pair, the following relations should hold:

$$\tilde{X}(r_1,r_2) = \sum_{k_1=0}^{M_1}\sum_{k_2=0}^{M_2} X(k_1,k_2) W_1^{-k_1 r_1} W_1^{-k_2 r_2}, \quad (13)$$

$$X(k_1,k_2) = \frac{1}{R}\sum_{r_1=0}^{M_1}\sum_{r_2=0}^{M_2} \tilde{X}(r_1,r_2) W_1^{k_1 r_1} W_1^{k_2 r_2}, \quad (14)$$

where

$$W_i = e^{\frac{2\pi j}{M_i+1}}, \quad \forall i = 1,2, \quad (15)$$

$$R = (M_1+1)(M_2+1), \quad (16)$$

and X, $\tilde{X}$ are discrete argument matrix-valued functions. The relation (13) is the forward Fourier transform of $X(k_1,k_2)$, while (14) is the inverse Fourier transform of $\tilde{X}(r_1,r_2)$.

The great advantage of FFT methods is their reduced complexity. The complexity of 1D DFT on a matrix $M \in \mathbb{R}^{1\times R}$ is $\mathcal{O}(R^2)$, while the FFT has a complexity of $\mathcal{O}(R\log R)$. Similarly, the complexity of the DFT of a matrix $M \in \mathbb{R}^{m_1\times m_2}$ is $\mathcal{O}(\prod_{i=1}^2 m_i^2)$ which, using the FFT, reduces to $\mathcal{O}((\prod_{i=1}^2 m_i)(\sum_{i=1}^2 \log m_i))$. The inverse DFT is of the same complexity as the forward one.

In the following, we propose a new algorithm for the calculation of the minimal polynomial of $A(s)$ using discrete Fourier transforms. From (5) it is easily seen that the greatest powers of the variables s and z in the minimal polynomial $p(s,z)$ are

$$\begin{aligned} \deg_z(p(z,s)) &= b_0 := m\,(\le r), \\ \deg_s(p(z,s)) &\le b_1 := mq\,(\le rq). \end{aligned} \quad (17)$$

Thus, the polynomial $p(z,s)$ can be written as

$$p(z,s) = \sum_{k_0=0}^{b_0}\sum_{k_1=0}^{b_1} (p_{k_0k_1})\left(z^{k_0}s^{k_1}\right) \quad (18)$$

and numerically computed via interpolation using the following R_1 points:

$$u_i(r_j) = W_i^{-r_j}, \quad i = 0,1 \text{ and } r_j = 0,1,\dots,b_i, \quad (19)$$

$$W_i = e^{\frac{2\pi j}{b_i+1}}, \quad (20)$$

where

$$R_1 = (b_0+1)(b_1+1). \quad (21)$$

In order to evaluate the coefficients $p_{k_0k_1}$, define

$$\tilde{p}_{r_0r_1} = p\left(u_0(r_0), u_1(r_1)\right), \quad (22)$$

where we use an $O\left(r^3\right)$ algorithm for the computation of the minimal polynomial of the above constant matrix $A(u_1(r_1))$ (Augot and Camion, 1997). From (18), (20) and (22) we get

$$\tilde{p}_{r_0r_1} = \sum_{l_0=0}^{b_0}\sum_{l_1=0}^{b_1} (p_{l_0l_1})\left(W_0^{-r_0l_0}W_1^{-r_1l_1}\right).$$

Notice that $[p_{l_0l_1}]$ and $[\tilde{p}_{r_0r_1}]$ form a DFT pair and thus using (14) we derive the coefficients of (18), i.e.,

$$p_{l_0l_1} = \frac{1}{R_1}\sum_{r_0=0}^{b_0}\sum_{r_1=0}^{b_1} \tilde{p}_{r_0r_1} W_0^{r_0l_0} W_1^{r_1l_1}, \quad (23)$$

where $l_i = 0,\dots,b_i$ and $i = 0,1$.

Having in mind the above theoretical deliberations, we will continue by describing the algorithm as an outline for computation.

Algorithm 3. *DFT computation of the minimal polynomial*

Step 1. *Calculate the number of interpolation points* b_i *using (17).*

Step 2. *Compute* R_1 *points* $u_i(r_j)$ *for* $i = 0,1$ *and* $r_j = 0,1,\dots,b_i$ *in (19).*

Step 3. *Determine the values at* $u_0(r_0)$ *of the minimal polynomials of the constant matrices* $A(u_1(r_1))$ *and thus construct the values* $\tilde{p}_{r_0r_1}$ *in (22).*

Step 4. *Use the inverse DFT (23) for the points* $\tilde{p}_{r_0r_1}$ *in order to construct the values* $p_{l_0l_1}$.

The above algorithm can also be used for the computation of the *characteristic polynomial* of a matrix polynomial by making necessary changes in Step 3 (the computation of *characteristic polynomial* of $A(u_1(r_1))$ instead of the *minimal polynomial*). The upper bound for the complexity of the above algorithm is $O(r^4 q^2)$ if we use DFT techniques or $O(r^4 q \log(q))$ if we use FFT techniques, and is better than the *CHACM* method for the characteristic polynomial of $A(s)$ while being comparable to Algorithm 2 when the minimal polynomial has a much smaller degree in z than the characteristic polynomial.

Example 3. Consider the polynomial matrix $A(s)$ of Example 2. Then by applying Algorithm 3 we have the following results:

Step 1. Calculate the number of interpolation points b_i by (17).

$$b_0 = \deg_z p(z,s) \le r = 3,$$

$$b_1 = \deg_s a(z,s) \le rq = 3.$$

Step 2. Compute

$$R_1 = \prod_{i=0}^{1}(b_i+1) = (3+1)(3+1) = 16$$

points $u_i(r_j) = W_i^{-r_j}, W_i = e^{\frac{2\pi j}{b_i+1}}$, $i = 0,1$ and $r_j = 0,1,\ldots,b_j$ in (19). We get

$$u_0(0) = W_0^0 = 1,$$

$$u_0(1) = W_0^{-1} = e^{-\frac{2\pi j}{3+1}} = e^{-\frac{\pi j}{2}},$$

$$u_0(2) = W_0^{-2} = e^{-2\frac{2\pi j}{3+1}} = e^{-\pi j},$$

$$u_0(3) = W_0^{-3} = e^{-3\frac{2\pi j}{3+1}} = e^{-\frac{3\pi j}{2}},$$

$$u_1(0) = W_1^0 = 1,$$

$$u_1(1) = W_1^{-1} = e^{-\frac{2\pi j}{3+1}} = e^{-\frac{2\pi j}{4}},$$

$$u_1(2) = W_1^{-2} = e^{-2\frac{2\pi j}{3+1}} = e^{-\frac{4\pi j}{4}},$$

$$u_1(3) = W_1^{-3} = e^{-3\frac{2\pi j}{3+1}} = e^{-\frac{6\pi j}{4}}.$$

Step 3. Determine the minimal polynomials of the constant matrices $A(u_1(r_1))$:

$$p(z,u_1(0)) = z^2 - 2z + 1,$$

$$p(z,u_1(1)) = z^2 + 2jz - 1,$$

$$p(z,u_1(2)) = z^2 + 2z + 1,$$

$$p(z,u_1(3)) = z^2 - 2jz - 1,$$

and then the values of each polynomial at $u_0(r_0)$,

$$\begin{aligned}\tilde{p}_{0,0} &= p(u_0(0),u_1(0)) \\ &= \left(z^2-2z+1\right)_{z=1} = 0,\end{aligned}$$

$$\begin{aligned}\tilde{p}_{1,0} &= p(u_0(1),u_1(0)) \\ &= \left(z^2-2z+1\right)_{z=e^{-\pi j/2}} = 2j,\end{aligned}$$

$$\begin{aligned}\tilde{p}_{2,0} &= p(u_0(2),u_1(0)) \\ &= \left(z^2-2z+1\right)_{z=e^{-\pi j}} = 4,\end{aligned}$$

$$\begin{aligned}\tilde{p}_{3,0} &= p(u_0(3),u_1(0)) \\ &= \left(z^2-2z+1\right)_{z=e^{-3\pi j/2}} = -2j,\end{aligned}$$

$$\begin{aligned}\tilde{p}_{0,1} &= p(u_0(0),u_1(1)) \\ &= \left(z^2+2jz-1\right)_{z=1} = 2j,\end{aligned}$$

$$\begin{aligned}\tilde{p}_{1,1} &= p(u_0(1),u_1(1)) \\ &= \left(z^2+2jz-1\right)_{z=e^{-\pi j/2}} = 0,\end{aligned}$$

$$\begin{aligned}\tilde{p}_{2,0} &= p(u_0(2),u_1(1)) \\ &= \left(z^2+2jz-1\right)_{z=e^{-\pi j}} = -2j,\end{aligned}$$

$$\begin{aligned}\tilde{p}_{3,1} &= p(u_0(3),u_1(1)) \\ &= \left(z^2+2jz-1\right)_{z=e^{-3\pi j/2}} = -4,\end{aligned}$$

$$\begin{aligned}\tilde{p}_{0,2} &= p(u_0(0),u_1(2)) \\ &= \left(z^2+2z+1\right)_{z=1} = 4,\end{aligned}$$

$$\begin{aligned}\tilde{p}_{1,2} &= p(u_0(1),u_1(2)) \\ &= \left(z^2+2z+1\right)_{z=e^{-\pi j/2}} = -2j,\end{aligned}$$

$$\begin{aligned}\tilde{p}_{2,2} &= p(u_0(2),u_1(2)) \\ &= \left(z^2+2z+1\right)_{z=e^{-\pi j}} = 0,\end{aligned}$$

$$\begin{aligned}\tilde{p}_{3,2} &= p(u_0(3),u_1(2)) \\ &= \left(z^2+2z+1\right)_{z=e^{-3\pi j/2}} = 2j,\end{aligned}$$

$$\begin{aligned}\tilde{p}_{0,3} &= p(u_0(0),u_1(3)) \\ &= \left(z^2-2jz-1\right)_{z=1} = -2j,\end{aligned}$$

$$\begin{aligned}\tilde{p}_{1,3} &= p(u_0(1),u_1(3)) \\ &= \left(z^2-2jz-1\right)_{z=e^{-\pi j/2}} = -4,\end{aligned}$$

$$\begin{aligned}\tilde{p}_{2,3} &= p(u_0(2),u_1(3)) \\ &= \left(z^2-2jz-1\right)_{z=e^{-\pi j}} = 2j,\end{aligned}$$

$$\begin{aligned}\tilde{p}_{3,3} &= p(u_0(3),u_1(3)) \\ &= \left(z^2-2jz-1\right)_{z=e^{-3\pi j/2}} = 0,\end{aligned}$$

and thus construct the values $\tilde{p}_{r_0 r_1}$ in (22).

Step 4. Use the inverse DFT (23) for the points $\tilde{p}_{r_0 r_1}$ in order to construct the values $p_{l_0 l_1} = \frac{1}{16}\sum_{r_0=0}^{3}\sum_{r_1=0}^{3}\tilde{p}_{r_0 r_1}W_0^{r_0 l_0}W_1^{r_1 l_1}$.

$$p_{0,0}=0,\quad p_{0,1}=0,\quad p_{0,2}=1,\quad p_{0,3}=0,$$
$$p_{1,0}=0,\quad p_{1,1}=-2,\quad p_{1,2}=0,\quad p_{1,3}=0,$$
$$p_{2,0}=1,\quad p_{2,1}=0,\quad p_{2,2}=0,\quad p_{2,3}=0,$$
$$p_{3,0}=0,\quad p_{3,1}=0,\quad p_{3,2}=0,\quad p_{3,3}=0,$$

and thus the *minimal polynomial* is

$$p(z,s) = s^2 - 2sz + z^2 = (z-s)^2 .$$

In the case when we are interested in the characteristic polynomial of the matrix $A(s)$, we have to change Steps 3 and 4 as follows:

Step 3a. Determine the characteristic polynomials of the constant matrices $A(u_1(r_1))$:

$$p(z,u_1(0)) = z^3 - 3z^2 + 3z - 1,$$
$$p(z,u_1(1)) = z^3 - 3z^2 e^{-\frac{1}{2}\pi j} + 3ze^{-\pi j} - e^{-\frac{3}{2}\pi j},$$
$$p(z,u_1(2)) = z^3 - 3z^2 e^{-\pi j} + 3ze^{-2\pi j} - e^{-3\pi j},$$
$$p(z,u_1(3)) = z^3 - 3z^2 e^{-\frac{3}{2}\pi j} + 3ze^{-3\pi j} - e^{-\frac{9}{2}\pi j},$$

and then the values of each polynomial at $u_0(r_0)$,

$$\tilde{p}_{0,0}=0,\qquad \tilde{p}_{1,0}=2-2j,$$
$$\tilde{p}_{2,0}=-8,\qquad \tilde{p}_{3,0}=2+2j,$$
$$\tilde{p}_{0,1}=-2+2j,\qquad \tilde{p}_{1,1}=0,$$
$$\tilde{p}_{2,0}=2+2j,\qquad \tilde{p}_{3,1}=-8j,$$
$$\tilde{p}_{0,2}=8,\qquad \tilde{p}_{1,2}=-2-2j,$$
$$\tilde{p}_{2,2}=0,\qquad \tilde{p}_{3,2}=-2+2j,$$
$$\tilde{p}_{0,3}=-2-2j,\qquad \tilde{p}_{1,3}=8j,$$
$$\tilde{p}_{2,3}=2-2j,\qquad \tilde{p}_{3,3}=0,$$

and thus construct the values $\tilde{p}_{r_0 r_1}$ in (22).

Step 4a. Use the inverse DFT (23) for the points $\tilde{p}_{r_0 r_1}$ in order to construct the values $p_{l_0 l_1} = \frac{1}{16}\sum_{r_0=0}^{3}\sum_{r_1=0}^{3}\tilde{p}_{r_0 r_1}W_0^{r_0 l_0}W_1^{r_1 l_1}$,

$$p_{0,0}=0,\quad p_{0,1}=0,\quad p_{0,2}=0,\quad p_{0,3}=-1,$$
$$p_{1,0}=0,\quad p_{1,1}=0,\quad p_{1,2}=3,\quad p_{1,3}=0,$$
$$p_{2,0}=0,\quad p_{2,1}=-3,\quad p_{2,2}=0,\quad p_{2,3}=0,$$
$$p_{3,0}=1,\quad p_{3,1}=0,\quad p_{3,2}=0,\quad p_{3,3}=0,$$

and thus the *characteristic polynomial* is

$$p(z,s) = z^3 - 3z^2 s + 3zs^2 - s^3.$$

Note that, when using the DFT method or the CHACM method, it is not easy to find the family of annihilating polynomials of degree 3 in z, as we have already done in Example 2. ♦

4. Conclusions

Two algorithms for the computation of the minimal polynomial and the characteristic polynomial of univariate matrices have been developed. The proposed algorithms are easily implemented on a digital computer and are very useful in many problems, such as the computation of the powers of polynomial matrices, the evaluation of the controllability and observability grammians of polynomial matrix descriptions, the calculation of the state-transition matrix, which is used for the evaluation of the solution of homogeneous matrix differential equations of the form $A(\rho)\beta(t) = 0$ with $A(\rho) \in \mathbb{R}[\rho]^{r\times r}$. The algorithm presented in Section 2 is based on the solution of linear matrix equations. Its main advantage is that (a) it creates a family of annihilating polynomials of the same degree with the characteristic polynomial, and (b) has a complexity comparable with the DFT method presented in Section 3 in the case where the degree of the minimal polynomial in z is lower enough in contrast to the degree of the characteristic polynomial. Its main disadvantage is that the upper bound for its complexity is big enough since it uses large-scale matrices. The algorithm presented in Section 3 is based on DFT techniques and has a lower upper bound complexity than the ones in Section 2. An extension of these algorithms to the n-variable case can be easily done (Tzekis and Karampetakis, 2005).

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