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ANALYTIC RESULTS FOR OSCILLATORY SYSTEMS WITH EXTREMAL DYNAMIC PROPERTIES

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The maximal value of the error is the most important criterion in system design. It is also the most difficult one. For that reason there exist many other criteria. The extreme value of the error represents the attainable accuracy which can be obtained and the corresponding extreme time gives information about how fast the transients are. The extreme values of the error and the corresponding time are treated here as functions of the roots of the characteristic equation. The proposed analytical formulae allow designing systems with prescribed dynamic properties.

Keywords: extremal dynamic properties, oscillatory systems, extremal time.

1. Introduction

Oscillations can be observed in electrical, mechanical and many other types of systems. Analytical results allow deep inspection and understanding of the system behavior. The proposed method allows the design of a system with required values of the amplitude and period of the oscillations.

2. Problem statement

Let us consider the linear differential equation determining error in a linear system of the n-th order with lumped and constant parameters:

$$x^{(n)}(t) + a_1 x^{(n-1)}(t) + \cdots + a_{n-1}x^{(1)}(t) + a_n x(t) = 0. \quad (1)$$

The initial conditions are determined by the force function and the system's parameters.

Let us assume, in general, that

$$x^{(i)}(0) = c_{i+1} \neq 0, \quad i = 0, 1, \ldots, n-1.$$

The characteristic equation of (1) is

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1}s + a_n = 0. \quad (2)$$

The solution of Eqn. (1) has the form

$$x(t) = \sum_{k=1}^{m} A_k e^{s_k} t + \sum_{k=1}^{p} \Big[B_k \cos(\omega_k t) + C_k \sin(\omega_k t) \Big] e^{\alpha_k t}, \quad (3)$$

where $A_k, B_k, c_k, s_k, \alpha_k, \omega_k$ are real numbers, s_k are real roots and $\alpha_k + j\omega_k = r_k$, $\alpha_k - j\omega_k = \hat{r}_k$ $(k = 1, 2, \ldots, p)$ are complex conjugate roots.

The necessary condition for the error $x(t)$ to attain an extremal value at $t = \tau$ is given by the relation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sum_{k=1}^{m} A_k s_k e^{s_k t} + \sum_{k=1}^{p} \Big[(-B_k \sin\omega_k\tau + C_k \cos\omega_k\tau)\omega_k + (B_k \cos\omega_k\tau + C_k \sin\omega_k\tau)\alpha_k \Big] e^{\alpha_k\tau} = 0. \quad (4)$$

The constants are determined from

$$x^{(i)}(0) = c_{i+1} = \sum_{k=1}^{m} A_k s_k^i + \sum_{k=1}^{p} \left[B_k \mathrm{Re}(r_k^i) + C_k \mathrm{Im}(r_k^i) \right], \quad i = 0, 1, \ldots, n-1. \quad (5)$$

The extreme value of the dynamic error is

$$x(\tau) = \sum_{k=1}^{m} A_k e^{s_k \tau} + \sum_{k=1}^{p} \left[B_k \cos(\omega_k \tau) + C_k \sin(\omega_k \tau) \right] e^{\alpha_k \tau}. \tag{6}$$

The extremum of the extreme value of the dynamic error given by Eqn. (6), computed with regard to the parameters s_k, α_k and ω_k, is obtained by equating the respective partial derivatives of $x(\tau)$ to zero.

Denoting by

$$\left(\frac{\partial x(\tau)}{\partial s_k}\right)^*, \quad \left(\frac{\partial x(\tau)}{\partial \alpha_k}\right)^*, \quad \left(\frac{\partial x(\tau)}{\partial \omega_k}\right)^*$$

the partial derivatives of the expression (6) for constant τ, we may write

$$\begin{cases} \dfrac{\partial x(\tau)}{\partial s_k} = \left(\dfrac{\partial x(\tau)}{\partial s_k}\right)^* + \dfrac{\partial x(\tau)}{\partial \tau}\dfrac{\partial \tau}{\partial s_k}, \\ \dfrac{\partial x(\tau)}{\partial \alpha_k} = \left(\dfrac{\partial x(\tau)}{\partial \alpha_k}\right)^* + \dfrac{\partial x(\tau)}{\partial \tau}\dfrac{\partial \tau}{\partial \alpha_k}, \\ \dfrac{\partial x(\tau)}{\partial \omega_k} = \left(\dfrac{\partial x(\tau)}{\partial \omega_k}\right)^* + \dfrac{\partial x(\tau)}{\partial \tau}\dfrac{\partial \tau}{\partial \omega_k}. \end{cases} \tag{7}$$

However, from Eqn. (4) we have

$$x^{(1)}(t)\Big|_{t=\tau} = 0,$$

and therefore

$$\begin{cases} \dfrac{\partial x(\tau)}{\partial s_k} = \left(\dfrac{\partial x(\tau)}{\partial s_k}\right)^*, \\ \dfrac{\partial x(\tau)}{\partial \alpha_k} = \left(\dfrac{\partial x(\tau)}{\partial \alpha_k}\right)^*, \\ \dfrac{\partial x(\tau)}{\partial \omega_k} = \left(\dfrac{\partial x(\tau)}{\partial \omega_k}\right)^*. \end{cases} \tag{8}$$

We obtain the following conditions:

$$\begin{cases} \sum\limits_{k=1}^{m} \dfrac{\partial A_k}{\partial s_j} e^{s_k\tau} + A_j \tau e^{s_j \tau} \\ \quad + \sum\limits_{k=1}^{p} \left(\dfrac{\partial B_k}{\partial s_j} \cos\omega_k\tau + \dfrac{\partial C_k}{\partial s_j} \sin\omega_k\tau \right) e^{\alpha_k\tau} = 0, \\ \qquad\qquad j = 1, 2, \dots, m, \\ \sum\limits_{k=1}^{m} \dfrac{\partial A_k}{\partial \alpha_j} e^{s_k\tau} \\ \quad + \sum\limits_{k=1}^{p} \left(\dfrac{\partial B_k}{\partial \alpha_j} \cos\omega_k\tau + \dfrac{\partial C_k}{\partial \alpha_j} \sin\omega_k\tau \right) e^{\alpha_k\tau} \\ \quad + (B_j \cos\omega_j\tau + C_j \sin\omega_j\tau) e^{\alpha_j\tau}\tau = 0, \\ \sum\limits_{k=1}^{m} \dfrac{\partial A_k}{\partial \omega_j} e^{s_k\tau} \\ \quad + \sum\limits_{k=1}^{p} \left(\dfrac{\partial B_k}{\partial \omega_j} \cos\omega_k\tau + \dfrac{\partial C_k}{\partial \omega_j} \sin\omega_k\tau \right) e^{\alpha_k\tau} \\ \quad + (C_j \cos\omega_j\tau - B_j \sin\omega_j\tau) e^{\alpha_j\tau}\tau = 0, \\ \qquad\qquad j = 1, 2, \dots, p. \end{cases} \tag{9}$$

In this way, we have a system of n linear and homogeneous equations with n unknowns which are

$$e^{s_k\tau}, \quad e^{\alpha_k\tau} \sin\omega_k\tau, \quad e^{\alpha_k\tau} \cos\omega_k\tau.$$

The determinant of the system (9) must vanish if there are nontrivial solutions. The same determinant (after being reflected about one of the main diagonals) is

$$|D + A\tau|, \tag{10}$$

where D and A are matrices determined by the following equations:

$$\begin{cases} D = \sum\limits_{j=1}^{m}\sum\limits_{k=1}^{m} \dfrac{\partial A_j}{\partial s_k} E_{jk} \\ \quad + \sum\limits_{j=1}^{p}\sum\limits_{k=1}^{m} \left(\dfrac{\partial B_j}{\partial s_k} E_{m+2j-1,k} + \dfrac{\partial C_j}{\partial s_k} E_{m+2j,k} \right) \\ \quad + \sum\limits_{j=1}^{m}\sum\limits_{k=1}^{p} \left(\dfrac{\partial A_j}{\partial \alpha_k} E_{j,m+2k-1} + \dfrac{\partial A_j}{\partial \omega_k} E_{j,m+2k} \right) \\ \quad + \sum\limits_{j=1}^{p}\sum\limits_{k=1}^{p} \Bigg[\left(\dfrac{\partial B_j}{\partial \alpha_k} E_{m+2j-1,m+2k-1} \\ \quad + \dfrac{\partial B_j}{\partial \omega_k} E_{m+2j-1,m+2k} \right) \\ \quad + \left(\dfrac{\partial C_j}{\partial \alpha_k} E_{m+2j,m+2k-1} + \dfrac{\partial C_j}{\partial \omega_k} E_{m+2j,m+2k} \right) \Bigg], \\ \\ A = \sum\limits_{j=1}^{m} A_j E_{jj} \\ \quad + \sum\limits_{j=1}^{p} \left[B_j \left(E_{m+2j-1,m+2j-1} - E_{m+2j,m+2j} \right) \\ \quad + C_j \left(E_{m+2j-1,m+2j} + E_{m+2j,m+2j-1} \right) \right], \end{cases} \tag{11}$$

$$\begin{cases} E_{jk} = \left(e_{\mu,\nu}^{(jk)} \right), \quad \mu, \nu = 1, \dots, n, \\ e_{\mu,p}^{(jk)} = \delta_{\mu j}\delta_{\nu k} = \begin{cases} 1 & \text{for } \mu = j, \ \nu = k \\ 0 & \text{otherwise.} \end{cases} \end{cases} \tag{12}$$

Finally, we have

$$|D + A\tau| = 0, \tag{13}$$

for the unknown τ and the system (9) yields (after some algebraic manipulations) the following equation:

$$(-1)^n \tau^n \prod_{k=1}^{m} A_k \prod_{k=1}^{p} \left(B_k^2 + C_k^2 \right) = 0.$$

We obtain the following necessary condition.

Theorem 1. (Górecki and Turowicz, 1965) *The necessary condition for the extremal extremum $x(\tau)$ as the function of $(\tau, s_1, s_2, \dots, s_n)$ is*

$$(-1)^n \tau^n \prod_{k=1}^{m} A_k \prod_{k=1}^{p} \left(B_k^2 + C_k^2 \right) = 0. \tag{14}$$

The relation (14) can be fulfilled if at least one of the conditions is met:

$$\tau = 0, \tag{15}$$

which means

$$c_2 = 0, \tag{16}$$

or

$$A_k = 0 \tag{17}$$

or

$$B_k^2 + C_k^2 = 0. \tag{18}$$

The conditions (16) or (17) lead to a reduced order of Eqn. (1).

It might be asked whether the time τ, corresponding to the extreme value of the dynamic error, attains an extreme value with respect to the parameters s_k, α_k, ω_k. To investigate this, we assume that

$$\begin{aligned} &\frac{\partial \tau}{\partial s_k} = 0, && k = 1, \dots, m,\\ &\frac{\partial \tau}{\partial \alpha_k} = \frac{\partial \tau}{\partial \omega_k} = 0, && k = 1, \dots, \text{p}. \end{aligned} \tag{19}$$

We compute the partial derivatives of Eqn. (9), taking into account Eqn. (19):

$$\begin{aligned} &\sum_{k=1}^{m} \frac{\partial A_k}{\partial s_j} s_k e^{s_k \tau} + (1 + s_j \tau) A_j e^{s_j \tau}\\ &+\sum_{k=1}^{p} \left[\left(\frac{\partial B_k}{\partial s_j} \cos \omega_k \tau + \frac{\partial C_k}{\partial s_j} \sin \omega_k \tau \right) \alpha_k \right.\\ &+ \left. \left(\frac{\partial C_k}{\partial s_j} \cos \omega_k \tau - \frac{\partial B_k}{\partial s_j} \sin \omega_k \tau \right) \right] e^{s_k \tau} = 0,\\ &\qquad j = 1, \dots, m, \end{aligned} \tag{20}$$

$$\begin{aligned} &\sum_{k=1}^{m} \frac{\partial A_k}{\partial \alpha_j} s_k e^{s_k \tau}\\ &+ \sum_{k=1}^{p} \left(\frac{\partial B_k}{\partial \alpha_j} \cos \omega_k \tau + \frac{\partial C_k}{\partial \alpha_j} \sin \omega_k \tau \right) \alpha_k\\ &+ [(B_j \cos \omega_j \tau + C_j \sin \omega_j \tau)(1 + \alpha_j \tau)\\ &+ (C_j \cos \omega_j \tau - B_j \sin \omega_j \tau) \omega_j \tau] e^{\alpha_j \tau} = 0,\\ &\qquad j = 1, \dots, p, \end{aligned} \tag{21}$$

$$\begin{aligned} &\sum_{k=1}^{m} \frac{\partial A_k}{\partial \omega_j} s_k e^{s_k \tau}\\ &+ \sum_{k=1}^{p} \left[\left(\frac{\partial B_k}{\partial \omega_j} \cos \omega_k \tau + \frac{\partial C_k}{\partial \omega_j} \sin \omega_k \tau \right) \alpha_k \right.\\ &+ \left. \left(\frac{\partial C_k}{\partial \omega_j} \cos \omega_k \tau - \frac{\partial B_k}{\partial \omega_j} \sin \omega_k \tau \right) \omega_k \right] e^{\alpha_k \tau}\\ &+ [(C_j \cos \omega_j \tau - B_j \sin \omega_j \tau)(1 + \alpha_j \tau)\\ &- (B_j \cos \omega_j \tau + C_j \sin \omega_j \tau) \omega_j \tau] e^{\alpha_j \tau} = 0,\\ &\qquad j = 1, \dots, p. \end{aligned} \tag{22}$$

Let

$$\begin{aligned} F = &\sum_{\mu=1}^{m} s_\mu E_{\mu,\mu}\\ &+ \sum_{\mu=1}^{p} [\alpha_\mu (E_{m+2\mu-1,m+2\mu-1} + E_{m+2\mu,m+2\mu-1})\\ &+ \omega_\mu (E_{m+2\mu-1,m+2\mu} - E_{m+2\mu,m+2\mu-1})]. \end{aligned} \tag{23}$$

Using the relations (11), Eqns. (20)–(23) yield, after equating the determinant to zero,

$$|FD + A + FA\tau| = 0, \tag{24}$$

$$\begin{aligned} &(-1)^p \prod_{k=1}^{m} A_k \prod_{k=1}^{p} (B_k^2 + C_k^2)\\ &\times \prod_{k=1}^{m} s_k \prod_{k=1}^{p} (\alpha_k^2 + \omega_k^2)\\ &\times \tau^{n-1} \left[\tau + \sum_{k=1}^{m} \frac{1}{s_k} + \sum_{k=1}^{p} \left(\frac{1}{r_k} + \frac{1}{\hat{r}_k} \right) \right] = 0. \end{aligned}$$

We obtain the following necessary condition for the extreme time $\tau(s_1, \dots, s_n)$.

Theorem 2. (Górecki and Turowicz, 1965) *The necessary condition for the extreme time τ as the function of $s_1, \dots, s_n$ is*

$$\begin{aligned} &(-1)^p \prod_{k=1}^{m} A_k \prod_{k=1}^{p} (B_k^2 + C_k^2)\\ &\times \prod_{k=1}^{m} s_k \prod_{k=1}^{p} (\alpha_k^2 + \omega_k^2)\\ &\times \tau^{n-1} \left[\tau + \sum_{k=1}^{m} \frac{1}{s_k} + \sum_{k=1}^{p} \left(\frac{1}{r_k} + \frac{1}{\hat{r}_k} \right) \right] = 0. \end{aligned} \tag{25}$$

The relation (25) can be fulfilled if at least one of the conditions is satisfied:

$$\tau = 0,$$

which means

$$c_2 = 0, \tag{26}$$

or

$$\begin{cases} A_k = 0,\\ B_k^2 + C_k^2 = 0 \end{cases} \tag{27}$$

or

$$\begin{cases} s_k = 0,\\ \alpha_k^2 + \omega_k^2 = 0 \end{cases} \tag{28}$$

or, finally and most interestingly,

$$\tau = -\left[\sum_{k=1}^{m} \frac{1}{s_k} + \sum_{k=1}^{p} \left(\frac{1}{r_k} + \frac{1}{\hat{r}_k} \right) \right]. \tag{29}$$

Using Vieta's formulae, τ from (29) is equal to

$$\tau = \frac{a_{n-1}}{a_n}, \tag{30}$$

where a_{n-1} and a_n are the coefficients of Eqn. (2).

The set of equations (20)–(22) gives also another necessary condition for the extreme time $\tau(s_1, \ldots, s_n)$, which was presented by Górecki and Turowicz (1966).

Theorem 3. *The necessary condition for the extreme time $\tau(s_1, \ldots, s_n)$ is*

$$D_n(\tau) = \begin{vmatrix} c_1 & c_2 & c_3 & c_4 & \ldots & c_n \\ -\frac{a_{n-2}}{a_n} & \tau & -1 & 0 & \ldots & 0 \\ -\frac{a_{n-3}}{a_n} & 0 & \tau & -2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -\frac{a_1}{a_n} & 0 & 0 & 0 & \ldots & 2-n \\ -\frac{1}{a_n} & 0 & 0 & 0 & \ldots & \tau \end{vmatrix} = 0. \tag{31}$$

It is obvious from the condition (31) that there may exist $n-1$ values of τ. Taking into account that $\tau = a_{n-1}/a_n$, we eventually obtain from Eqn. (31) $n-2$ values of τ. In general if all $\tau_i > 0 \ \ (i = 1, 2, \ldots, n-1)$ exist, then all the ratios $c_i/c_1 \ \ (i = 2, 3, \ldots, n-1)$ can be determined univocally.

The solution of the algebraic equation (31) for a higher degree may be obtained using additional assumptions (see Górecki, 2009; Górecki and Zaczyk, 2010).

After substitution of $\tau = a_{n-1}/a_n$ into Eqn. (31), we obtain the relation between the initial conditions $c_{i+1}, \ i = 0, 1, \ldots, n-1$, and coefficients $a_k, \ \ k = 1, 2, \ldots, n$.

$$D_n = \begin{vmatrix} c_1 & c_2 & c_3 & c_4 & \ldots \\ a_{n-2} & -a_{n-1} & a_n & 0 & \ldots \\ a_{n-3} & 0 & -a_{n-1} & 2a_n & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & \ldots \end{vmatrix} = 0. \tag{32}$$

3. Problem solution

Theorem 4. (Sędziwy, 1969) *The sufficient conditions for the extreme $\tau(s_1, \ldots, s_n)$ are*

$$\frac{\mathrm{d}^2\tau}{\mathrm{d}s_k^2} \neq 0, \qquad k = 1, \ldots, n, \tag{33}$$

$$\frac{\mathrm{d}^2\tau}{\mathrm{d}s_k \mathrm{d}s_j} = 0, \qquad k \neq j, \ \ k = 1, \ldots, n. \tag{34}$$

The Hessian $H_n \neq 0$, where

$$H_k = \begin{vmatrix} \frac{\mathrm{d}^2\tau}{\mathrm{d}s_1^2} & 0 & \ldots & 0 \\ 0 & \frac{\mathrm{d}^2\tau}{\mathrm{d}s_2^2} & \ldots & 0 \\ \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 0 & \frac{\mathrm{d}^2\tau}{\mathrm{d}s_k^2} \end{vmatrix} \neq 0\,, \tag{35}$$
$$k = 1, 2, \ldots, n$$

If $H_{2k-1} < 0$ and $H_{2k} > 0$ for $k = 1, 2, \ldots, n$, then τ attains the maximum value with respect to $s_1, \ldots, s_n$. If

$$H_{2k-1} > 0 \quad \text{and} \quad H_{2k} > 0 \tag{36}$$

for $k = 1, 2, \ldots, n$, then τ attains the minimum value with respect to $s_1, \ldots, s_n$.

Theorem 5. *The conditions for the existence of $\tau_1(s_1, \ldots, s_n, c_1, \ldots, c_{n-1})$ are*

$$x^{(1)}(\tau) = 0, \tag{37}$$

$$D_n(a_1, \ldots, a_n, c_1, \ldots, c_{n-1}) = 0. \tag{38}$$

These two equations, (37) and (38), are linear with respect to the initial conditions $c_1, \ldots, c_{n-1}$. It is easy to solve them.

Theorem 6. *The conditions for the existence of $\tau_2, \tau_3, \ldots, \tau_{n-2}$ are*

$$x^{(1)}(\tau) = 0, \tag{39}$$

$$D_n(\tau) = 0, \tag{40}$$

where $\tau_1 = a_{n-1}/a_n$.

4. Particular cases

We illustrate the theorems in the particular cases of the equations.

4.1. Second-order equation ($n = 2$).

Let us consider the second order differential equations

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + a_1\frac{\mathrm{d}x}{\mathrm{d}t} + a_2 x = 0, \tag{41}$$

with the initial conditions

$$x(0) = c_1, \quad x^{(1)}(0) = c_2.$$

The characteristic equation of Eqn. (41) is

$$s^2 + a_1 s + a_2 = 0 \ \ , \qquad a_1, a_2 > 0. \tag{42}$$

We denote by s_1, s_2 the roots of this equation and consider three cases:

1. $s_1 \neq s_2$ real and negative,
2. $s_1 = s_2$ real and negative,
3. $s_1 = \alpha + j\omega, \ \ s_2 = \alpha - j\omega$ complex with $\alpha < 0$.

4.1.1. First case: $s_1 \neq s_2$. The solution of Eqn. (41) is

$$x(t) = \frac{s_2 c_1 - c_2}{s_2 - s_1} e^{s_1 t} + \frac{s_1 c_1 - c_2}{s_1 - s_2} e^{s_2 t}. \tag{43}$$

The derivative of $x(t)$ is equal to

$$\begin{aligned} x^{(1)}(t) = &\frac{s_1(s_2 c_1 - c_2)}{s_2 - s_1} e^{s_1 t} \\ &+ \frac{s_2(s_1 c_1 - c_2)}{s_1 - s_2} e^{s_2 t} \end{aligned} \tag{44}$$

The necessary condition for the extremum $x(t)$ is

$$x^{(1)}(\tau) = 0. \tag{45}$$

From the relation (44), using the condition (45), we obtain

$$e^{(s_1 - s_2)\tau} = \frac{s_2(s_1 c_1 - c_2)}{s_1(s_2 c_1 - c_2)}. \tag{46}$$

The necessary conditions for τ as the function of (s_1, s_2) attains an extremum are

$$\frac{\mathrm{d}\tau}{\mathrm{d}s_1} = \frac{1}{s_2 - s_1}\left(\tau - \frac{c_2}{s_1(s_1 c_1 - c_2)}\right) = 0, \tag{47}$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}s_2} = \frac{1}{s_2 - s_1}\left(\tau - \frac{c_2}{s_2(s_2 c_1 - c_2)}\right) = 0. \tag{48}$$

It is easy to show that there may be at most one value of extreme τ. In consequence, it is required that

$$\tau = \frac{c_2}{s_1(s_1 c_1 - c_2)} = \frac{c_2}{s_2(s_2 c_1 - c_2)}. \tag{49}$$

From (49) we obtain

$$s_1 + s_2 = \frac{c_2}{c_1}, \quad \frac{c_2}{c_1} < 0. \tag{50}$$

Substitution c_2 from (50) into the relation (49) gives

$$\begin{aligned} \tau &= \frac{c_2}{s_2(s_2 c_1 - c_2)} \\ &= \frac{(s_1 + s_2)c_1}{s_2[s_2 c_1 - (s_1 + s_2)c_1]} \\ &= -\frac{s_1 + s_2}{s_1 s_2} = -\left(\frac{1}{s_1} + \frac{1}{s_2}\right). \end{aligned} \tag{51}$$

Sufficient condition for $\tau(s_1, s_2)$. After differentiating (47) and (48), we obtain

$$\frac{\mathrm{d}^2\tau}{\mathrm{d}s_1^2} = \frac{c_2(2s_1 c_1 - c_2)}{s_1^2(s_1 c_1 - c_2)^2}, \tag{52}$$

$$\frac{\mathrm{d}^2\tau}{\mathrm{d}s_2^2} = \frac{c_2(2s_2 c_1 - c_2)}{s_2^2(s_2 c_1 - c_2)^2}, \tag{53}$$

$$\frac{\mathrm{d}^2\tau}{\mathrm{d}s_1 \mathrm{d}s_2} = -\frac{1}{(s_2 - s_1)^2}\frac{\mathrm{d}\tau}{\mathrm{d}s_2}, \tag{54}$$

but $\mathrm{d}\tau/\mathrm{d}s_2 = 0$ (see (48)) and

$$\frac{\mathrm{d}^2\tau}{\mathrm{d}s_1 \mathrm{d}s_2} = 0. \tag{55}$$

The Hessian for $\tau = -\left(\frac{1}{s_1} + \frac{1}{s_2}\right)$ is equal to

$$\begin{aligned} H &= \begin{vmatrix} \frac{\mathrm{d}^2\tau}{\mathrm{d}s_1^2} & \frac{\mathrm{d}^2\tau}{\mathrm{d}s_1 \mathrm{d}s_2} \\ \frac{\mathrm{d}^2\tau}{\mathrm{d}s_1 \mathrm{d}s_2} & \frac{\mathrm{d}^2\tau}{\mathrm{d}s_2^2} \end{vmatrix} \\ &= \begin{vmatrix} \frac{c_2(2s_1 c_1 - c_2)}{s_1^2(s_1 c_1 - c_2)^2} & 0 \\ 0 & \frac{c_2(2s_2 c_1 - c_2)}{s_2^2(s_2 c_1 - c_2)^2} \end{vmatrix} \\ &= \frac{c_2^2(2s_1 c_1 - c_2)(2s_2 c_1 - c_2)}{s_1^2 s_2^2(s_1 c_1 - c_2)^2(s_2 c_1 - c_2)^2}, \end{aligned} \tag{56}$$

and, taking into account (50), we finally have

$$H = \left[\frac{(s_2^2 - s_1^2)}{s_1^2 s_2^2}\right]^2 > 0$$

This means that if there exists an extremum $\tau(s_1, s_2)$, $s_1 \neq s_2$, then it has to be a minimum.

Existence condition. Substituting c_2 from the relation (50) into the relation (46), we obtain

$$\tau = \frac{1}{s_2 - s_1} \ln\left(\frac{s_1}{s_2}\right)^2. \tag{57}$$

Comparing with τ from (51), we have the equation

$$\ln\left(\frac{s_1}{s_2}\right)^2 = \left(\frac{s_2}{s_1} - \frac{s_1}{s_2}\right). \tag{58}$$

The only solution of Eqn. (58) is

$$s_1 = s_2 = s, \tag{59}$$

which is in contradiction with the assumption that $s_1 \neq s_2$.

We deduce that there does not exist an extremum τ for real $s_1 \neq s_2$.

4.1.2. Second case: $s_1 = s_2 = s < 0$. The solution of Eqn. (41) is

$$x(t) = [c_1 + (c_2 - s c_1)t]e^{st} \tag{60}$$

and its derivative is

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = [c_2 + (c_2 - s c_1)st]e^{st}. \tag{61}$$

From the necessary condition $x^{(1)}(t) = 0$ and (61) we obtain

$$\tau = \frac{c_2}{(s c_1 - c_2)s}. \tag{62}$$

The derivative is

$$\frac{\mathrm{d}\tau}{\mathrm{d}s} = -\frac{(2sc_1 - c_2)c_2}{s^2(sc_1 - c_2)^2} \tag{63}$$

From the condition $\mathrm{d}\tau/\mathrm{d}s = 0$ we finally have $c_2 = 0$. In consequence, $\tau_1 = 0$ or

$$s = \frac{1}{2}\frac{c_2}{c_1} < 0 \tag{64}$$

and

$$\tau_2 = -\frac{2}{s} = -4\frac{c_1}{c_2} > 0, \tag{65}$$

$$x(\tau_2) = -c_1 e^{-2}, \quad c_1 > 0 \tag{66}$$

$$\frac{\mathrm{d}^2\tau}{\mathrm{d}s^2} = -32\left(\frac{c_1}{c_2}\right)^3, \quad c_1c_2 < 0. \tag{67}$$

In conclusion, τ has a minimum with respect to s.

4.1.3. Third case: $s_1 = \alpha + j\omega, s_2 = \alpha - j\omega$ are complex and $\alpha < 0$. From the relation (46), we have

$$e^{2j\omega\tau} = \frac{[(\alpha^2+\omega^2)c_1 - \alpha c_2] + j\omega c_2}{[(\alpha^2+\omega^2)c_1 - \alpha c_2] - j\omega c_2}. \tag{68}$$

From the relation (68), we obtain

$$\cos(2\omega\tau) = \frac{[(\alpha^2+\omega^2)c_1 - \alpha c_2]^2 - \omega^2 c_2^2}{[(\alpha^2+\omega^2)c_1 - \alpha c_2]^2 + \omega^2 c_2^2}, \tag{69}$$

$$\sin(2\omega\tau) = \frac{2\omega c_2[(\alpha^2+\omega^2)c_1 - \alpha c_2]}{[(\alpha^2+\omega^2)c_1 - \alpha c_2]^2 + \omega^2 c_2^2}. \tag{70}$$

After division of (69) by (70), we find

$$\cot(2\omega\tau) = \frac{[(\alpha^2+\omega^2)c_1 - \alpha c_2]^2 - \omega^2 c_2^2}{2\omega c_2[(\alpha^2+\omega^2)c_1 - \alpha c_2]}. \tag{71}$$

From the necessary condition

$$\frac{\mathrm{d}\tau}{\mathrm{d}\alpha} = 0, \tag{72}$$

we have

$$\frac{2j\omega c_2(2\alpha c_1 - c_2)}{[(c_2 - c_1\alpha) + j\omega c_1]^2(\alpha + j\omega)^2} = 0. \tag{73}$$

From (73) we deduce that $c_2 = 0$, then $\tau = 0$ or $\omega = 0$ or

$$\alpha = \frac{1}{2}\frac{c_2}{c_1}, \quad \frac{c_2}{c_1} < 0. \tag{74}$$

After using (74) in (71), we obtain

$$\cot(2\omega\tau) = \frac{(\omega^2-\alpha^2)^2 - 4\alpha^2\omega^2}{4\alpha\omega(\omega^2-\alpha^2)}. \tag{75}$$

From the necessary condition

$$\frac{\mathrm{d}\tau}{\mathrm{d}\omega} = 0, \tag{76}$$

after differentiating (68), we have

$$-2\tau\sin(2\omega\tau) = -4\frac{\omega c_2^2[c_1(\alpha^2-\omega^2) - c_2\alpha]}{[(c_2 - c_1\alpha)^2 + c_1^2\omega^2]^2} \times \frac{[c_1(\alpha^2+\omega^2) - c_2\alpha]}{(\alpha^2+\omega^2)}. \tag{77}$$

After elimination of c_2, using (74), we get

$$-2\tau\sin(2\omega\tau) = -16\frac{(\alpha-\omega)(\alpha+\omega)\omega\alpha^2}{(\alpha^2+\omega^2)^3} \tag{78}$$

and

$$2\tau\cos(2\omega\tau) = 2\frac{c_2[c_1(\alpha^2-\omega^2) - c_2\alpha]}{(\alpha^2+\omega^2)^2} \times \frac{[c_1(\alpha^2+\omega^2) - c_2(\alpha+\omega)]}{[(c_2 - c_1\alpha)^2 + c_1^2\omega^2]^2} \times [c_1(\alpha^2+\omega^2) + c_2(\alpha-\omega)]. \tag{79}$$

After elimination of c_2 from (74),

$$2\tau\cos(2\omega\tau) = -4\frac{\alpha(\alpha^2 - 2\alpha\omega - \omega^2)(\alpha^2 + 2\alpha\omega - \omega^2)}{(\alpha^2+\omega^2)^3}. \tag{80}$$

From (77) and (80),

$$4\tau^2 = 4\frac{c_2^2[c_1(\alpha^2-\omega^2) - c_2\alpha]^2}{(\alpha^2+\omega^2)[(c_2 - c_1\alpha)^2 + c_1^2\omega^2]^2}. \tag{81}$$

After elimination of c_2,

$$\tau^2 = \frac{(2\alpha)^2}{(\alpha^2+\omega^2)^2}, \tag{82}$$

and, finally, for $\tau > 0$,

$$\tau = -\frac{2\alpha}{\alpha^2+\omega^2}, \quad \alpha < 0. \tag{83}$$

The determinant (56) in this case for $s_1 = \alpha + j\omega$ and $s_2 = \alpha - j\omega$ is

$$H = \left[\frac{(s_2^2 - s_1^2)}{s_1^2 s_2^2}\right]^2 = -\left[\frac{4\alpha\omega}{\alpha^2+\omega^2}\right]^2 < 0. \tag{84}$$

It is obvious that τ has a maximum with respect to ω.

Sufficient condition. After dividing both the sides of (79) by (77), we have

$$\cot(2\omega\tau) = \frac{1}{4}\frac{(\alpha^2 - 2\alpha\omega - \omega^2)(\alpha^2 + 2\alpha\omega - \omega^2)}{\alpha\omega(\alpha-\omega)(\alpha+\omega)}. \tag{85}$$

Comparing (71) with (85), for $\alpha = \frac{1}{2}c_2/c_1$ we obtain

$$\omega = \pm\alpha. \tag{86}$$

Substitution of (86) into (83) gives

$$\tau = -\frac{1}{\alpha}, \tag{87}$$

which, together with (74), yields

$$\tau = -2\frac{c_1}{c_2}, \quad \frac{c_1}{c_2} < 0. \tag{88}$$

4.2. Third-order equation ($n = 3$). Consider the following equation (Górecki and Zaczyk, 2013):

$$\frac{\mathrm{d}^3x(t)}{\mathrm{d}t^3} + a_1\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + a_2\frac{\mathrm{d}x(t)}{\mathrm{d}t} + a_3x(t) = 0. \quad (89)$$

The initial conditions are

$$\begin{cases} x(0) = c_1, \\ x^{(1)}(0) = c_2, \\ x^{(2)}(0) = c_3. \end{cases}$$

The characteristic equation is

$$s^3 + a_1s^2 + a_2s + a_3 = 0. \quad (90)$$

We assume that the roots of (90) are

$$s_1, \qquad s_2 = \alpha + j\omega, \qquad s_3 = \alpha - j\omega,$$

where $\alpha < 0$.

The solution of Eqn. (89) is

$$\begin{aligned} x(t) = &\frac{c_3 - c_2(s_2+s_3) + c_1s_2s_3}{(s_1-s_2)(s_1-s_3)}e^{s_1t} \\ &+ \frac{c_3 - c_2(s_3+s_1) + c_1s_3s_1}{(s_2-s_3)(s_2-s_1)}e^{s_2t} \\ &+ \frac{c_3 - c_2(s_1+s_2) + c_1s_1s_2}{(s_3-s_1)(s_3-s_2)}e^{s_3t}. \end{aligned} \quad (91)$$

The derivative of $x(t)$ is equal to

$$\begin{aligned} x^{(1)}(t) = &\frac{s_1[c_3 - c_2(s_2+s_3) + c_1s_2s_3]}{(s_1-s_2)(s_1-s_3)}e^{s_1t} \\ &+ \frac{s_2[c_3 - c_2(s_3+s_1) + c_1s_3s_1]}{(s_2-s_3)(s_2-s_1)}e^{s_2t} \\ &+ \frac{s_3[c_3 - c_2(s_1+s_2) + c_1s_1s_2]}{(s_3-s_1)(s_3-s_2)}e^{s_3t}. \end{aligned} \quad (92)$$

The necessary condition for the extremum $x(t)$ is

$$x^{(1)}(t)\Big|_{t=\tau} = 0. \quad (93)$$

After substitution of s_1, $s_2 = \alpha + j\omega$, and $s_3 = \alpha - j\omega$ into (92) and using (93), Eqn. (92) takes the form

$$\begin{aligned} &x^{(1)}(\tau) \\ &= -\frac{1}{2}j\left[\frac{-4js_1c_2\alpha\omega + 2js_1\omega^3c_1 + 2js_1c_1\alpha^2\omega}{(\alpha-j\omega-s_1)(\alpha+j\omega-s_1)\omega}\right. \\ &\quad \left.+\frac{2jc_3\omega s_1}{(\alpha-j\omega-s_1)(\alpha+j\omega-s_1)\omega}\right]e^{s_1\tau} \\ &\quad -\frac{1}{2}j\left[\frac{-c_3\alpha s_1 + s_1c_1\omega^2\alpha - jc_3\omega s_1 - js_1\omega^3c_1}{(\alpha-j\omega-s_1)(\alpha+j\omega-s_1)\omega}\right. \\ &\quad +\frac{s_1^2c_2\alpha + c_3\alpha^2 - js_1c_1\alpha^2\omega + jc_2\omega^3 + js_1^2c_2\omega - c_2\alpha^3}{(\alpha-j\omega-s_1)(\alpha+j\omega-s_1)\omega} \\ &\quad \left.+\frac{jc_2\alpha^2\omega - c_2\omega^2\alpha - s_1^2c_1\alpha^2 - s_1^2c_1\omega^2 + s_1c_1\alpha^3}{(\alpha-j\omega-s_1)(\alpha+j\omega-s_1)\omega}\right] \\ &\quad \times e^{(\alpha+j\omega)\tau} \\ &\quad -\frac{1}{2}j\left[\frac{c_2\omega^2\alpha - c_3\omega^2 + c_2\alpha^3 + c_3\alpha s_1 - jc_3\omega s_1}{(\alpha-j\omega-s_1)(\alpha+j\omega-s_1)\omega}\right. \\ &\quad -\frac{s_1c_1\omega^2\alpha + jc_2\omega^3 - js_1\omega^3c_1 - js_1c_1\alpha^2\omega - s_1^2c_2\alpha}{(\alpha-j\omega-s_1)(\alpha+j\omega-s_1)\omega} \\ &\quad +\frac{js_1^2c_2\omega + s_1^2c_1\alpha^2 + jc_2\alpha^2\omega}{(\alpha-j\omega-s_1)(\alpha+j\omega-s_1)\omega} \\ &\quad \left.-\frac{s_1c_1\alpha^3 + s_1^2c_1\omega^2 - c_3\alpha^2}{(\alpha-j\omega-s_1)(\alpha+j\omega-s_1)\omega}\right] \\ &\quad \times e^{(\alpha-j\omega)\tau} = 0. \end{aligned} \quad (94)$$

The derivatives of τ, determined by Eqn. (94), with respect to s_1, α and ω, yield the necessary conditions for the extreme τ:

$$\frac{\mathrm{d}\tau}{\mathrm{d}s_1} = 0, \quad (95)$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}\alpha} = 0, \quad (96)$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}\omega} = 0. \quad (97)$$

We get

$$\begin{aligned} &e^{(-\alpha+s_1)\tau}\cos(\omega\tau) \\ &= \left(\frac{\tau\alpha s_1^2 - \tau s_1\alpha^2 + \tau s_1\omega^2}{s_1(\tau^2\omega^2\alpha^2 + (\alpha+\tau\omega^2)^2)}\right. \\ &\quad +\frac{0.5\tau^2s_1^2\alpha^2 + 0.5\tau^2s_1^2\omega^2 - \tau^2s_1\alpha^3 - \tau^2s_1\omega^2\alpha}{s_1(\tau^2\omega^2\alpha^2 + (\alpha+\tau\omega^2)^2)} \\ &\quad \left.+\frac{0.5\tau^2\alpha^4 + \tau^2\omega^2\alpha^2 + 0.5\tau^2\omega^4 + s_1\alpha}{s_1(\tau^2\omega^2\alpha^2 + (\alpha+\tau\omega^2)^2)}\right) \\ &\quad \times(\alpha+\tau\omega^2), \end{aligned} \quad (98)$$

$$
\begin{aligned}
&e^{(-\alpha+s_1)\tau}\sin(\omega\tau)\\
&=\Bigg(\frac{\tau\alpha s_1^2-\tau s_1\alpha^2+\tau s_1\omega^2}{s_1(\tau^2\omega^2\alpha^2+(\alpha+\tau\omega^2)^2)}\\
&\quad+\frac{0.5\tau^2 s_1^2\alpha^2+0.5\tau^2 s_1^2\omega^2-\tau^2 s_1\alpha^3-\tau^2 s_1\omega^2\alpha}{s_1(\tau^2\omega^2\alpha^2+(\alpha+\tau\omega^2)^2)}\\
&\quad+\frac{0.5\tau^2\alpha^4+\tau^2\omega^2\alpha^2+0.5\tau^2\omega^4+s_1\alpha}{s_1(\tau^2\omega^2\alpha^2+(\alpha+\tau\omega^2)^2)}\Bigg)(\alpha\tau\omega),
\end{aligned}
\tag{99}
$$

$$
e^{2j\omega\tau}=\frac{j\alpha+j\tau\omega^2-\tau\omega\alpha}{j\alpha+j\tau\omega^2+\tau\omega\alpha}. \tag{100}
$$

From Eqn. (100), we have

$$
\cos(2\omega\tau)=\frac{(\alpha+\tau\omega^2)^2-\alpha^2\omega^2\tau^2}{(\alpha+\tau\omega^2)^2+\alpha^2\omega^2\tau^2} \tag{101}
$$

and

$$
\sin(2\omega\tau)=\frac{2(\alpha+\tau\omega^2)\alpha\omega\tau}{(\alpha+\tau\omega^2)^2+\alpha^2\omega^2\tau^2}. \tag{102}
$$

The relations (98) and (99) lead to the assumption that

$$
s_1=\alpha. \tag{103}
$$

In this case, the necessary condition for the extreme τ is

$$
\begin{aligned}
\tau&=-\left(\frac{1}{s_1}+\frac{1}{\alpha+j\omega}+\frac{1}{\alpha-j\omega}\right)\\
&=-\frac{3\alpha^2+\omega^2}{\alpha(\alpha^2+\omega^2)}.
\end{aligned}
\tag{104}
$$

Substitution of τ from the relation (104) into the relation (102) gives

$$
\begin{aligned}
&\sin(2\omega\tau)\\
&\quad=-2\frac{\alpha\omega(3\alpha^2+\omega^2)(\alpha^4-2\alpha^2\omega^2-\omega^4)}{(\alpha^2+\omega^2)(\alpha^4+3\alpha^2\omega^2+\omega^4)}.
\end{aligned}
\tag{105}
$$

One of the solutions of Eqn. (105) is

$$
\omega=\pm\alpha\sqrt{\sqrt{2}-1}. \tag{106}
$$

Then

$$
\sin(2\omega\tau)=0. \tag{107}
$$

Substitution of (106) into (101) gives

$$
\cos(2\omega\tau)=1. \tag{108}
$$

Taking into account (103) and (106) in the relation (104), we finally obtain

$$
\tau=-\frac{1+\sqrt{2}}{\alpha},\qquad \alpha<0. \tag{109}
$$

Substituting $s_1=\alpha,\ s_{2,3}=\alpha\pm j\sqrt{\sqrt{2}-1}\,\alpha$ into (90), we finally obtain that

$$
\alpha=\frac{a_3(9-4\sqrt{2})-a_1a_2}{2a_1^2+2a_2(1-2\sqrt{2})}. \tag{110}
$$

Sufficient conditions. Calculation of the second derivatives of τ with respect to s_1, s_2, s_3 gives the following results for $\tau=-\left(\frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{s_3}\right)$:

$$
\begin{aligned}
\frac{\mathrm{d}^2\tau}{\mathrm{d}s_1^2}&=\exp\left(-\frac{s_1s_3+s_1s_2+s_2s_3}{s_1s_2}\right)\\
&\quad\times\frac{-s_2s_3c_1+s_3c_2-c_3+s_2c_2}{s_1^3s_2^2(s_1s_3-s_3^2+s_1s_2+s_2s_3)}\\
&\quad\times(s_1s_3+s_1s_2+s_2s_3)^2,
\end{aligned}
\tag{111}
$$

$$
\frac{\mathrm{d}^2\tau}{\mathrm{d}s_1\mathrm{d}s_2}=0, \tag{112}
$$

$$
\frac{\mathrm{d}^2\tau}{\mathrm{d}s_1\mathrm{d}s_3}=0, \tag{113}
$$

$$
\begin{aligned}
\frac{\mathrm{d}^2\tau}{\mathrm{d}s_2^2}&=\exp\left(-\frac{s_1s_3+s_1s_2+s_2s_3}{s_1s_2}\right)\\
&\quad\times\frac{-s_1s_3c_1+s_3c_2-c_3+s_1c_2}{s_2^3s_1^2(s_1s_3-s_3^2+s_1s_2+s_2s_3)}\\
&\quad\times(s_1s_3+s_1s_2+s_2s_3)^2,
\end{aligned}
\tag{114}
$$

$$
\frac{\mathrm{d}^2\tau}{\mathrm{d}s_2\mathrm{d}s_3}=0, \tag{115}
$$

$$
\begin{aligned}
\frac{\mathrm{d}^2\tau}{\mathrm{d}s_3^2}&=\exp\left(-\frac{s_1s_3+s_1s_2+s_2s_3}{s_1s_2}\right)\\
&\quad\times\frac{-s_1s_2c_1+s_1c_2-c_3+s_2c_2}{s_2^2s_1^2s_3(s_1s_3-s_3^2+s_1s_2+s_2s_3)}\\
&\quad\times(s_1s_3+s_1s_2+s_2s_3)^2.
\end{aligned}
\tag{116}
$$

The Hessian is equal to

$$
H=\begin{vmatrix}
\frac{\mathrm{d}^2\tau}{\mathrm{d}s_1^2} & 0 & 0\\
0 & \frac{\mathrm{d}^2\tau}{\mathrm{d}s_2^2} & 0\\
0 & 0 & \frac{\mathrm{d}^2\tau}{\mathrm{d}s_3^2}
\end{vmatrix}. \tag{117}
$$

From Eqns. (111), (114) and (116), we obtain that

$$
\begin{aligned}
&H_1\\
&=\exp\left(-\frac{(s_1s_3+s_1s_2+s_2s_3)(s_1+s_2+s_3)}{s_1s_2s_3}\right)\\
&\quad\times\frac{(s_1s_3+s_1s_2+s_2s_3)^6}{s_1^5s_2^5s_3^5(-s_3^2+s_1s_3+s_2s_3+s_1s_2)}\\
&\quad\times\frac{(c_3+s_2s_3c_1-s_2c_2-s_3c_2)}{s_1^2-s_1s_2-s_1s_3-s_2s_3)}\\
&\quad\times\frac{(c_3+s_1s_3c_1-s_1c_2-s_3c_2)}{(-s_2^2+s_2s_3+s_1s_3+s_1s_2)}\\
&\quad\times(c_3+s_1s_2c_1-s_1c_2-s_2c_2),
\end{aligned}
\tag{118}
$$

or, after symmetrization,

$$H_1^3 = \left(\frac{-a_2^6\left(a_3^2c_1^3 + c_1^2c_3a_3a_1 + 2c_1^2c_2a_2a_3\right.}{a_3^5(-4a_2^3 + a_3^2 + 2a_1a_2a_3 + a_2^2a_1^2)} + \frac{3c_1c_3c_2a_3 + c_1c_3c_2a_1a_2 + c_1c_2^2a_3a_1 + c_1c_2^2a_2^2}{a_3^5(-4a_2^3 + a_3^2 + 2a_1a_2a_3 + a_2^2a_1^2)} + \frac{-c_2^3a_3 + c_2^3a_1a_2 + c_1c_3^2a_2}{a_3^5(-4a_2^3 + a_3^2 + 2a_1a_2a_3 + a_2^2a_1^2)} + \frac{c_3c_2^2a_2 + c_3c_2^2a_1^2 + 2a_1c_3^2c_2 + c_3^3)}{a_3^5(-4a_2^3 + a_3^2 + 2a_1a_2a_3 + a_2^2a_1^2)} \right) \times \exp\left(-\frac{a_1a_2}{a_3}\right),$$

$$H_2 = H_1^2, \tag{119}$$

$$H_3 = H_1^3. \tag{120}$$

Sufficient conditions. From (118), (119) and (120), we finally find that

$$H = \begin{vmatrix} H_1 & 0 & 0 \\ 0 & H_1 & 0 \\ 0 & 0 & H_1 \end{vmatrix}. \tag{121}$$

From (121) we deduce that, if

$$H_1 > 0, \tag{122}$$

it is a minimum τ with respect to s_1, s_2, s_3, and if

$$H_1 < 0, \tag{123}$$

τ has a maximum, according to (36), with respect to s_1, s_2, s_3,

$$H = \begin{vmatrix} \frac{d^2\tau}{ds_1^2} & 0 & 0 \\ 0 & \frac{d^2\tau}{d\alpha^2} & 0 \\ 0 & 0 & \frac{d^2\tau}{d\omega^2} \end{vmatrix} = \begin{vmatrix} \frac{d^2\tau}{d\alpha^2} & 0 & 0 \\ 0 & \frac{d^2\tau}{d\alpha^2} & 0 \\ 0 & 0 & \frac{d^2\tau}{d\omega^2} \end{vmatrix} = \left(\frac{d^2\tau}{d\alpha^2}\right)^2 \frac{d^2\tau}{d\omega^2}. \tag{124}$$

This indicates that for α there is a minimum of τ and for ω there is a maximum of τ.

Existence conditions. Substituting τ from (109) and ω from (106) into (94), we obtain the relation

$$x^{(1)}(t)\Big|_{t=\tau} = 0.0800187074\,\alpha^2 - 0.07571678\,\alpha\frac{c_2}{c_1} + 0.01954085\frac{c_3}{c_1} = 0. \tag{125}$$

The second equation for the determined c_2/c_1 and $c_3/c_1, c_1 \neq 0$ is obtained from

$$a_2^2 + (a_3 + a_1a_2)\frac{c_2}{c_1} + a_2\frac{c_3}{c_1} = 0. \tag{126}$$

4.3. Fourth-order equation $(n = 4)$.

Consider

$$\frac{d^4x(t)}{dt^4} + a_1\frac{d^3x(t)}{dt^3} + a_2\frac{d^2x(t)}{dt^2} + a_3\frac{dx(t)}{dt} + a_4x(t) = 0. \tag{127}$$

The initial conditions are

$$x(0) = c_1, \quad x^{(1)}(0) = c_2,$$
$$x^{(2)}(0) = c_3, \quad x^{(3)}(0) = c_4.$$

The characteristic equation is

$$s^4 + a_1s^3 + a_2s^2 + a_3s + a_4 = 0. \tag{128}$$

The first derivative of the solution of Eqn. (127) is

$$\begin{aligned} \frac{dx}{dt}\Big|_{t=\tau} &= -\frac{(s_2c_3 + s_3c_3 - c_4 - s_3s_4c_2 - s_2s_3c_2}{(s_1 - s_3)(s_1 - s_2)(s_1 - s_4)} + \frac{s_4c_3 + s_2s_3s_4c_1 - s_2s_4c_2)s_1e^{s_1\tau}}{(s_1 - s_3)(s_1 - s_2)(s_1 - s_4)} \\ &+ \frac{(s_1c_3 + s_3c_3 - s_1s_3c_2 - c_4 - s_3s_4c_2}{(s_2 - s_3)(s_1 - s_2)(s_2 - s_4)} + \frac{s_4c_3 + s_1s_3s_4c_1 - s_1s_4c_2)s_2e^{s_2\tau}}{(s_2 - s_3)(s_1 - s_2)(s_2 - s_4)} \\ &- \frac{(s_1c_3 + s_4c_3 - c_4 - s_1s_2c_2 - s_1s_4c_2}{(s_3 - s_4)(s_2 - s_3)(s_1 - s_3)} + \frac{s_1s_2s_4c_1 - s_2s_4c_2 + s_2c_3)s_3e^{s_3\tau}}{(s_3 - s_4)(s_2 - s_3)(s_1 - s_3)} \\ &+ \frac{(-s_1s_3c_2 + s_3c_3 + s_2c_3 + s_1c_3 + s_1s_2s_3c_1}{(s_3 - s_4)(s_2 - s_4)(s_1 - s_4)} + \frac{s_2s_3c_2 - s_1s_2c_2 - c_4)s_4e^{s_4\tau}}{(s_3 - s_4)(s_2 - s_4)(s_1 - s_4)} = 0. \end{aligned} \tag{129}$$

Derivatives of τ determined by (129) with respect to s_1, s_2, s_3 and s_4 give the following necessary conditions:

$$\begin{aligned} e^{(s_1-s_4)\tau} &= -s_4(-s_3s_2^2s_1\tau^2 + s_2^2s_3^2\tau^2 \\ &- s_1s_2^2\tau - s_3s_2^2\tau + s_1^2s_2\tau + 2s_2s_3 \\ &- s_1s_2s_3^2\tau^2 - s_2s_3^2\tau + 2s_1s_2 + s_1^2s_2s_3\tau^2 \\ &+ 2s_1s_2s_3\tau + 2s_1s_3 + s_1^2s_3\tau - s_1s_3^2\tau) = 0, \end{aligned} \tag{130}$$

$$\begin{aligned} e^{(s_2-s_4)\tau} \\ &= s_4(s_1^2 s_2 s_3 \tau^2 + s_1^2 s_3 \tau \\ &\quad - s_1^2 s_3^2 \tau^2 + s_1^2 s_2 \tau - s_1 s_2^2 s_3 \tau^2 - 2 s_1 s_2 \\ &\quad - 2 s_1 s_2 s_3 \tau - 2 s_1 s_3 - s_1 s_2^2 \tau + s_1 s_2 s_3^2 \tau^2 \\ &\quad + s_1 s_3^2 \tau + s_2 s_3^2 \tau - 2 s_2 s_3 - s_2^2 s_3 \tau) = 0, \end{aligned} \tag{131}$$

$$\begin{aligned} e^{(s_3-s_4)\tau} \\ &= -s_4(-s_1^2 s_2 s_3 \tau^2 - s_1^2 s_3 \tau \\ &\quad + s_1^2 s_2^2 \tau^2 - s_1^2 s_2 \tau + 2 s_1 s_2 s_3 \tau + 2 s_1 s_2 \\ &\quad + s_1 s_3^2 \tau + s_1 s_2 s_3^2 \tau^2 + 2 s_1 s_3 - s_1 s_2^2 s_3 \tau^2 \\ &\quad - s_1 s_2^2 \tau - s_2^2 s_3 \tau + s_2 s_3^2 \tau + 2 s_2 s_3) = 0. \end{aligned} \tag{132}$$

We assume that

$$s_1 = s_2 = \alpha, \quad s_3 = \alpha + j\omega, \quad s_4 = \alpha - j\omega.$$

The optimal value of τ is

$$\begin{aligned} \tau &= -\left(\frac{1}{s_1} + \frac{1}{s_2} + \frac{2\alpha}{\alpha^2+\omega^2}\right) \\ &= -2\frac{s_1\alpha + \alpha^2 + \omega^2}{s_1(\alpha^2+\omega^2)}. \end{aligned} \tag{133}$$

From Eqn. (132), we obtain (134) and its solution gives

$$\omega = \pm\alpha\ \sqrt[4]{3}. \tag{135}$$

In the special case, when

$$s_1 = s_3 = \alpha + j\omega, \quad s_2 = s_4 = \alpha - j\omega,$$

we obtain

$$\tau = -4\frac{\alpha}{\alpha^2+\omega^2}, \tag{136}$$

and from the equation

$$\begin{aligned} \sin(2\omega\tau) = -8 &\frac{(-\omega^2 + 3\alpha^2)}{(9\alpha^4 + 42\alpha^2\omega^2 + \omega^4)} \\ &\times \frac{(-\omega^4 - 14\alpha^2\omega^2 + 3\alpha^4)\alpha\omega}{(\alpha^2+\omega^2)^2} \end{aligned} \tag{137}$$

we have that

$$\omega = \pm\alpha\ \sqrt{3}. \tag{138}$$

4.4. Fifth-order equation ($n = 5$).

Consider

$$\begin{aligned} &\frac{\mathrm{d}^5 x(t)}{\mathrm{d}t^5} + a_1\frac{\mathrm{d}^4 x(t)}{\mathrm{d}t^4} + a_2\frac{\mathrm{d}^3 x(t)}{\mathrm{d}t^3} \\ &\quad + a_3\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + a_4\frac{\mathrm{d}x(t)}{\mathrm{d}t} + a_5 x(t) = 0. \end{aligned} \tag{139}$$

We assume one real root and a double pair of the complex roots:

$$\begin{cases} s_1 = \alpha, \quad s_2 = s_4 = \alpha + j\omega, \\ s_3 = s_5 = \alpha - j\omega. \end{cases} \tag{140}$$

In the same way, we obtain (141), from which we have

$$\omega = \pm\alpha\ . \tag{142}$$

Last example of the fifth-order equation. We assume that

$$\begin{cases} s_1 = s_2 = s_3 = \alpha, s_4 = \alpha + j\omega, \\ s_5 = \alpha - j\omega. \end{cases} \tag{143}$$

In this case, we obtain (144) and its solution is

$$\omega = 0.7606336797\,\alpha. \tag{145}$$

5. Basic results

Theorem 7. *If the characteristic equation (2) has complex-conjugate roots, then the optimal time τ can be computed numerically from the system of equations (106), (135), (138), (142), (145).*

Theorem 8. *The optimal times $\tau_i > 0$, $i = 1, 2, \ldots, (n-2)$, $n \geq 3$ are determined by $D_n(\tau) = 0$ (31), if they exist, and the equation $\frac{\mathrm{d}x(t)}{\mathrm{d}t}|_\tau = 0$. Here (4) gives $(n-1)$ linear algebraic equations for the initial conditions $c_2/c_1, c_3/c_1, \ldots, c_{n-1}/c_1$, $c_1 \neq 0$. This set of equations represents the solution of the problem.*

6. Numerical examples

6.1. Third-order equation.

Consider

$$\frac{\mathrm{d}^3 x(t)}{\mathrm{d}t^3} + a_1\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + a_2\frac{\mathrm{d}x(t)}{\mathrm{d}t} + a_3 x(t) = 0. \tag{146}$$

We assume that

$$\begin{cases} s_1 = \alpha = -1, \\ s_2 = \alpha + j\omega, \\ s_3 = \alpha - j\omega, \end{cases} \tag{147}$$

and, according to the relation (106), we have

$$\omega = \pm\alpha\sqrt{\sqrt{2}-1} = \pm 0.6435942526. \tag{148}$$

From (109), we get

$$\tau = -\frac{1+\sqrt{2}}{\alpha} = 2.414213563, \tag{149}$$

$$\begin{aligned} \left.\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right|_{t=\tau} &= -0.07330051053 c_3 \\ &\quad - 0.2840244129 c_2 \\ &\quad - 0.3001615506 c_1 = 0. \end{aligned} \tag{150}$$

From the relation (126) we get

$$D_3 = a_2^2 c_1 + (a_3 + a_1 a_2) c_2 + a_2 c_3 = 0$$

$$\sin(2\omega\tau) = -4\frac{(-\omega^4+3\alpha^4)\alpha\omega(2\alpha^2+\omega^2)}{(\alpha^2+\omega^2)^2}\frac{(-5\omega^6-17\alpha^2\omega^4-13\alpha^4\omega^2+3\alpha^6)}{(9\alpha^{10}+48\omega^2\alpha^8+106\omega^4\alpha^6+92\omega^6\alpha^4+33\alpha^2\omega^8+4\omega^{10})} \quad (134)$$

$$\sin(2\omega\tau) = -2\frac{(\alpha^2-\omega^2)(\omega^2+3\alpha^2)(\omega^2+5\alpha^2)\alpha\omega}{(\alpha^2+\omega^2)^2}\frac{(-2\omega^6-13\alpha^2\omega^4-24\alpha^4\omega^2+3\alpha^6)}{(9\alpha^{10}+63\alpha^8\omega^2+153\alpha^6\omega^4+82\alpha^4\omega^6+16\alpha^2\omega^8+\omega^{10})}. \quad (141)$$

$$\sin(2\omega\tau) = \frac{(-2(9\omega^{10}+12\alpha^2\omega^8-72\alpha^4\omega^6-172\alpha^6\omega^4-93\alpha^8\omega^2+12\alpha^{10})(-9\omega^6-22\alpha^2\omega^4-5\alpha^4\omega^2+12\alpha^6)(5\alpha^2+3\omega^2)\alpha\omega)}{((9\omega^6+18\alpha^2\omega^4+9\alpha^4\omega^2+4\alpha^6)(9\omega^{10}+69\alpha^2\omega^8+208\alpha^4\omega^6+297\alpha^6\omega^4+189\alpha^8\omega^2+36\alpha^{10})(\alpha^2+\omega^2)^2)} \quad (144)$$

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and

$$3.414213562c_3 + 11.65685425c_2 + 11.65685425c_1 = 0, \quad (151)$$

where

$$\begin{cases} a_1 = 3, \\ a_2 = 3.414213562, \\ a_3 = 1.414213562. \end{cases} \quad (152)$$

From (150) and (151), assuming $c_1 = 1$, we have

$$\begin{cases} c_2 = -1.477984236, \\ c_3 = 1.631940262. \end{cases} \quad (153)$$

We finally obtain

$$x(t) = 0.2177267e^{-t} + 0.7822733e^{-t}\cos(0.6435942526t) - 0.74267947e^{-t}\sin(0.6435942526t). \quad (154)$$

In Fig. 1 we present the optimal transient of $x(t)$.

6.2. Fourth-order equation. Consider

$$\frac{\mathrm{d}^4x(t)}{\mathrm{d}t^4} + a_1\frac{\mathrm{d}^3x(t)}{\mathrm{d}t^3} + a_2\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + a_3\frac{\mathrm{d}x(t)}{\mathrm{d}t} + a_4x(t) = 0. \quad (155)$$

We shall analyse two cases:

- one double real root and one pair of complex-conjugate roots,
- one double pair of complex roots.

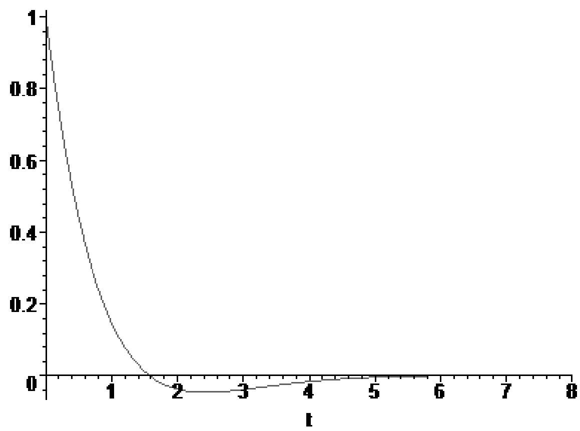

Fig. 1. Optimal transient of $x(t)$ (one real root and one pair of complex roots).

6.2.1. Case 1. Assume that

$$\begin{cases} s_1 = s_2 = \alpha = -1, \\ s_3 = \alpha + j\omega, \\ s_4 = \alpha - j\omega, \end{cases} \quad (156)$$

and, according to the relation (135), we have

$$\omega = \pm\alpha\,\sqrt[4]{3} = \pm 1.316074013. \quad (157)$$

From (133), we get

$$\tau = 2.732050808, \quad (158)$$

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=\tau} = -0.04383663c_4 - 0.224546377c_3 - 0.430314558c_2 - 0.314690486c_1 = 0. \quad (159)$$

Let $c_2 = 0$. Then

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = -0.0438366299c_4 - 0.224546377c_3 - 0.31469047c_1 = 0. \tag{160}$$

From the relation (32), we obtain

$$D_4 = c_1 a_3^3 + (a_3^2 a_2 + a_1 a_3 a_4 + 2a_4^2)c_2 + (2a_3 a_4 + a_1 a_3^3)c_3 + a_3^2 c_4 = 0. \tag{161}$$

For $c_2 = 0$, and $\alpha = -1$, $\omega = -1.316074013$, from (161) we have

$$D_4 = 415.8460971c_1 + 263.63586c_3 + 55.7128129c_4 = 0. \tag{162}$$

From (160) and (162), we finally have $c_3 = 0.7312184409c_1$, $c_4 = -10.92426443c_1$, and for $c_1 = 1$,

$$x(t) = -3.463269te^{-t} + 1.999519e^{-t} - 0.999519433\cos(1.316074t)e^{-t} + 3.39135125\sin(1.316074013t)e^{-t}. \tag{163}$$

In Fig. 2 we present the optimal transient of $x(t)$.

6.2.2. Case 2. Assume that

$$\begin{cases} s_1 = s_3 = \alpha + j\omega, \\ s_2 = s_4 = \alpha - j\omega. \end{cases} \tag{164}$$

Then the optimal time is

$$\tau = -4\frac{\alpha}{\alpha^2 + \omega^2}. \tag{165}$$

From (137) we obtain that

$$\omega = \pm\alpha\sqrt{3}. \tag{166}$$

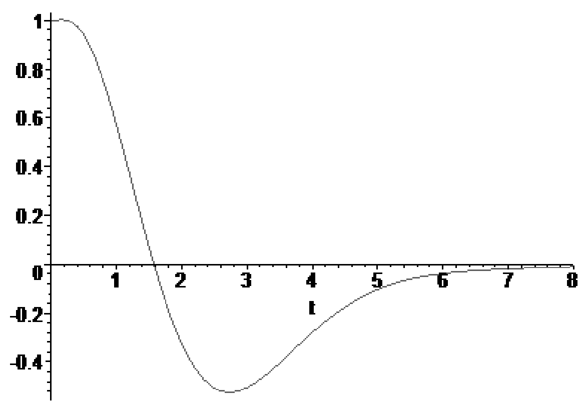

Fig. 2. Optimal transient of $x(t)$ for $c_2 = 0$ (double real root and one pair of complex roots).

For

$$\begin{cases} \alpha = -1, \\ \omega = \pm 1.732050808 \end{cases} \tag{167}$$

we get the coefficients

$$\begin{cases} a_1 = 4, \\ a_2 = 12, \\ a_3 = 16, \\ a_4 = 16, \end{cases} \tag{168}$$

and from (165),

$$\tau = 1. \tag{169}$$

In much the same way as in to the previous case, we assume $c_2 = 0$ and obtain that

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=\tau} = -0.7165473715c_1 + 0.1505743654c_3 + 0.06003569669c_4 = 0. \tag{170}$$

From (161), we get

$$4096.000008c_1 + 1536.000002c_3 + 256.000003c_4 = 0. \tag{171}$$

The solution of (170) and (171) is

$$\begin{cases} c_3 = -8.00000005c_1, \\ c_4 = 32.00000002c_1. \end{cases} \tag{172}$$

For $c_1 = 1$, we get

$$x(t) = -2\cos(1.732050808t)te^{-t} + \cos(1.732050808t)e^{-t} - 1.154700539\sin(1.7320508t)te^{-t} + 1.732050808\sin(1.7320508t)e^{-t}. \tag{173}$$

In Fig. 3 we present the transient of $x(t)$.

6.3. Fifth-order equation. Consider

$$\frac{\mathrm{d}^5x(t)}{\mathrm{d}t^5} + a_1\frac{\mathrm{d}^4x(t)}{\mathrm{d}t^4} + a_2\frac{\mathrm{d}^3x(t)}{\mathrm{d}t^3} + a_3\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + a_4\frac{\mathrm{d}x(t)}{\mathrm{d}t} + a_5x(t) = 0. \tag{174}$$

We consider the case of one real root and double pair of complex roots,

$$\begin{cases} s_1 = \alpha, \\ s_2 = s_4 = \alpha + j\omega, \\ s_3 = s_5 = \alpha - j\omega. \end{cases} \tag{175}$$

From (141), we have

$$\omega = \pm\alpha. \tag{176}$$

For $\alpha = -1$, and $c_2 = 0,\ c_3 = 0$, we obtain the following results:

$$\begin{cases} a_1 = 5, & a_2 = 12, \\ a_3 = 16, & a_4 = 12, \\ a_5 = 4, \end{cases} \tag{177}$$

and the optimal time

$$\tau = \frac{a_4}{a_5} = 3. \tag{178}$$

From the equations

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=\tau} = -0.3541478601c_1 - 0.1112703c_4 - 0.1109075c_5 = 0 \tag{179}$$

and from

$$D_5 = 20736c_1 + 10368c_4 + 1728c_5 = 0, \tag{180}$$

the solution is

$$\begin{cases} c_4 = -4.942537184c_1, \\ c_5 = 17.6552231c_1. \end{cases} \tag{181}$$

The optimal transient $x(t)$, for $c_1 = 1$, is

$$\begin{aligned} x(t) = {} & 1.885074362e^{-t} \\ & - 0.8850743624e^{-t}\cos(t) \\ & + 1.471268592e^{-t}\cos(t)t \\ & - 0.47126859e^{-t}\sin(t) \\ & + 0.0574628215e^{-t}\sin(t)t. \end{aligned} \tag{182}$$

In Fig. 4, $x(t)$ is presented.

In the same way, for $c_2 = 0, c_4 = 0$ we obtain

$$\begin{cases} c_3 = -1.145649523c_1, \\ c_5 = 6.33039236c_1, \end{cases} \tag{183}$$

and for $c_1 = 1$ the optimal transient is

$$\begin{aligned} x(t) = {} & 1.1651196176e^{-t} \\ & + 0.7184742831e^{-t}\cos(t)t \\ & - 0.1651961744e^{-t}\cos(t) \\ & - 0.1554228492e^{-t}\sin(t)t \\ & + 0.2815257151e^{-t}\sin(t), \end{aligned} \tag{184}$$

which is presented in Fig. 5.

For $c_4 = 0, c_5 = 0$ we obtain

$$\begin{cases} c_2 = -0.9458611703c_1, \\ c_3 = 0.5111482271c_1, \end{cases}$$

and for $c_1 = 1$ the transient is

$$\begin{aligned} x(t) = {} & 0.5223e^{-t} + 0.125e^{-t}\cos(t)t \\ & + 0.4777035489e^{-t}\cos(t) \\ & + 0.048564717e^{-t}\sin(t)t \\ & - 0.07086117224e^{-t}\sin(t), \end{aligned} \tag{185}$$

which is presented in Fig. 6.

7. Conclusion

Our basic theorems derive the solution of the problem of determining an optimal time τ. The presented examples of the differential equations of the order $n = 2, 3, 4, 5$ illustrate the solution method. We stress that for the differential equation of the n-th order it is in general necessary to determine $n-2$ values of $\tau_i > 0,\quad i = 1, 2, \ldots, n-1$.

Remark 1. The functions $e^s, \sin(s), \cos(s)$ are analytic in the whole domain and have all derivatives. For that reason it is sufficient to consider the real, negative roots s.

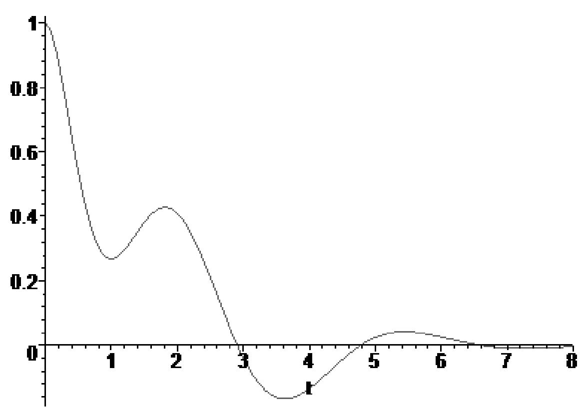

Fig. 3. Optimal transient of $x(t)$ for $c_2 = 0$ (double pair of complex roots).

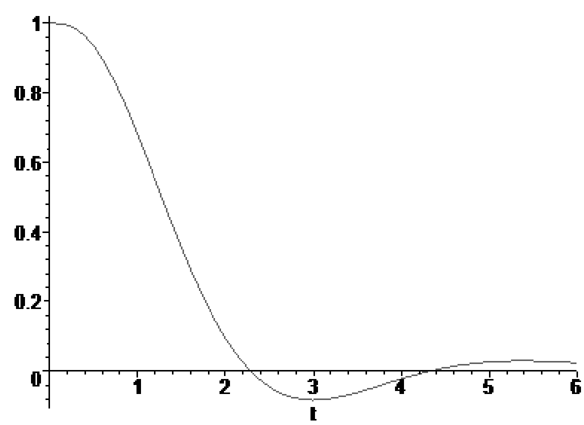

Fig. 4. Optimal transient of $x(t)$ for $c_2 = 0, c_3 = 0$.

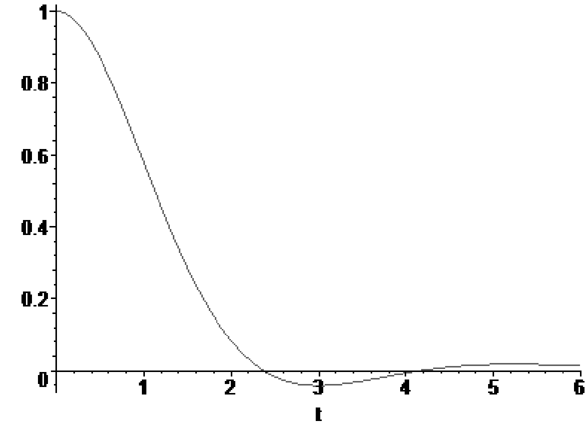

Fig. 5. Optimal transient of $x(t)$ for $c_2 = 0, c_4 = 0$.

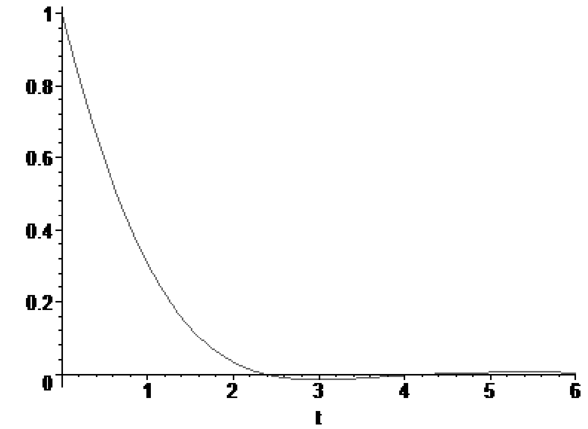

Fig. 6. Optimal transient of $x(t)$ for $c_4 = 0, c_5 = 0$.

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