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A SCHEME OF RESOURCE ALLOCATION AND STABILITY FOR PEER–TO–PEER FILE–SHARING NETWORKS

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Peer-to-peer (P2P) networks offer a cost-effective and easily deployable framework for sharing content. However, P2P file-sharing applications face a fundamental problem of unfairness. Pricing is regarded as an effective way to provide incentives to peers to cooperate. In this paper we propose a pricing scheme to achieve reasonable resource allocation in P2P file-sharing networks, and give an interpretation for the utility maximization problem and its sub-problems from an economic point of view. We also deduce the exact expression of optimal resource allocation for each peer, and confirm it with both simulation and optimization software. In order to realize the optimum in a decentralized architecture, we present a novel price-based algorithm and discuss its stability based on Lyapunov stability theory. Simulation results confirm that the proposed algorithm can attain an optimum within reasonable convergence times.

Keywords: peer-to-peer networks, fairness, pricing, stability, utility maximization.

1. Introduction

In recent years, peer-to-peer (P2P) file sharing has become a popular, cheap, and effective way to distribute content. It breaks through the limitation of the traditional client–server (CS) content distribution scheme, and generates most of the traffic on the current Internet (Li *et al*., 2012; Song *et al*., 2015a). In the CS mechanism, each client obtains the resource from a certain number of servers. Then the performance of CS systems can be degraded with an increase in the number of clients. By contrast, peers in P2P file-sharing networks exchange content with each other, and each peer can request several file fragments from remote ones at the same time. Thus in P2P networks a large number of peers can potentially lead to high throughput, large scalability and strong robustness. In recent years, P2P networks have been commonly used for distributed storage (Chen *et al*., 2014), cloud computing (e.g., Chmaj *et al*., 2012; Rho *et al*., 2014; Song *et al*., 2014) and social networking (Lin *et al*., 2014).

The system performance of P2P file-sharing networks fully relies on each peer's cooperation with others, due to self-organizing and self-managing features of P2P networks. However, P2P file-sharing applications face a fundamental problem of unfairness (Nishida and Nguyen, 2010). Many users are free-riding with peers consuming much download bandwidth but contributing little or no upload bandwidth. By unfairly taking the upload bandwidth of resourceable peers, free-riders cause slower download time for contributing peers and degrade the system performance. Thus many approaches have been implemented in P2Ps to address the problem of unfair bandwidth exchange, such as reputation systems (e.g., Satsiou and Tassiulas, 2010; Tseng and Chen, 2011; Qureshi *et al*., 2012; Song *et al*., 2015b), incentive mechanisms (e.g., Zhang *et al*., 2012; Meng and Li, 2013; Zhang and Antonopoulos, 2013), and pricing schemes (e.g., Eger and Killat, 2007; Zghaibeh and Harmantzis, 2008; Kumar *et al*., 2011; Analoui and Rezvani, 2011).

Pricing is regarded as an effective way to control computer networks since it can help to recover costs and provide incentives to users to cooperate (Zghaibeh

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and Harmantzis, 2008). Generally, pricing mechanisms mainly concentrate on motivating users to share resource with others and not declining other users' requests by relating their behaviors to a financial utility or mechanism. Several interesting pricing approaches have been presented for P2Ps in recent years, for example, the lottery-based scheme (Zghaibeh and Harmantzis, 2008), the Lagrangian multiplier method (Eger and Killat, 2007; Koutsopoulos and Iosifidis, 2010; Neely and Golubchik, 2011), the microeconomics-based approach (Kumar *et al.*, 2011; Analoui and Rezvani, 2011), auction-based mechanisms (Iosifidis and Koutsopoulos, 2010; Zuo and Zhang, 2013), and game-based ones (Park and Van Der Schaar, 2010; Nakano and Okaie, 2010; Okaie and Nakano, 2012; Kang and Wu, 2015; Lin *et al.*, 2015).

In this paper we propose a pricing scheme to achieve reasonable resource allocation in P2P file-sharing networks. The scheme is different from the aforementioned. We assume there is a service in P2Ps which is requested and offered by several peers at the same time. A service requester can request a service from several service providers in parallel. A service provider allocates its resource (e.g., upload bandwidth) to its requesters based on the prices paid by these requesters. Then, a spot-market for the scarce upload bandwidth of service providers emerges. The pricing scheme ensures efficient and fair allocation of the available resource to the serviced peers. We give an interpretation for the utility maximization problem and its sub-problems from an economic point of view. We also deduce the exact expression of optimal resource allocation for each peer, and confirm it with both simulation and optimization software. In order to implement fair resource allocation in a decentralized P2P architecture, we present a novel price-based algorithm which depends only on local information. Simulation results confirm that the proposed algorithm can attain an optimum within reasonable convergence times.

The rest of the paper is organized as follows. Section 2 reviews related work on resource allocation in P2Ps, Section 3 introduces the resource allocation model and its analysis, Section 4 proposes a novel price-based algorithm for resource allocation and discusses its stability, Section 5 gives some simulation examples to confirm the convergence of the algorithm, and finally, conclusions are summarized in Section 6.

2. Related work

In recent years, many researchers have investigated efficient resource allocation in peer-to-peer networks, mainly in the context of provision of incentive schemes as a means for encouraging peers' cooperation. Since P2P networks face the problem of free-riding, where peers only consume the resource but contribute little to the network, an incentive mechanism has to be established so as to encourage each peer to cooperate and serve other peers (e.g., Nishida and Nguyen, 2010; Zhang *et al.*, 2012; Meng and Li, 2013; Zhang and Antonopoulos, 2013).

The main idea behind these incentive schemes concentrates on peer cooperation enforcement so as to improve the performance of P2P networks such as reducing free-riding and achieving fairness. Besides these results, there is another type of research, where peers are assumed to cooperate with each other to improve the overall network performance. The motivation is driven by the theory of microeconomics. This approach takes into account economic issues when managing limited resources in P2P networks, and assesses the sustainability of the economic model by means of the pricing policy (Li and Liao, 2014; Li and Sun, 2016). For example, Analoui and Rezvani (2011) applied the theory of consumer-firm developed in microeconomics to resource allocation in overlay networks and proposed distributed algorithms for peers' "joining" and "leaving" operations. Kumar *et al.* (2011) designed a mechanism for pricing and resource allocation in P2P networks that allows users in a firm to effectively share computing resources. By applying the proposed pricing mechanism, no individual user has an incentive to over-utilize shared resources, thereby avoiding the "tragedy of the commons" for P2P networks.

To improve the overall performance of a P2P network, some researchers investigate the problem of maximizing the total network utility (social welfare), which is inspired by the theory of microeconomics and driven by the recent emerging framework of network utility maximization proposed for resource allocation in IP networks (Chiang *et al.*, 2007; Li *et al.*, 2015). For example, Eger and Killat (2007) applied resource pricing into P2P networks to ensure fair allocation of resources based on the congestion pricing principle known from IP networks. Zghaibeh and Harmantzis (2008) proposed a lottery-based pricing mechanism to enhance the sharing level in P2P networks and help increase the number of objects disseminated. The scheme is an extension of the traditional micropayment mechanism. Koutsopoulos and Iosifidis (2010) studied the problem of maximizing the total network utility (social welfare) through controlling bandwidth allocation and presented a framework for distributed bandwidth allocation in peer-to-peer networks by applying the Lagrangian method. Li and Sun (2016) formulated a fair resource allocation model for P2P networks and investigated the utility optimization problem by the Lagrangian method. In order to realize optimal resource allocation, they presented a novel price-based resource allocation scheme by applying the first order Lagrangian method and a low-pass filtering scheme.

Since peers in P2Ps use their access link bandwidth either to download content from other peers or to let other peers upload content from them, they face the dilemma of devoting their limited resource to their own benefit against acting altruistically and anticipating to be aided in the future. To address the challenges, auction- and game-based schemes can be used to balance their download and upload. Iosifidis and Koutsopoulos (2010) introduced a double-sided auction market framework where each peer announces one bid for buying and one for selling the resource, and proved that there exist bidding and charging strategies that maximize social welfare. Zuo and Zhang (2013) investigated the game theoretic mechanism for bandwidth allocation to incentivize a node's real bandwidth demands and defined an incentive-compatible pricing vector explicitly. By applying game theory, Park and Van Der Schaar (2010) investigated the issues of incentives in content production and sharing over P2P networks. Nakano and Okaie (2010) as well as Okaie and Nakano (2012) developed a rational model of a market-oriented service overlay network in which peers trade resources and services through a common currency called energy.

In this paper we consider a cooperative group of peers in P2P networks. That is, the peers act in order to maximize the total network utility (social welfare) and each peer exchanges with others the content required by them. In contrast to existing research works, we formulate the network utility maximization model for a fair resource allocation problem with different fairness concepts, and develop a pricing scheme to achieve fair resource allocation in P2P networks. We also present a novel resource allocation algorithm and discuss its performance using both analysis and simulation results. The algorithm can be applied to P2P file-sharing networks for which a upload bandwidth is a scarce resource.

3. Resource allocation model

3.1. Model description. Consider a set of peers and a set of services in a P2P network. Each peer is interested in one or several services and offers one or several services, or is interested and offers different services at the same time. A peer that offers at least one service to others consumes its resources. For file-sharing applications in P2Ps, the upload bandwidth of one peer is used to transmit a file or a fragment of a file to the remote peer interested in the file. Hence, upload bandwidth for this file is a scarce resource and other peers compete for it. We focus on resources such as uplink bandwidth which are divisible and where any allocation of resources has a benefit for a service requesting peer.

In order to differentiate between service providing and service requesting peers in the resource allocation model, we introduce a set of service providers P and a set of service requesters R. A peer is a member of P or R if it offers or requests at least one service, respectively. Thereby, a peer can be not only a service provider, but also a service requester. During the period of peer interaction we assume the P2P network is static, i.e., we do not consider the churn of this network due to peer arrivals or departures.

Define the set of peers which offer at least one service to peer r, the service requester, as the set of service providers $P(r)$ of peer r. Also, define the set of service requesters of the service provider p as $R(p)$. Note that $p \in P(r)$ if and only if $r \in R(p)$. Assume the flow rate of service requester r from its provider p is x_{pr}. Thus the total service rate of this requester is $y_r = \sum_{p\in P(r)} x_{pr}$. Then the requester r attains a utility $U_r(y_r)$ which depends on the total rate y_r. Meanwhile, the total upload rate of service provider p is $z_p = \sum_{r\in R(p)} x_{pr}$, which is constrained by the resource capacity C_p at the service provider. Thus resource allocation for P2P networks is modeled as the following optimization problem:

$$\max \quad \sum_{r:r\in R} U_r(y_r)$$

subject to

$$\begin{aligned} \sum_{p:p\in P(r)} x_{pr} &= y_r, \quad r \in R, \\ \sum_{r:r\in R(p)} x_{pr} &\le C_p, \quad p \in P, \end{aligned} \tag{1}$$

over

$$x_{pr} \ge 0, \quad r \in R, \quad p \in P.$$

The objective in this optimization problem is to maximize the aggregated utility of the service rate y_r over all service requesters in the network. For service requester r, the service rate y_r is the sum of the rates x_{pr} that service provider p offers, which is described by the equality in the optimization problem. Meanwhile, the service rate of provider p is constrained by the capacity of these providers, i.e., C_p, which is described by the inequality in the optimization problem above.

We are interested in the class of utility functions in the following form:

$$U_r(y_r) = w_r \frac{y_r^{1-\alpha}}{1-\alpha}, \tag{2}$$

where w_r is considered the willingness to pay service requester r. This family of utility functions is used to characterize a large class of fairness concepts and has been investigated extensively in IP networks (e.g., Chiang *et al.*, 2007; Shakkottai and Srikant, 2007; Li *et al.*, 2014).

Option 1. Throughput maximization is achieved as the fairness parameter $\alpha \to 0$. The objective of the resource allocation problem is

$$\max \sum_{r:r\in R} w_r y_r.$$

It is a greedy optimization problem that favors requesters with higher willingness to pay, but may leave requesters with lower willingness to pay unserved.

Option 2. Proportional fairness is obtained as $\alpha \to 1$. The objective of the resource allocation problem is

$$\max \sum_{r:r\in R} w_r \log y_r.$$

This is a well-known variant of utility-based fairness, and has been receiving increasing attention in recent years (e.g., Chiang *et al.*, 2007; Shakkottai and Srikant, 2007; Li *et al.*, 2014).

Option 3. Harmonic mean fairness corresponds to $\alpha = 2$. The objective is

$$\max \sum_{r:r\in R} -\frac{w_r}{y_r} \quad \text{or} \quad \min \sum_{r:r\in R} \frac{w_r}{y_r}.$$

This variant of fairness can be reduced to the minimization of transmission delay for files and has also gained much attention recently.

Option 4. Max-min fairness corresponds to $\alpha \to \infty$. The objective is

$$\max \min_{r\in R} m_r,$$

where m_r is the resource threshold of service requester r. It aims at serving all service requesters, but is constrained by the requesters with lowest resource thresholds.

Proportional fair and harmonic mean fair resource allocations are trade-offs between throughput maximization and max-min fair allocations.

3.2. Model analysis. As to the resource allocation model (1) with the utility functions (2), the objective is strictly concave with respect to y_r, but is not strictly concave with respect to x_{pr}. Meanwhile, the constraints are linear, hence the constraint set of this optimization problem is convex. Thus, based on convex optimization theory (Bertsekas, 2003), we can obtain the following result.

Theorem 1. *For the resource allocation model (1) with the utility functions (2), the optimal rate allocation of each service requester, i.e., y_r^*, is unique and can be obtained. However, the optimal rate of each requester from its providers, i.e., x_{pr}^*, is not unique.*

Next, we will analyze this model and obtain the optimal rate allocation of each service requester, i.e., the optimum of the optimization problem (1). First, we obtain the following Lagrangian of the optimization problem (1):

$$\begin{aligned} &L(x, y; \lambda, \mu) \\ &\quad = \sum_{r:r\in R} \left(U_r(y_r) + \lambda_r \left(\sum_{p:p\in P(r)} x_{pr} - y_r \right) \right) \\ &\qquad + \sum_{p:p\in P} \mu_p \left(C_p - \sum_{r:r\in R(p)} x_{pr} \right), \end{aligned} \tag{3}$$

where $\lambda = (\lambda_r, r \in R)$ is the price vector with element λ_r, which is considered the price per unit bandwidth paid by service requester r, and $\mu = (\mu_p, p \in P)$ is the price vector with element μ_p, which is considered the price per unit bandwidth charged by service provider p.

The Lagrangian (3) can be rewritten as

$$\begin{aligned} &L(x, y; \lambda, \mu) \\ &\quad = \sum_{r:r\in R} (U_r(y_r) - \lambda_r y_r) \\ &\qquad + \sum_{r:r\in R} \sum_{p:p\in P(r)} x_{pr} (\lambda_r - \mu_p) + \sum_{p:p\in P} \mu_p C_p. \end{aligned} \tag{4}$$

Note that the first term in (4) is separable in y_r, and the second term is separable in x_{pr}. Then the objective function of the dual problem is

$$\begin{aligned} D(\lambda, \mu) \quad &= \max_{x,y} L(x, y; \lambda, \mu) \\ &= \sum_{r:r\in R} \mathcal{R}_r(\lambda_r) + \sum_{r:r\in R} \sum_{p:p\in P(r)} \mathcal{P}_{pr}(\lambda_r, \mu_p) \\ &\quad + \sum_{p:p\in P} \mu_p C_p, \end{aligned}$$

where

$$\mathcal{R}_r(\lambda_r) = \max_{y_r} U_r(y_r) - \lambda_r y_r, \tag{5}$$

$$\mathcal{P}_{pr}(\lambda_r, \mu_p) = \max_{x_{pr}} x_{pr} (\lambda_r - \mu_p). \tag{6}$$

We can interpret the sub-problems (5) and (6) from an economic point of view.

For the sub-problem (5), service requester r tries to maximize its own utility, which depends on the total flow rate y_r. Meanwhile, the requester has to pay for using bandwidth. Recall that λ_r is the price per unit bandwidth paid by requester r; then $\lambda_r y_r$ is the total cost that requester r has to pay. Thus, the problem (5) is an optimization problem that each requester is to maximize its own profit.

For the sub-problem (6), the product $x_{pr}\lambda_s$ is the cost paid by requester r for using bandwidth x_{pr}. Since μ_p is the price per unit bandwidth charged by service provider p, $x_{pr}\mu_p$ is the cost charged by provider p. Hence, the problem (6) is an optimization problem where each provider is to maximize its own revenue.

Then the dual problem is

$$\min\ D(\lambda,\mu)$$

subject to (7)

$$\lambda_r \geq 0,\quad \mu_p \geq 0,\quad r \in R,\quad p \in P.$$

The *primal problem* (1) is to maximize the aggregated utility of the service rate over all requesters with the constraints of providers' capacities. The *dual problem* above is modeled as an optimization problem with prices charged by providers, which is to minimize the total price under the constraints that requesters are guaranteed certain levels of satisfaction.

3.3. Optimal resource allocation. Let $(x^*, y^*, \lambda^*, \mu^*)$ be the optimal primal and dual variables. Let $\partial L(x, y; \lambda, \mu)/\partial y_r = 0$. We can obtain the optimal rate of service requester r,

$$y_r^* = \left(\frac{w_r}{\lambda_r}\right)^{1/\alpha}. \tag{8}$$

Substituting (8) into (3), we can obtain

$$\begin{aligned}
&\widehat{L}(x;\lambda,\mu) \\
&= \sum_{r:r\in R}\Bigg(U_r\left(\frac{w_r}{\lambda_r}\right)^{1/\alpha} - \lambda_r\left(\frac{w_r}{\lambda_r}\right)^{1/\alpha} \\
&\quad + \lambda_r \sum_{p:p\in P(r)} x_{pr}\Bigg) \\
&\quad + \sum_{p:p\in P}\mu_p\Bigg(C_p - \sum_{r:r\in R(p)} x_{pr}\Bigg) \\
&= \sum_{r:r\in R}\Bigg(\frac{\alpha}{1-\alpha}\frac{w_r^{1/\alpha}}{\lambda_r^{(1-\alpha)/\alpha}} + \lambda_r \sum_{p:p\in P(r)} x_{pr}\Bigg) \\
&\quad + \sum_{p:p\in P}\mu_p\Bigg(C_p - \sum_{r:r\in R(p)} x_{pr}\Bigg).
\end{aligned} \tag{9}$$

Let $\partial \widehat{L}(x;\lambda,\mu)/\partial\lambda_r = 0$. Then we obtain the optimal price paid by requester r,

$$\lambda_r^* = \frac{w_r}{\left(\sum_{p:p\in P(r)} x_{pr}\right)^{\alpha}}. \tag{10}$$

Substituting (10) into (9), we get

$$\begin{aligned}
\overline{L}(x;\mu) &= \sum_{r:r\in R}\frac{w_r}{1-\alpha}\Bigg(\sum_{p:p\in P(r)} x_{pr}\Bigg)^{1-\alpha} \\
&\quad + \sum_{p:p\in P}\mu_p\Bigg(C_p - \sum_{r:r\in R(p)} x_{pr}\Bigg),
\end{aligned} \tag{11}$$

If, at the optimum, requester r attains non-zero resource allocation from provider p, i.e., $x_{pr} > 0$, let $\partial\overline{L}(x;\mu)/\partial x_{pr} = 0$, then the optimal total rate of requester r can also be

$$y_r^* = \sum_{p:p\in P(r)} x_{pr} = \left(\frac{w_r}{\mu_p}\right)^{1/\alpha}, \tag{12}$$

where μ_p is the price charged by provider p.

From the analysis above of the price charged by the service providers, we can obtain the following result.

Theorem 2. *At the optimum of the resource allocation model (1), if requester r attains non-zero resource allocation from its two service providers $p_1, p_2 \in P(r)$, then the prices charged by the providers are both equal to that paid by requester r, that is, if $x_{p_1r}^* > 0$ and $x_{p_2r}^* > 0$, then $\mu_{p_1}^* = \mu_{p_2}^* = \lambda_r^*$.*

Indeed, from (8) and (12), the following equality is satisfied:

$$\mu_{p_1}^* = \mu_{p_2}^* = \frac{w_r}{\left(\sum\limits_{p:p\in P(r)} x_{pr}^*\right)^{\alpha}} = \lambda_r^*, \tag{13}$$

and thus this result is obtained.

A bipartite graph can be composed of the two sets R and P. An edge denotes a service between a provider and a requester. If the bipartite graph is not connected, optimization can be run for every disjoint connected subgraph. Then we separate the whole P2P network into κ regions. Each region consists of a subset P_κ of service providers and a subset R_κ of service requesters. In each region the prices charged by service providers are all equal at the optimum. Suppose the price charged by service providers in region κ is $\mu_p = \mu_\kappa, \forall p \in P_\kappa$.

The Lagrangian (11) can be rewritten as

$$\begin{aligned}
&\overline{L}(x;\mu) \\
&= \sum_{\kappa}\Bigg(\sum_{r:r\in R_\kappa}\frac{w_r}{1-\alpha}\Bigg(\sum_{p:p\in P(r)} x_{pr}\Bigg)^{1-\alpha} \\
&\quad + \sum_{p:p\in P_\kappa}\mu_p\Bigg(C_p - \sum_{r:r\in R(p)} x_{pr}\Bigg)\Bigg) \\
&= \sum_{\kappa}\Bigg(\sum_{r:r\in R_\kappa}\Bigg(\frac{w_r}{1-\alpha}\Bigg(\sum_{p:p\in P(r)} x_{pr}\Bigg)^{1-\alpha} \\
&\quad - \mu_\kappa \textstyle\sum_{p:p\in P(r)} x_{pr}\Bigg) + \mu_\kappa \displaystyle\sum_{p:p\in P_\kappa} C_p\Bigg)
\end{aligned}$$

$$\overset{(a)}{=} \sum_{\kappa}\Bigg(\sum_{r:r\in R_\kappa}\Bigg(\frac{w_r}{1-\alpha}\Big(\frac{w_r}{\mu_\kappa}\Big)^{(1-\alpha)/\alpha} - \mu_\kappa\Big(\frac{w_r}{\mu_\kappa}\Big)^{1/\alpha}\Bigg) + \mu_\kappa \sum_{p:p\in P_\kappa} C_p\Bigg)$$
$$= \sum_{\kappa}\Bigg(\sum_{r:r\in R_\kappa}\frac{\alpha}{1-\alpha}\frac{w_r^{1/\alpha}}{\mu_\kappa^{(1-\alpha)/\alpha}} + \mu_\kappa \sum_{p:p\in P_\kappa} C_p\Bigg),$$

where (a) follows from (12).

Setting $\partial\overline{L}(x;\mu)/\partial\mu_\kappa = 0$, we obtain the optimal price charged by service providers in region κ,

$$\mu_\kappa^* = \left(\frac{\sum_{r:r\in R_\kappa} w_r^{1/\alpha}}{\sum_{p:p\in P_\kappa} C_p}\right)^{\alpha}. \tag{14}$$

Substituting (14) into (12), the optimal flow rate of requester r is

$$y_r^* = \sum_{p:p\in P(r)} x_{pr} = w_r^{1/\alpha}\frac{\sum_{p:p\in P_\kappa} C_p}{\sum_{r:r\in R_\kappa} w_r^{1/\alpha}}. \tag{15}$$

From (8) and (15), the optimal prices paid by requesters are

$$\lambda_r^* = \lambda_s^* = \lambda_\kappa^* = \mu_\kappa^*, \quad r,s\in R_\kappa. \tag{16}$$

The total flow rate of a service requester depends on the fairness parameter, i.e., α, the willingness of each requester, to pay i.e., w_r, and the total upload capacity of service providers, i.e., $\sum_{p:p\in P_\kappa} C_p$, in the requester's region. It is also obvious from (15) that the optimal flow rate of each requester is unique, which has been mentioned in Theorem 1. Furthermore, other kinds of fairness and resource allocation among requesters can also be achieved if we choose different values of parameter α in the class of utility functions (2).

4. Resource allocation algorithm

4.1. Algorithm description. To obtain optimal resource allocation in P2P networks, we present the following rate allocation algorithm, which is a price-based fluid flow model.

The algorithm is iterative in each separate region. At time t, each service provider p updates its rate allocation $x_{pr}(t)$ for requester r according to

$$\frac{\mathrm{d}}{\mathrm{d}t}x_{pr}(t) = \theta x_{pr}(t)\left(\mathrm{Pay}_r(t) - \mathrm{Cost}_p(t)\right)^+_{x_{pr}(t)-\varepsilon}, \tag{17}$$

$$\mathrm{Pay}_r(t) = C_p\lambda_r^{\frac{1}{\alpha}}(t), \tag{18}$$

$$\mathrm{Cost}_p(t) = \sum_{s:s\in R(p)} x_{ps}(t)\lambda_s^{\frac{1}{\alpha}}(t). \tag{19}$$

Meanwhile, each service requester r updates its price paid for its using bandwidth according to

$$\lambda_r(t) = \frac{w_r}{\max\{\eta, y_r^\alpha(t)\}}, \tag{20}$$

$$y_r(t) = \sum_{p:p\in P(r)} x_{pr}(t). \tag{21}$$

Here $a = (b)_c^+$ means $a = b$ if $c > 0$ and $a = \max\{0, b\}$ if $c = 0$, $\theta > 0$ is the step size of algorithm, $\varepsilon > 0, \eta > 0$ are small constants so that the rate $x_{pr}(t)$ of service requester r does not fall below ε and the price $\lambda_r(t)$ is bounded by w_r/η.

In the algorithm above, $\mathrm{Pay}_r(t)$ can be regarded as the total pay of requester r if it attains the total bandwidth of provider p, and $\mathrm{Cost}_p(t)$ can be considered the actual total cost charged by provider p when it allocates its bandwidth to requesters including r. At the equilibrium the prices paid by requesters are all expected to be equal, i.e., $\lambda_r^* = \lambda_s^*$.

4.2. Equilibrium. Next we consider the proposed algorithm (17)–(21) and analyze its equilibrium. By substituting (18) and (19) into (17) and setting (17) to zero, we can get the equilibrium (x^*, λ^*), that is,

$$C_p\lambda_r^{*\frac{1}{\alpha}} = \sum_{s:s\in R(p)} x_{ps}^*\lambda_s^{*\frac{1}{\alpha}}. \tag{22}$$

Meanwhile, from (20) and (21), at the equilibrium

$$\lambda_r^* = \frac{w_r}{y_r^{*\alpha}} = \frac{w_r}{\Big(\sum_{p:p\in P(r)} x_{pr}^*\Big)^{\alpha}}. \tag{23}$$

Then

$$\lambda_r^{*\frac{1}{\alpha}} = \frac{1}{C_p}\sum_{s:s\in R(p)} x_{ps}^*\lambda_s^{*\frac{1}{\alpha}}, \tag{24}$$

and so

$$\begin{aligned}\sum_{p:p\in P_\kappa} C_p &= \sum_{p:p\in P_\kappa}\frac{\sum_{s:s\in R(p)} x_{ps}^*\lambda_s^{*\frac{1}{\alpha}}}{\lambda_r^{*\frac{1}{\alpha}}}\\ &= \frac{1}{\lambda_r^{*\frac{1}{\alpha}}}\sum_{s:s\in R_\kappa}\sum_{p:p\in P(s)} x_{ps}^*\lambda_s^{*\frac{1}{\alpha}}\\ &\overset{(b)}{=} \frac{y_r^*}{w_r^{\frac{1}{\alpha}}}\sum_{s:s\in R_\kappa}\lambda_s^{*\frac{1}{\alpha}}\sum_{p:p\in P(s)} x_{ps}^*\\ &\overset{(c)}{=} \frac{y_r^*}{w_r^{\frac{1}{\alpha}}}\sum_{s:s\in R_\kappa} w_s^{\frac{1}{\alpha}},\end{aligned}$$

where (b) and (c) follow from (23). Thus the total flow rate of service requester r is

$$y_r^* = w_r^{\frac{1}{\alpha}} \frac{\sum_{p:p\in P_\kappa} C_p}{\sum_{s:s\in R_\kappa} w_s^{\frac{1}{\alpha}}}. \quad (25)$$

Then from (23) the price paid by requester r is

$$\lambda_r^* = \left(\frac{\sum_{s:s\in R_\kappa} w_s^{\frac{1}{\alpha}}}{\sum_{p:p\in P_\kappa} C_p} \right)^{\alpha}. \quad (26)$$

Obviously, the equilibrium (25), (26) is identical to the optimum (14)–(16) of the resource allocation problem (1).

Meanwhile, we also observe that the prices paid by requesters are all equal, i.e.,

$$\lambda_r^* = \lambda_s^* = \lambda_\kappa^*,$$

and from (22) we can obtain

$$\sum_{s:s\in R(p)} x_{ps}^* = C_p. \quad (27)$$

Hence, at the equilibrium of the dynamic system, the total flow rate offered by service provider p is just equal to the capacity of this provider, as long as the service provider has at least one service requester. This can be understood from the fact that each service requester is selfish and tries to fully use the resource of service providers to increase its own satisfaction.

4.3. Stability. We know that the equilibrium of the dynamic system (17)–(21) is identical to the optimum of the resource allocation problem (1). Now we investigate the asymptotic stability of the proposed algorithm based on Lyapunov stability theory for continuous dynamic systems without delays, and obtain the following theorem.

Theorem 3. *The equilibrium point (25), (26) of the dynamic system (17)–(21) is asymptotically stable.*

The proof, which is based on Lyapunov stability theory, is presented in Appendix.

Therefore, for the dynamic system (17)–(21) the equilibrium (25), (26) is asymptotically stable. Thus, all trajectories along (17)–(21) ultimately converge to the optimum (14)–(16) of the resource allocation problem (1).

4.4. Implementation. The algorithm can be implemented in each separate region. Actually, in practical implementation, each service provider calculates the prices paid by service requesters and updates its rate allocation according to the discrete form of the proposed algorithm above. That is, at times $t = 1, 2, \ldots,$ each service provider updates its rate allocation according to the following discrete-form algorithm:

$$\begin{aligned} x_{pr}[t+1] = \Big(&(1-\xi)x_{pr}[t] + \xi\widetilde{x}_{pr}[t] \\ &+ \xi\theta x_{pr}[t](\mathrm{Pay}_r[t] \\ &- \mathrm{Cost}_p[t])\Big)^{+}_{x_{pr}[t]-\varepsilon}, \end{aligned} \quad (28)$$

$$\widetilde{x}_{pr}[t+1] = (1-\xi)\widetilde{x}_{pr}[t] + \xi x_{pr}[t], \quad (29)$$

$$\mathrm{Pay}_r[t] = C_p \lambda_r^{\frac{1}{\alpha}}[t], \quad (30)$$

$$\mathrm{Cost}_p[t] = \sum_{s:s\in R(p)} x_{ps}[t]\lambda_s^{\frac{1}{\alpha}}[t]. \quad (31)$$

Here, we introduce another augmented variable $\widetilde{x}_{pr}[t]$, which is regarded as the optimal estimation of flow rate $x_{pr}[t]$. Thus, we slightly improve the primal algorithm (17) by applying the concept of a low-pass filtering scheme, where ξ is the parameter for low-pass filtering. The augmented variable is assisted solely to remove possible oscillation due to the fact that the model is not strictly concave and optimal resource allocation is not necessarily unique, but not to change optimal resource allocation.

Each service requester updates its price according to

$$\lambda_r[t] = \frac{w_r}{\max\{\eta, y_r^{\alpha}[t]\}}, \quad (32)$$

$$y_r[t] = \sum_{p:p\in P(r)} x_{pr}[t]. \quad (33)$$

The implementation of the algorithm can be described in the following steps:

Step 1: Initialize the variables and parameters. Initialize the step size θ and the positive constants ε, η in the algorithm. At time t for each service requester r, initialize its flow rate $x_{pr}[t]$ from service provider p.

Step 2: Calculate the prices paid by service requesters. At time t, service requester r collects its total flow rate $y_r[t]$ from all available service providers according to (33) and calculates the price $\lambda_r[t]$ it should pay for using resource according to (32).

Step 3: Calculate the total cost charged by service providers. Consider one kind of the concept of fairness in P2Ps and choose the fairness parameter α. At time t, calculate the total cost $\mathrm{Cost}_p[t]$ charged by service provider p according to (31) and the total pay $\mathrm{Pay}_r[t]$ of service requester r according to (30).

Step 4: Update rate allocation. At time $t+1$, each service provider p updates its rate allocation $x_{pr}[t+1]$ for service requester r according to (28) and (29).

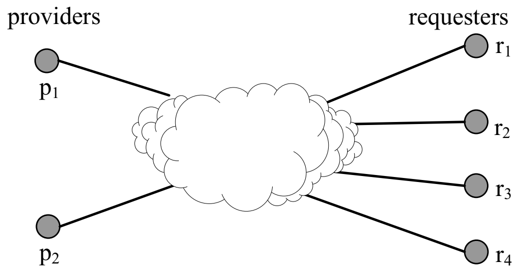

Fig. 1. Simple P2P network.

Step 5: Set the stop criterion. When the equilibrium of the algorithm is achieved, the iteration procedure can be stopped and optimal resource allocation is obtained.

In each iteration, each service requester individually calculates the price that it should pay for using bandwidth and communicates the price to its providers. Each service provider updates rate allocation for service requesters according to the prices that they pay. The iteration procedure above repeats until the equilibrium point is achieved.

Note that a kind of fairness concept can be realized by using the algorithm if different utility functions are chosen accordingly (corresponding to different parameters α in (2)).

5. Numerical examples

In this section, we investigate the performance of the proposed resource allocation scheme. We consider a simple P2P network consisting of two service providers and four service requesters as shown in Fig. 1. In this P2P network we do not differentiate between the different types of services (e.g., elastic and inelastic services) and assume there is only one service offered in this P2P network.

Suppose the capacities of service providers are $C = (C_1, C_2) = (20, 30)$ Mbps, and the willingness to pay service requesters is $w = (w_1, w_2, w_3, w_4) = (10, 20, 30, 40)$. The parameter ξ for low-pass filtering is chosen as 0.2. The small positive constants are $\varepsilon = \eta = 0.001$ in the algorithm.

5.1. Resource allocation. We first investigate the convergence of the resource allocation scheme and consider harmonic mean fairness among competing service requesters, i.e., $\alpha = 2$. In this rate allocation, we choose the step size $\theta = 0.02$.

The solutions obtained by using the proposed algorithm and the equalities (14)–(16) are listed in Table 1. The optimal solution solved by the nonlinear programming software LINGO is also presented in this table. We can easily observe from Table 1 that the algorithm is convergent to the optimum of resource allocation problem (1). We also obtain from Table 1 that the optimal rate of each service requester from its providers, i.e., x^*_{pr}, is not unique; however, the total rate of each service requester, i.e., y^*_r, is unique, which has been proved in Theorem 1.

Simulation results of the algorithm for harmonic mean fairness are also shown in Fig. 2, where (a), (b), (c) and (d) are the optimal flow rates of service requesters, and (e) is the optimal price paid by these requesters. It can be seen that the proposed algorithm efficiently converges to the optimum of the fair resource allocation model for a P2P network within reasonable convergence times. The optimal prices offered by service requesters are also presented and they are all equal at the optimum. Indeed, for each service requester the price paid by this requester is equivalent to that charged by one of its providers. This result has been discussed in theory in Section 3.3.

5.2. Convergence speed. Now we investigate the convergence speed of the resource allocation algorithm. The simulation setup is identical with the aforementioned, except that we choose different step sizes. It is very important to emphasize that the size of the network (i.e., the number of peers) does not affect the convergence speed of the algorithm. In fact, the convergence speed mainly depends on parameters such as the step size other than the number of cooperating peers. We depict the objective (i.e., $\sum_{r:r\in R} w_r/y_r$) evolution in Fig. 3, when different step sizes are selected. Obviously, the convergence speed of the algorithm is proportional to the step size. Roughly speaking, the step size in the algorithm should be small enough so as to ensure convergence, but not too small, such that the convergence would unnecessarily very slow. On the other hand, the step size should also be not too big, since then the algorithm may not converge efficiently in the neighborhood of the optimum. By choosing an appropriate step size, the

Table 1. Optimal resource allocation: harmonic mean fairness.

Variable	x_{11}	x_{21}	x_{12}	x_{22}
Algorithm	3.4028	4.7322	4.6413	6.8633
LINGO	3.7454	4.3897	1.9295	9.5751
Variable	x_{13}	x_{23}	x_{14}	x_{24}
Algorithm	5.5828	8.5075	6.3730	9.8971
LINGO	8.2254	5.8649	6.0997	10.1703
Variable	y_1	y_2	y_3	y_4
Algorithm	8.1350	11.5046	14.0903	16.2701
LINGO	8.1351	11.5046	14.0903	16.2700
Eqs. (14)–(16)	8.1350	11.5046	14.0902	16.2699
Variable	λ_1	λ_2	λ_3	λ_4
Algorithm	0.1511	0.1511	0.1511	0.1511
Eqs. (14)–(16)	0.1511	0.1511	0.1511	0.1511

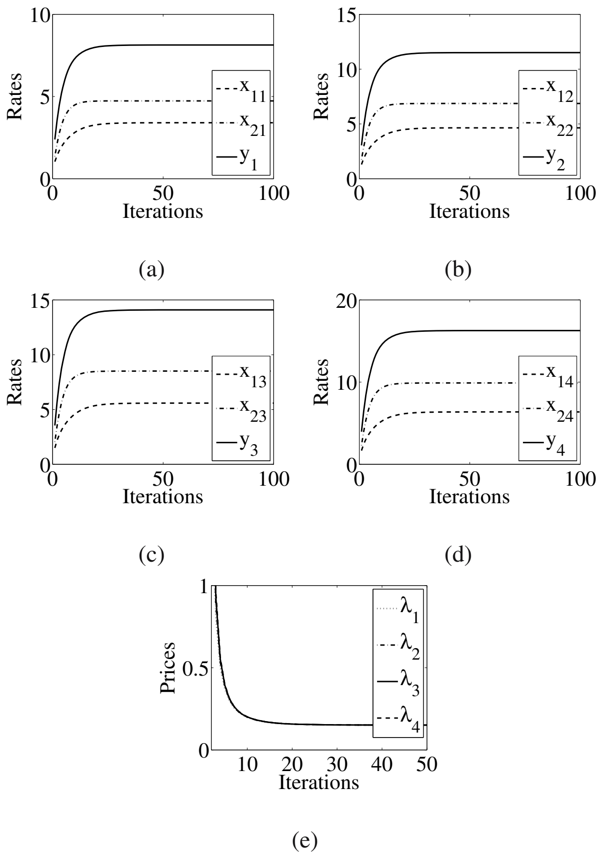

Fig. 2. Optimal resource allocation: harmonic mean fairness. Rates for requester 1 (a), rates for requester 2 (b), rates for requester 3 (c), rates for requester 4 (d), prices paid by requesters (e).

optimum can be achieved within reasonable convergence iterations.

5.3. Large scale networks. Now we consider the performance of the algorithm with different numbers of peers. The capacities of service providers are all assumed to be 20 Mbps, and the willingness to pay service requesters is 10. In Fig. 4 we depict objective (i.e., $\sum_{r:r\in R} w_r/y_r$) evolution with different numbers of peers (service providers and requesters). We observe that the size of the network (i.e., the number of peers) does not affect the convergence speed of the algorithm. The final objective increases with the number of peers but, in all cases, this value is reached with almost the same number of iterations (e.g., 50 iterations). This is rather expected. The algorithm is synchronously operated, and different peers run the separate optimization steps in parallel. Thus the number of peers does not alter the convergence speed obviously. In fact, the convergence speed mainly depends on algorithm parameters other than the number of peers. As shown in Fig. 3, the impact of the step sizes on the performance of the proposed scheme is very crucial.

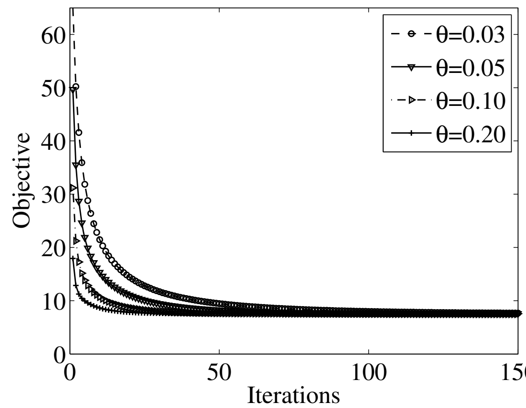

Fig. 3. Performance of the resource allocation algorithm with different step sizes.

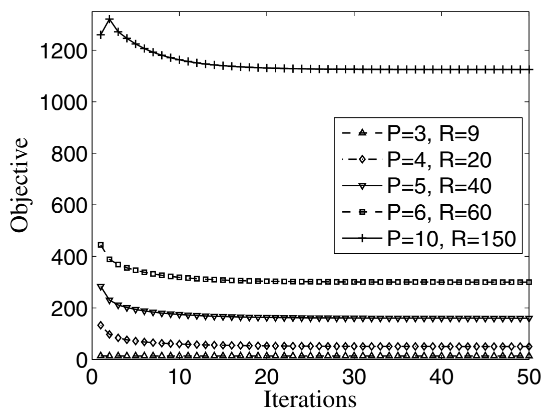

Fig. 4. Performance of the resource allocation algorithm with different number of peers.

6. Conclusions

In today's Internet-based social communities, P2P networks offer a popular, cheap, and effective way to distribute content. While P2P networks have many advantages such as scalability, resilience, and effectiveness in coping with dynamics and heterogeneity, they have intrinsic incentive problems of unfairness in that the transfer of content always incurs costs to uploaders but benefits only downloaders. Many peers free-ride by contributing little or no upload bandwidth while consuming much download bandwidth. Many approaches have been implemented to address the problem of unfair bandwidth exchange, such as reputation systems, incentive mechanisms, and pricing schemes.

In this paper we consider a pricing scheme and fair resource allocation in P2P networks. In contrast to existing research work, we formulate the network utility maximization model for the fair resource allocation problem with different fairness concepts and develop a pricing scheme to achieve fair resource allocation in P2P networks. We give an interpretation for the utility maximization problem and its sub-problems from an economic point of view. We also deduce the exact expression of optimal resource allocation for each peer, and confirm it with both simulation and optimization software. In order to realize the fair resource allocation in a decentralized P2P architecture, we present a novel distributed price-based resource allocation algorithm. Both the analysis and simulation results confirm that the proposed algorithm is efficient in solving the resource allocation model and can achieve the optimum within reasonable convergence times.

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References

Analoui, M. and Rezvani, M. (2011). Microeconomics-based resource allocation in overlay networks by using non-strategic behavior modeling, *Communications in Nonlinear Science and Numerical Simulation* **16**(1): 493–508.

Bertsekas, D. (2003). *Nonlinear Programming*, Athena Scientific, Belmont, MA.

Boyce, W.E. and DiPrima, R.C. (2005). *Elementary Differential Equations and Boundary Value Problems*, John Wiley & Sons, Hoboken, NJ.

Chen, G., Hu, T., Jiang, D., Lu, P., Tan, K.-L., Vo, H. and Wu, S. (2014). Bestpeer++: A peer-to-peer based large-scale data processing platform, *IEEE Transactions on Knowledge and Data Engineering* **26**(6): 1316–1331.

Chiang, M., Low, S., Calderbank, A. and Doyle, J. (2007). Layering as optimization decomposition: A mathematical theory of network architectures, *Proceedings of the IEEE* **95**(1): 255–312.

Chmaj, G., Walkowiak, K., Tarnawski, M. and Kucharzak, M. (2012). Heuristic algorithms for optimization of task allocation and result distribution in peer-to-peer computing systems, *International Journal of Applied Mathematics and Computer Science* **22**(3): 733–748, DOI: 10.2478/v10006-012-0055-0.

Eger, K. and Killat, U. (2007). Fair resource allocation in peer-to-peer networks (extended version), *Computer Communications* **30**(16): 3046–3054.

Iosifidis, G. and Koutsopoulos, I. (2010). Double auction mechanisms for resource allocation in autonomous networks, *IEEE Journal on Selected Areas in Communications* **28**(1): 95–102.

Kang, X. and Wu, Y. (2015). Incentive mechanism design for heterogeneous peer-to-peer networks: A Stackelberg game approach, *IEEE Transactions on Mobile Computing* **14**(5): 1018–1030.

Koutsopoulos, I. and Iosifidis, G. (2010). A framework for distributed bandwidth allocation in peer-to-peer networks, *Performance Evaluation* **67**(4): 285–298.

Kumar, C., Altinkemer, K. and De, P. (2011). A mechanism for pricing and resource allocation in peer-to-peer networks, *Electronic Commerce Research and Applications* **10**(1): 26–37.

Li, S. and Sun, W. (2016). A mechanism for resource pricing and fairness in peer-to-peer networks, *Electronic Commerce Research*, DOI: 10.1007/s10660-016-9211-1.

Li, S., Sun, W. and Hua, C. (2014). Fair resource allocation and stability for communication networks with multipath routing, *International Journal of Systems Science* **45**(11): 2342–2353.

Li, S., Sun, W. and Tian, N. (2015). Resource allocation for multi-class services in multipath networks, *Performance Evaluation* **92**(1): 1–23.

Li, Y., Gruenbacher, D. and Scoglio, C. (2012). Evaluating stranger policies in P2P file-sharing systems with reciprocity mechanisms, *Computer Networks* **56**(4): 1470–1485.

Li, Z. and Liao, Q. (2014). Network pricing: Can both ISP and P2P benefit?, *International Journal of Network Management* **24**(6): 433–449.

Lin, F., Zhou, X., Lv, X. and Song, W. (2015). Novel pre-pushing scheme for peer-assisted streaming network based on multi-leader multi-follower Stackelberg model, *Wireless Personal Communications* **80**(1): 289–301.

Lin, P., Chung, P.-C. and Fang, Y. (2014). P2P-ISN: A peer-to-peer architecture for heterogeneous social networks, *IEEE Network* **28**(1): 56–64.

Meng, X. and Li, T. (2013). A dynamic load balancing scheme with incentive mechanism in heterogeneous structured P2P networks, *Computers and Electrical Engineering* **39**(7): 2124–2134.

Nakano, T. and Okaie, Y. (2010). Cooperative resource pricing in service overlay networks for mobile agents, *IEICE Transactions on Communications* **E93-B**(7): 1927–1930.

Neely, M. and Golubchik, L. (2011). Utility optimization for dynamic peer-to-peer networks with tit-for-tat constraints, *Proceedings of the IEEE International Conference on Computer Communications, INFOCOM'11, Shanghai, China*, pp. 1458–1466.

Nishida, H. and Nguyen, T. (2010). A global contribution approach to maintain fairness in P2P networks, *IEEE Transactions on Parallel and Distributed Systems* **21**(6): 812–826.

Okaie, Y. and Nakano, T. (2012). Network formation games in non-cooperative service overlay networks, *Computer Systems Science and Engineering* **27**(1): 41–49.

Park, J. and Van Der Schaar, M. (2010). Pricing and incentives in peer-to-peer networks, *Proceedings of the IEEE International Conference on Computer Communications, INFOCOM'10, San Diego, CA, USA*, pp. 1–9.

Qureshi, B., Min, G. and Kouvatsos, D. (2012). A distributed reputation and trust management scheme for mobile peer-to-peer networks, *Computer Communications* **35**(5): 608–618.

Rho, S., Chang, H., Kim, S. and Lee, Y. (2014). An efficient peer-to-peer and distributed scheduling for cloud and grid computing, *Peer-to-Peer Networking and Applications* **8**(5): 863–871.

Satsiou, A. and Tassiulas, L. (2010). Reputation-based resource allocation in P2P systems of rational users, *IEEE Transactions on Parallel and Distributed Systems* **21**(4): 466–479.

Shakkottai, S. and Srikant, R. (2007). Network optimization and control, *Foundations and Trends in Networking* **2**(3): 271–379.

Song, F., Huang, D., Zhou, H., Zhang, H. and You, I. (2014). An optimization-based scheme for efficient virtual machine placement, *International Journal of Parallel Programming* **42**(5): 853–872.

Song, F., Li, R. and Zhou, H. (2015a). Feasibility and issues for establishing network-based carpooling scheme, *Pervasive and Mobile Computing* **24**(1): 4–15.

Song, F., Zhang, Y., An, Z., Zhou, H. and You, I. (2015b). The correlation study for parameters in four tuples, *International Journal of Ad Hoc and Ubiquitous Computing* **19**(1): 38–49.

Tseng, Y.-M. and Chen, F.-G. (2011). A free-rider aware reputation system for peer-to-peer file-sharing networks, *Expert Systems with Applications* **38**(3): 2432–2440.

Zghaibeh, M. and Harmantzis, F. (2008). A lottery-based pricing scheme for peer-to-peer networks, *Telecommunication Systems* **37**(4): 217–230.

Zhang, K. and Antonopoulos, N. (2013). A novel bartering exchange ring based incentive mechanism for peer-to-peer systems, *Future Generation Computer Systems* **29**(1): 361–369.

Zhang, Z., Chen, S., Mo, Z. and Yoon, M. (2012). An efficient incentive scheme with a distributed authority infrastructure in peer-to-peer networks, *Journal of Parallel and Distributed Computing* **72**(12): 1741–1752.

Zuo, F. and Zhang, W. (2013). An auction based distribute mechanism for P2P adaptive bandwidth allocation, *IEICE Transactions on Information and Systems* (12): 2704–2712.

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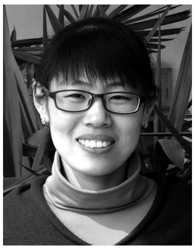

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Appendix
Proof of Theorem 3

We consider the proposed algorithm (17)–(21) for the fair rate allocation among competing service requesters and investigate the stability based on Lyapunov stability theory for continuous dynamic systems without delays.

Define the following Lyapunov function:

$$\begin{aligned} V(t) &= V_1(t) + V_2(t) \\ &= \sum_{r:r\in R_\kappa} \int_{y_r(t)}^{y_r^*} \left(\frac{w_r^{\frac{1}{\alpha}}}{\upsilon} - \lambda_\kappa^{*\frac{1}{\alpha}} \right) \mathrm{d}\upsilon \\ &+ \sum_{p:p\in P_\kappa} \lambda_\kappa^{*\frac{1}{\alpha}} (C_p - \xi_p(t)), \end{aligned}$$

where

$$\xi_p(t) = \sum_{r:r\in R(p)} x_{pr}(t), \quad p \in P_\kappa,$$
$$\lambda_\kappa^* = \lambda_r^*, \quad \forall r \in R_\kappa.$$

Since $y_r(t) \geq 0$, $y_r^* \geq 0$, the first term of the Lyapunov function is

$$\begin{aligned} &\int_{y_r(t)}^{y_r^*} \left(\frac{w_r^{\frac{1}{\alpha}}}{\upsilon} - \lambda_\kappa^{*\frac{1}{\alpha}} \right) \mathrm{d}\upsilon \\ &= w_r^{\frac{1}{\alpha}} (\log y_r^* - \log y_r(t)) - \lambda_\kappa^{*\frac{1}{\alpha}} (y_r^* - y_r(t)) \\ &= w_r^{\frac{1}{\alpha}} \left(\frac{y_r(t)}{y_r^*} - 1 - \log \frac{y_r(t)}{y_r^*} \right) \geq 0. \end{aligned}$$

Obviously, $V_1(t) = 0$ if and only if $y_r(t) = y_r^*$ (i.e., $\lambda_r(t) = \lambda_r^* = \lambda_\kappa^*$). Meanwhile, the second part of the Lyapunov function is $V_2(t) \geq 0$ since $\xi_p(t) = \sum_{r:r\in R(p)} x_{pr}(t) \leq C_p$, and $V_2(t) = 0$ if and only if $\xi_p(t) = \sum_{r:r\in R(p)} x_{pr}^* = C_p$. Thus, the Lyapunov function $V(t)$ is a positive definite function, and it is zero only at the equilibrium point (x^*, λ^*) (i.e., $\lambda_r(t) = \lambda_r^* = \lambda_\kappa^*$, $\xi_p(t) = \sum_{r:r\in R(p)} x_{pr}^* = C_p$).

The derivative of $V(t)$ along the trajectories of the dynamic system (17)–(21) is

$$\begin{aligned} &\frac{\mathrm{d}V(t)}{\mathrm{d}t} \\ &= \sum_{r:r\in R_\kappa} \frac{\partial V(t)}{\partial y_r(t)} \frac{\mathrm{d}y_r(t)}{\mathrm{d}t} \\ &\quad + \sum_{p:p\in P_\kappa} \frac{\partial V(t)}{\partial \xi_p(t)} \frac{d\xi_p(t)}{\mathrm{d}t} \\ &= - \sum_{r:r\in R_\kappa} \left(\frac{w_r^{\frac{1}{\alpha}}}{y_r(t)} - \lambda_\kappa^{*\frac{1}{\alpha}} \right) \sum_{p:p\in P(r)} \frac{\mathrm{d}x_{pr}(t)}{\mathrm{d}t} \\ &\quad - \sum_{p:p\in P_\kappa} \lambda_\kappa^{*\frac{1}{\alpha}} \sum_{r:r\in R(p)} \frac{\mathrm{d}x_{pr}(t)}{\mathrm{d}t} \\ &= - \sum_{r:r\in R_\kappa} \left(\lambda_r^{\frac{1}{\alpha}}(t) - \lambda_\kappa^{*\frac{1}{\alpha}} \right) \sum_{p:p\in P(r)} \frac{\mathrm{d}x_{pr}(t)}{\mathrm{d}t} \\ &\quad - \sum_{p:p\in P_\kappa} \sum_{r:r\in R(p)} \lambda_\kappa^{*\frac{1}{\alpha}} \frac{\mathrm{d}x_{pr}(t)}{\mathrm{d}t} \\ &= - \sum_{r:r\in R_\kappa} \sum_{p:p\in P(r)} \theta \lambda_r^{\frac{1}{\alpha}}(t) x_{pr}(t) \\ &\quad \times \left(C_p \lambda_r^{\frac{1}{\alpha}}(t) - \sum_{s:s\in R(p)} x_{ps}(t) \lambda_s^{\frac{1}{\alpha}}(t) \right) \\ &= - \sum_{r:r\in R_\kappa} \sum_{p:p\in P(r)} \theta C_p \lambda_r^{\frac{2}{\alpha}}(t) x_{pr}(t) \\ &\quad + \sum_{p:p\in P_\kappa} \sum_{r:r\in R(p)} \theta x_{pr}(t) \lambda_r^{\frac{1}{\alpha}}(t) \sum_{s:s\in R(p)} x_{ps}(t) \lambda_s^{\frac{1}{\alpha}}(t) \\ &= - \sum_{r:r\in R_\kappa} \sum_{p:p\in P(r)} \theta \lambda_r^{\frac{2}{\alpha}}(t) x_{pr}(t) (C_p - x_{pr}(t)) \\ &\quad + \sum_{p:p\in P_\kappa} \sum_{r:r\in R(p)} \sum_{s:s\in R(p)\setminus\{r\}} \theta x_{pr}(t) \lambda_r^{\frac{1}{\alpha}}(t) \\ &\quad \times x_{ps}(t) \lambda_s^{\frac{1}{\alpha}}(t). \end{aligned}$$

Add

$$\sum_{r:r\in R_\kappa} \sum_{p:p\in P(r)} \theta \lambda_r^{\frac{2}{\alpha}}(t) x_{pr}(t) \sum_{s:s\in R(p)\setminus\{r\}} x_{ps}(t)$$

to the first part of the derivative above, and subtract the same term from the second part. Then

$$\begin{aligned} &\frac{\mathrm{d}V(t)}{\mathrm{d}t} \\ &= - \sum_{r:r\in R_\kappa} \sum_{p:p\in P(r)} \theta \lambda_r^{\frac{2}{\alpha}}(t) x_{pr}(t) \\ &\quad \times \left(C_p - \sum_{s:s\in R(p)} x_{ps}(t) \right) \\ &\quad + \sum_{p:p\in P_\kappa} \sum_{r:r\in R(p)} \sum_{s:s\in R(p)\setminus\{r\}} \theta \Big(x_{pr}(t) \lambda_r^{\frac{1}{\alpha}}(t) \\ &\quad \times x_{ps}(t) \lambda_s^{\frac{1}{\alpha}}(t) - \lambda_r^{\frac{2}{\alpha}}(t) x_{pr}(t) x_{ps}(t) \Big), \end{aligned}$$

and hence

$$\begin{aligned} &\frac{\mathrm{d}V(t)}{\mathrm{d}t} \\ &= - \sum_{r:r\in R_\kappa} \sum_{p:p\in P(r)} \theta \lambda_r^{\frac{2}{\alpha}}(t) x_{pr}(t) \end{aligned}$$

$$\times \left(C_p - \sum_{s:s\in R(p)} x_{ps}(t) \right) - \sum_{p:p\in P_\kappa} \sum_{r:r\in R(p)} \sum_{s:s\in R(p)\setminus\{r\}} \frac{\theta x_{pr}(t) x_{ps}(t)}{2} \times \left(\lambda_r^{\frac{1}{\alpha}}(t) - \lambda_s^{\frac{1}{\alpha}}(t) \right)^2 .$$

Thus, $\mathrm{d}V(t)/\mathrm{d}t \le 0$ since $\sum_{s:s\in R(p)} x_{ps}(t)) \le C_p, p \in P_\kappa$, and $\mathrm{d}V(t)/\mathrm{d}t = 0$ if and only if $\lambda_r(t) = \lambda_s(t) = \lambda_r^* = \lambda_\kappa^*$, $\sum_{s:s\in R(p)} x_{ps}^* = C_p$ (i.e., the equilibrium point (x^*, λ^*)). Therefore, from Lyapunov stability theory (Boyce, 2005) the equilibrium point (25)–(26) of the dynamic system (17)–(21) is asymptotically stable.

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