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THE LIMIT OF INCONSISTENCY REDUCTION IN PAIRWISE COMPARISONS

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This study provides a proof that the limit of a distance-based inconsistency reduction process is a matrix induced by the vector of geometric means of rows when a distance-based inconsistent pairwise comparisons matrix is transformed into a consistent PC matrix by stepwise inconsistency reduction in triads. The distance-based inconsistency indicator was defined by Koczkodaj (1993) for pairwise comparisons. Its convergence was analyzed in 1996 (regretfully, with an incomplete proof) and finally completed in 2010. However, there was no interpretation provided for the limit of convergence despite its considerable importance. This study also demonstrates that the vector of geometric means and the right principal eigenvector are linearly independent for the pairwise comparisons matrix size greater than three, although both vectors are identical (when normalized) for a consistent PC matrix of any size.

Keywords: pairwise comparison, inconsistency reduction, convergence limit, decision making.

1. Introduction

In modern science, we often compare entities in pairs without even realizing it. For example, when someone asserts *My car is 11 years old,* we compare the car age to one year (the unit of time). This is a pair: one entity (year) is a unit and another entity is a magnitude of life duration. When we have no unit (e.g., for software quality), we can construct a pairwise comparisons (PCs) matrix to express our assessments based on relative comparisons of its attributes (such as safety or reliability).

The first documented use of PCs is attributed to Llull (1299) in the 13th century. It may surprise some readers that the Nobelist Kenneth Arrow used *pair* or *pairs* 25 times in his work dated 1950. Needless to say, that paper contains his famous impossibility theorem (Arrow, 1950). The importance of pairwise comparisons for computer science has been recently evidenced in one of the flagship ACM publications (Faliszewski *et al.*, 2010). Finally, there are a considerable number of customizations, but the authors specifically refuse to discuss them here since this study is on the theory of pairwise comparisons and does not endorse any customization.

Needless to say, inaccurate pairwise comparisons lead to an inconsistency of a PC matrix and, in consequence, to an inadequate hierarchy of the compared alternatives. Hence, it is crucial to reduce the inconsistency of the initial PC matrix introducing, however, as little changes as possible.

In this study, we prove that the limit of the inconsistency reduction algorithm, introduced by Koczkodaj (1993) and mathematically analyzed by Koczkodaj and Szarek (2010), is the (normalized) vector of geometric means of rows. We stress the foundation nature of our research and its independence of any customization of pairwise comparisons. To the best of our knowledge, this has never been done. In our study, it is stressed that the vectors of geometric means of rows of the input and output PC matrices are invariant for the "automatic" inconsistency reduction by the orthogonal projections. This finding is of considerable importance and greatly simplifies the process of ranking entities.

The outline of the article is as follows. Section 2 includes the basic concepts of PC theory. Section 3

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focuses on the notion of inconsistency. Section 4 compares geometric means with the right principal eigenvector method. The inconsistency reduction process is presented in Section 5. The conclusions are stated in the last section.

Finally, we stress that vector normalization is not really of great importance for convergence or its limit. However, the vector of weights needs to be normalized when we compare different methods since they may differ by a constant (called the scaling of the eigenvector). Usually, "normalized" is dropped in our presentation unless it may lead to an ambiguity problem. Linear dependence of vectors can be used instead of the "scaling" of a vector.

2. Basic concepts of pairwise comparisons

In this study, we assume that the *pairwise comparisons matrix* (PC matrix here) is a square matrix $M = [m_{ij}]$, $n \times n$, such that $m_{ij} > 0$ for every $i, j = 1, \dots, n$, where m_{ij} expresses a relative preference of an entity E_i over E_j. An entity could be any object, attribute of it or a stimulus.

A PC matrix M is called *reciprocal* if $m_{ij} = 1/m_{ji}$ for every $i, j = 1, \dots, n$ (in such a case, $m_{ii} = 1$ for every $i = 1, \dots, n$):

$$M = \begin{bmatrix} 1 & m_{12} & \cdots & m_{1n} \\ \frac{1}{m_{12}} & 1 & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m_{1n}} & \frac{1}{m_{2n}} & \cdots & 1 \end{bmatrix}.$$

A PC matrix M is called *consistent* (or *transitive*) if

$$m_{ik} \cdot m_{kj} = m_{ij} \tag{1}$$

for every $i, j, k = 1, 2, \dots, n$.

Three PC matrix elements (m_{ij}, m_{ik}, m_{jk}) for $i < j < k$ form *a triad*. We will denote by $\mathcal{T}_M$ the set of all triads in M.

Note that each vector $(v_1, \dots, v_n)$ with positive coordinates generates a consistent PC matrix $[v_i/v_j]$. Reversely, from a consistent PC matrix $[m_{ij}]$ we can obtain the so-called *vector of weights* v by dividing any column by its norm. For example,

$$v = \frac{1}{\sqrt{\sum_{i=1}^{n} m_{i1}^2}} (m_{11}, \dots, m_{n1}).$$

We will refer to Eqn. (1) as a "consistency condition." While every consistent PC matrix is reciprocal, the converse is false in general. If the consistency condition does not hold, the PC matrix is inconsistent (or intransitive). In several studies (e.g., Kendall and Smith, 1940), the inconsistency in pairwise comparisons, based on triads or "cycles of three elements" (as specified in the consistency condition), was defined and examined.

There are two types of pairwise comparisons: multiplicative (with entries as ratios) and additive (with entries as differences). A multiplicative PC matrix can be converted into an additive PC matrix by the logarithmic mapping and an additive PC matrix can be converted into a multiplicative PC matrix by the exponential mapping, as it is presented in Section 4.

3. Inconsistency in pairwise comparisons

The fundamental challenge of the pairwise comparisons method is inconsistency. It frequently occurs when we are dealing with predominantly subjective assessments, as objective measurements do not usually require using pairwise comparisons. For objective data, the exact values of ratios can be computed and inserted into a PC matrix (if needed). We approximate a given inconsistent $n \times n$ PC matrix M by a consistent PC matrix M' of the same size as PC matrix M. It is reasonable to expect that the approximating PC matrix M' is somehow minimally different from the given PC matrix M. Evidently, it is an optimization problem. By "minimally", we usually assume the minimal distance between M and M'. This is worth noting that the PC matrix $M' = [v_i/v_j]$ is consistent for all (even random) values v_i. This is yet another compelling reason for considering approximation as an optimization process. Evidently, random values v_i are not satisfactory, so we need to find optimal values as it is always possible for a given metric or distance.

The approximation problem is reduced to minimizing the distance between M and M'. For the Euclidean norm, the normalized vector v of geometric means generates, according to Jensen (1984), a consistent PC matrix M' by $[v_i/v_j]$.

Needless to say, inconsistent assessments lead to inaccuracy, but for each inconsistent PC matrix there is a consistent approximation which can be computed by different methods. One of them is obtained by means of a vector v of geometric means of rows. The approximating matrix $[v_i/v_j]$ is consistent. The distance-based *inconsistency indicator* for a single triad (hence a PC matrix of size $n = 3$) was proposed by Koczkodaj (1993) as the minimum distance to the nearest consistent triad:

$$ii = \min\left(\left|1 - \frac{y}{xz}\right|, \left|1 - \frac{xz}{y}\right|\right),$$

and expanded for the entire PC matrix of size $n > 3$ as

$$ii(A) = \max_{(x,y,z)\in\mathcal{T}_A}\left(\min\left(\left|1 - \frac{y}{xz}\right|, \left|1 - \frac{xz}{y}\right|\right)\right).$$

In its simplest form, it is

$$ii(A) = \max_{(x,y,z)\in\mathcal{T}_A} \left(1 - \min\left(\frac{y}{xz}, \frac{xz}{y}\right)\right),$$

which is equivalent to

$$ii(A) = \max_{(x,y,z)\in\mathcal{T}_A} \left(1 - e^{-\left|\ln\left(\frac{y}{xz}\right)\right|}\right).$$

It is important to notice here that the distance-based inconsistency allows us to localize the most inconsistent triad in a PC matrix. This fact is of considerable importance for the inconsistency reduction process, since ii is a measurable characteristic of inconsistency reduction.

For a given PC matrix A, let us denote by $EV(A)$ the normalized principle right eigenvector of A, by $AM(A)$ the vector of arithmetic means of rows and by $GM(A)$ the vector of geometric means. A vector of geometric means of rows of a given PC matrix A is transformed into a vector of arithmetic means by the logarithmic mapping. The arithmetic mean has several properties that make it useful for a measure of central tendency. Colloquially, measures of central tendency are often called averages. For values $w_1, \ldots, w_n$, we have a mean $\bar{w}$, for which

$$(w_1 - \bar{w}) + \cdots + (w_n - \bar{w}) = 0.$$

We may assume that the values below it are balanced by the values above the mean since $w_i - \bar{w}$ is the distance from a given number to the mean. The mean is the only value for which the residuals (deviations from the estimate) sum up to zero. When we are restricted to using a single value for representing a set of known values $w_1, \ldots, w_n$, the arithmetic mean is the choice since it minimizes the sum of squared deviations from the typical value—the sum of $(w_i - \bar{w})^2$. In other words, the sample mean is also the best predictor in the sense of having the lowest root mean squared error. Means were analyzed by Aczel (1948) as well as Aczel and Saaty (1983). It is difficult to state when exactly the logarithmic mapping was used for PC matrices as an alternative method for scaling priorities in hierarchical structures. Usually, this is attributed to Jensen (1984).

4. Geometric means and the eigenvector

For a PC matrix A with positive coordinates, we define the PC matrix $B = \mu(A)$ such that $b_{ij} = \ln(a_{ij})$. Reversely, for a PC matrix B, we define the PC matrix $A = \varphi(B)$ such that $a_{ij} = \exp(b_{ij})$. By $\ln(x_1, \ldots, x_n)$ and $\exp(x_1, \ldots, x_n)$ we denote vectors $(\ln(x_1), \ldots, \ln(x_n))$ and $(\exp(x_1), \ldots, \exp(x_n))$, respectively. This implies that

$$\ln(GM(A)) = AM(\mu(A)) \tag{2}$$

and

$$\exp(AM(B)) = GM(\varphi(B)). \tag{3}$$

If A is consistent, then elements of $B = \mu(A)$ satisfy

$$b_{ik} + b_{kj} = b_{ij}$$

for every $i, j, k = 1, 2, \ldots, n$. We call such a PC matrix *additively consistent*.

Note that the set $APC := \{\mu(A) :\ A \text{ is a PC matrix}\} = \{B :\ b_{ij} + b_{ji} = 0\}$ is a $\frac{1}{2}(n^2 - n)$-dimensional vector space. For $B, B' \in APC$, we can easily define the Euclidean distance

$$\rho(B, B') = \sqrt{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (b_{ij} - b'_{ij})^2}.$$

The set ACM of all additively consistent $n \times n$ PC matrices is a linear subspace of APC.

Throughout the paper, by a distance of two PC matrices A and A' we will understand

$$d(A, A') = \rho(\mu(A), \mu(A')). \tag{4}$$

Let us consider an additive triad (x, y, z) (given or received by logarithmic mapping). It is consistent if $y = x + z$, which is equivalent to the inner product of $v = (x, y, z)$ by the vector $e = (1, -1, 1)$, giving 0. This indicates that v and e are perpendicular vectors. In other words, if a triad is inconsistent, an orthogonal projection onto the subspace perpendicular to the vector $e = (1, -1, 1)$ in the space $\mathbb{R}^3$ makes it consistent. Such projections can be expressed by

$$\tilde{v} = v - v \circ \frac{e}{e \circ e}\, e,$$

where $u_1 \circ u_2$ is the inner product of vectors u_1 and u_2 in $\mathbb{R}^3$.

We have

$$e \circ e = 3, \quad v \circ e = x - y + z,$$

and hence $\tilde{v} = Pv$, where

$$P = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Thus, the final formulas take the form

$$\tilde{b}_{ik} = \frac{1}{3}(2b_{ik} + b_{ij} - b_{kj}), \tag{5}$$

$$\tilde{b}_{ij} = \frac{1}{3}(b_{ik} + 2b_{ij} + b_{kj}), \tag{6}$$

$$\tilde{b}_{kj} = \frac{1}{3}(-b_{ik} + b_{ij} + 2b_{kj}). \tag{7}$$

Remark 1. A given additively inconsistent triad (b_{ik}, b_{ij}, b_{kj}) can be transformed into a consistent triad by replacing only one element in the following three ways:

$$(b_{ik}, b_{ik} + b_{kj}, b_{kj}),$$
$$(b_{ij} - b_{kj}, b_{ij}, b_{kj}),$$
$$(b_{ik}, b_{ij}, b_{ij} - b_{ik}).$$

Taking the average of the above three triads, we get a triad expressed by (5)–(7).

For a multiplicatively inconsistent triad (a_{ik}, a_{ij}, a_{kj}), its transformation to the consistent triad $(\tilde{a}_{ik}, \tilde{a}_{ij}, \tilde{a}_{kj})$ is given by

$$\tilde{a}_{ik} = a_{ik}^{2/3} a_{ij}^{1/3} a_{kj}^{-1/3}, \tag{8}$$
$$\tilde{a}_{ij} = a_{ik}^{1/3} a_{ij}^{2/3} a_{kj}^{1/3}, \tag{9}$$
$$\tilde{a}_{kj} = a_{ik}^{-1/3} a_{ij}^{1/3} a_{kj}^{2/3}. \tag{10}$$

The above formulas were used by Koczkodaj *et al*. (2015) for a Monte Carlo experimentation with the convergence of inconsistency by the sequence of inconsistency reductions of the most inconsistent triad. The n-th step of the algorithm used by Holsztynski and Koczkodaj (1996) transforms the most inconsistent triad $(b_{ik}, b_{ij}, b_{kj}) = (x, y, z)$ of a given PC matrix B_n into $\tilde{v} = (\tilde{b}_{ik}, \tilde{b}_{ij}, \tilde{b}_{kj}) = (\tilde{x}, \tilde{y}, \tilde{z})$ according to the formulas (5)–(7). Obviously, we also replace (b_{ki}, b_{ji}, b_{jk}) with $-\tilde{v}$, leaving the rest of the entries unchanged. Let B_{n+1} denote the PC matrix after the transformation. The coordinates of $AM(B_n)$ may change only for the i-th, the j-th and the k-th position. However, a simple calculation shows that

$$\tilde{x} + \tilde{y} = x + y \tag{11}$$
$$-\tilde{x} + \tilde{z} = -x + z \tag{12}$$
$$-\tilde{y} - \tilde{z} = -y - z. \tag{13}$$

This proves that

$$AM(B_n) = AM(B_{n+1}). \tag{14}$$

Consequently,

$$GM(A_n) = GM(A_{n+1}), \tag{15}$$

where $A_n = \varphi(B_n)$.

Theorem 1. *For a given additive PC matrix*

$$B = \begin{bmatrix} 0 & b_{12} & \cdots & b_{1n} \\ -b_{12} & 0 & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -b_{1n} & -b_{2n} & \cdots & 0 \end{bmatrix}$$

and its orthogonal projection B' onto the space of additively consistent matrices, we have

$$AM(B) = AM(B').$$

Proof. Theorem 4 by Holsztynski and Koczkodaj (1996) states that the sequence of PC matrices B_n is convergent to the orthogonal projection of $B_1 = B$ onto the linear space of additively consistent matrices. Equations (14) mean that AM is invariant at each step of the algorithm, so AM of B', which is the limit of B_n, also must be the same. ■

As a consequence, we obtain what follows.

Theorem 2. *We have*

$$GM(A) = GM(A'),$$

where $A = \varphi(B)$ is the original PC matrix and $A' = \varphi(B')$ is the limit consistent matrix.

Proof. The statement follows immediately from the previous theorem, since

$$\begin{aligned} GM(A) &= GM(\varphi(B)) = \exp(AM(B)) \\ &= \exp(AM(B')) = GM(\varphi(B')) \\ &= GM(A'). \end{aligned}$$

■

The following two examples illustrate how the above works in practice.

Example 1. Consider an example of a PC matrix:

$$\begin{bmatrix} 1 & 2 & 5 \\ \frac{1}{2} & 1 & 3 \\ \frac{1}{5} & \frac{1}{3} & 1 \end{bmatrix}$$

After an orthogonal transformation, we get

$$\begin{bmatrix} 1 & 1.882072 & 5.313293 \\ 0.531329 & 1 & 2.823108 \\ 0.188207 & 0.35422 & 1 \end{bmatrix}.$$

The first PC matrix is inconsistent since $2 \cdot 3 \neq 5$. The vector of geometric means is identical in both cases and it equals

$$\begin{bmatrix} 2.15443469 \\ 1.14471424 \\ 0.40548013 \end{bmatrix}.$$

♦

Example 2. Consider an additive PC matrix:

$$\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}.$$

The linear space of all such matrices is isomorphic with $\mathbb{R}^6$. The following system of equations must hold for the above additive PC matrix to be additively consistent:

$$U_1 : a + d = b,$$
$$U_2 : d + f = e,$$
$$U_3 : a + e = c,$$
$$U_4 : b + f = c.$$

Each equation describes a subspace U_i of the dimension 5. Subspace W of the consistent matrices is the intersection of dimension 3. A PC matrix can be made consistent with the greedy algorithm utilized by Holsztynski and Koczkodaj (1996). It transforms the most inconsistent triad into a consistent one by the orthogonal projection on U_i. The proof of convergence for this stepwise inconsistency reduction was proposed by Holsztynski and Koczkodaj (1996), but it was incomplete. It was finally done by Koczkodaj and Szarek (2010). Another possible solution is the orthogonal projection on W. As a result, we obtain the matrix

$$\begin{bmatrix} 0 & A & B & C \\ -A & 0 & D & E \\ -B & -D & 0 & F \\ -C & -E & -F & 0 \end{bmatrix},$$

where

$$\begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2a + b + c - d - e \\ a + 2b + c + d - f \\ a + b + 2c + e + f \\ -a + b + 2d + e - f \\ -a + c + d + 2e + f \\ -b + c - d + e + 2f \end{bmatrix}.$$

♦

Equations (2), (3) and Theorem 1 imply that it is sufficient to compute $GM(M)$ for any reciprocal PC matrix M to obtain the vector of weights of the closest consistent PC matrix. In general, $GM(M)$ is not equal to $EV(M)$, which is the right principal eigenvector corresponding to the principal eigenvalue of M (even if both are normalized).

Remark 2. Assume that

$$M = \begin{bmatrix} 1 & a & b \\ \frac{1}{a} & 1 & c \\ \frac{1}{b} & \frac{1}{c} & 1 \end{bmatrix}$$

is a reciprocal matrix. Then

$$GM(M) = EV(M).$$

Proof. Set

$$v := GM(M) = \begin{bmatrix} \sqrt[3]{ab} \\ \sqrt[3]{\frac{c}{a}} \\ \frac{1}{\sqrt[3]{bc}} \end{bmatrix}.$$

Then, for

$$\lambda = 1 + \sqrt[3]{\frac{ac}{b}} + \sqrt[3]{\frac{b}{ac}},$$

we have $Mv = \lambda v$, and this completes the proof.

■

The conclusion of the above remark cannot be generalized to PC matrices of higher degrees.

Example 3. For the PC matrix

$$M = \begin{bmatrix} 1 & 2 & 1 & 3 \\ \frac{1}{2} & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ \frac{1}{3} & 1 & \frac{1}{2} & 1 \end{bmatrix}$$

we calculate the vector v of geometric means of row elements

$$v := GM(M) = \begin{bmatrix} \sqrt[4]{6} \\ \sqrt[4]{\frac{1}{2}} \\ \sqrt[4]{2} \\ \sqrt[4]{\frac{1}{6}} \end{bmatrix}.$$

There is no λ such that $Mv = \lambda v$, and it follows that $GM(M) \neq EV(M)$. Vector $GM(M)$ generates the consistent PC matrix which is the closest to M. Consequently, $EV(M)$ does not. ♦

5. Inconsistency reduction process

For a given inconsistent PC matrix A, a consistent PC matrix can be computed by one transformation $[v_i/v_j]$, where $v_i = GM_i(A)$ and $GM_i(A)$ denotes the geometric mean of the i-th row of PC matrix A. There is probably no better way of providing explanations why the one step approximation is not as good as many of them than the following:

> *Arriving at one goal is the starting point to another.*
>
> John Dewey, 1859–1952
> (American philosopher)

The triad-by-triad reduction for the case of $n = 3$ is needed in practice since examination of individual triads facilitates their improvement by not only "mechanistic computations", but by additional data acquisition, too. For $n = 3$, only one orthogonal transformation is needed for achieving the consistency. For $n > 3$, the number of orthogonal transformations may be indefinite. However, the convergence is very quick, as evidenced by Koczkodaj *et al.* (2015). Usually, fewer than 10 steps are needed for most applications to reduce the inconsistency below the threshold of $1/3$ assumed for applications.

For an inconsistent PC matrix, we find the most inconsistent triad (according to Kii) and, by using the three expressions for the orthogonal projection, we make it consistent. The idea of the algorithm is illustrated by Fig. 1. The starting point x_0 corresponds to the initial PC matrix. The lines U and V represent the linear subspaces of PC matrices with an additively consistent triad located in a fixed place. For example, they might be the sets

$$U = \{A : \ a_{13} + a_{36} = a_{16}\}$$

and

$$V = \{A : \ a_{23} + a_{36} = a_{26}\}.$$

We select the most inconsistent triad (in our case, $T_1 = (a_{13}, a_{16}, a_{36})$) and project the PC matrix orthogonally on U (point x_1). Obviously, we must transform this triad, which may result in increasing the inconsistency in another triad (say $T_2 = (a_{23}, a_{26}, a_{36})$). Thus, in the next step, T_2 may be the most inconsistent triad. We continue projecting x_1 on V, getting x_2. The projecting continues until the PC matrix is sufficiently consistent. Figure 1 illustrates the point, obtained by orthogonal projections, which is sufficiently close to $U \cap V$.

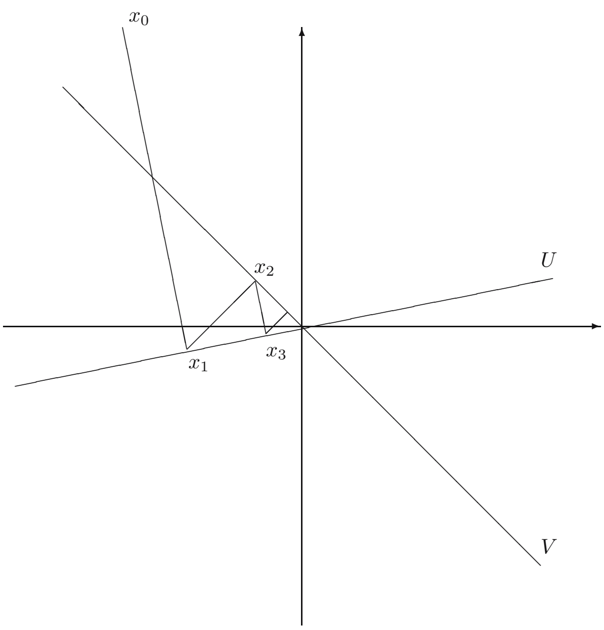

Fig. 1. Orthogonal projections.

Let us recall that only one transformation is needed to make a PC matrix 3×3 consistent since there is only one triad in such a matrix, hence no other triad can be influenced by this transformation. For $n > 3$, any transformation to all three values in one triad may propagate to other triads.

It is not evident for $n \to \infty$ how the propagation may go. However, the proof of convergence was provided by Bauschke and Borwein (1996) (unrelated to PC matrices) and independently by Koczkodaj and Szarek (2010) in the following theorem.

Theorem 3. (Koczkodaj and Szarek, 2010, Thm. 1) *Let $\mathcal{L}$ be a non-empty finite family of the linear subspaces of $\mathbb{R}^{\mathbb{N}}$. Let $W = \bigcap \mathcal{L}$ be the intersection of members of $\mathcal{L}$. Let $w : \mathbb{N} \to \mathcal{L}$ be a sequence such that for any $V \in \mathcal{L}$ the equality $w(n) = V$ holds for infinitely many $n \in \mathbb{N}$. Fix $x \in \mathbb{R}^{\mathbb{N}}$, and define $x_0 = x$ and*

$$x_n = p_{w(n)}(x_{n-1})$$

for $n \geq 1$. Then

$$\lim_{n\to\infty} x_n = p_W(x),$$

where p_V denotes the orthogonal projection on V.

Consequently, a consistent PC matrix is obtained from a vector of geometric means of rows of the original inconsistent PC matrix. When we apply the above theorem to the finite family $\mathcal{L}$ of $(n^2 - n)/2$ linear spaces of additive PC matrices B with a consistent triad (b_{ij}, b_{ik}, b_{jk}), for $1 \leq i < j < k \leq n$, the intersection $W = \bigcap \mathcal{L}$ is a linear space of all additively consistent matrices, and hence it is an infinite set.

Therefore, a natural question arises whether the inconsistency reduction algorithm obtains the closest solution. According to Theorem 4 of Holsztynski and Koczkodaj (1996), the answer is positive with respect to the metric defined in (4).

As indicated by Dong *et al.* (2008), the complexity of the presented computations is at most $O(n^3)$. From the theoretical point of view, the complexity of searches of a PC matrix of size n is $O(n^2)$, but for a triad (x, y, z) it may be $O(n^3)$ if each coordinate is changed independently. For the consistent case, $y = x \cdot z$, hence the complexity is still $O(n^2)$ when it comes to searches for all triads in a consistent PC matrix.

Example 4. Consider the PC matrix

$$M = \begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 1 & 1 & \underline{3.5} & 2.5 & \underline{1.5} \\ 0.5 & \frac{2}{7} & 1 & 1.4 & \underline{1.2} \\ \frac{1}{3} & \frac{2}{5} & \frac{5}{7} & 1 & 1.1 \\ 0.25 & \frac{2}{3} & \frac{5}{6} & \frac{10}{11} & 1 \end{bmatrix}.$$

The elements of the most inconsistent triad are underlined.

Evidently, vectors $GM(M)$ and $EV(M)$ are neither equal nor linearly dependent:

$$v_1 = GM(M) = \begin{bmatrix} 1.888175 \\ 1.673477 \\ 0.751696 \\ 0.636855 \\ 0.661081 \end{bmatrix},$$

and

$$v_2 = EV(M) = \begin{bmatrix} 0.670129 \\ 0.609415 \\ 0.267868 \\ 0.222817 \\ 0.241121 \end{bmatrix},$$

which, after normalization, gives

$$nv_1 = \begin{bmatrix} 0.336496 \\ 0.298234 \\ 0.133961 \\ 0.113495 \\ 0.117812 \end{bmatrix}$$

and

$$nv_2 = \begin{bmatrix} 0.333173 \\ 0.302988 \\ 0.133178 \\ 0.110779 \\ 0.119880 \end{bmatrix}.$$

A simple computation shows that the highest inconsistency indicator is reached for the underlined triad and it is equal to 0.642857143, so the inconsistency reduction begins with this triad.

Now, let us have a look at the first five iteration steps and the limit matrix, which is consistent. In each step, we indicate the recently changed triad in boxes, underline the most inconsistent triad to be corrected in the next iteration, and give its inconsistency index.

Step 1 ($\max ii = 0.574305$):

$$M = \begin{bmatrix} 1 & 1 & \underline{2} & 3 & \underline{4} \\ 1 & 1 & \boxed{2.483} & 2.5 & \boxed{2.114} \\ 0.5 & 0.403 & 1 & 1.4 & \boxed{\underline{0.851}} \\ 0.333 & 0.4 & 0.714 & 1 & 1.1 \\ 0.25 & 0.473 & 1.175 & 0.909 & 1 \end{bmatrix}.$$

Step 2 ($\max ii = 0.297384$):

$$M = \begin{bmatrix} 1 & \underline{1} & \boxed{2.659} & 3 & \boxed{\underline{3.009}} \\ 1 & 1 & 2.483 & 2.5 & \underline{2.114} \\ 0.376 & 0.403 & 1 & 1.4 & \boxed{1.132} \\ 0.333 & 0.4 & 0.714 & 1 & 1.1 \\ 0.332 & 0.473 & 0.884 & 0.909 & 1 \end{bmatrix}.$$

Step 3 ($\max ii = 0.280888$):

$$M = \begin{bmatrix} 1 & \boxed{1.125} & 2.659 & 3 & \boxed{2.675} \\ 0.889 & 1 & \underline{2.483} & \underline{2.5} & \boxed{3.378} \\ 0.376 & 0.403 & 1 & \underline{1.4} & 1.132 \\ 0.333 & 0.4 & 0.714 & 1 & 1.1 \\ 0.374 & 0.42 & 0.884 & 0.909 & 1 \end{bmatrix}.$$

Step 4 ($\max ii = 0.225233$):

$$M = \begin{bmatrix} 1 & 1.125 & 2.659 & 3 & 2.675 \\ 0.889 & 1 & \boxed{2.225} & \boxed{\underline{2.79}} & \underline{2.378} \\ 0.376 & 0.449 & 1 & \boxed{1.254} & 1.132 \\ 0.333 & 0.358 & 0.797 & 1 & \underline{1.1} \\ 0.374 & 0.42 & 0.884 & 0.909 & 1 \end{bmatrix}.$$

Step 5 ($\max ii = 0.117406$):

$$M = \begin{bmatrix} 1 & 1.125 & 2.659 & \underline{3} & \underline{2.675} \\ 0.889 & 1 & 2.225 & \boxed{2.563} & \boxed{2.589} \\ 0.376 & 0.449 & 1 & 1.254 & 1.132 \\ 0.333 & 0.39 & 0.797 & 1 & \boxed{\underline{1.01}} \\ 0.374 & 0.386 & 0.884 & 0.99 & 1 \end{bmatrix}.$$

Limit matrix ($\max ii = 0$):

$$M = \begin{bmatrix} 1 & 1.128 & 2.512 & 2.965 & 2.856 \\ 0.886 & 1 & 2.226 & 2.628 & 2.531 \\ 0.398 & 0.449 & 1 & 1.18 & 1.137 \\ 0.337 & 0.381 & 0.847 & 1 & 0.963 \\ 0.35 & 0.395 & 0.879 & 1.038 & 1 \end{bmatrix}.$$

It is easy to notice that the level of inconsistency (measured by ii) is reduced in each step and the vector of geometric means of rows equal to

$$(1.888175, 1.673477, 0.751696, 0.636855, 0.661081)$$

is invariant during the whole procedure. Evidently, it remains the same for the limit matrix. In general, the sequence $ii(A_n)$ does not have to be monotonic; however, its convergence to zero is guaranteed by Theorem 3. The right eigenvector of the given PC matrix is

$$v_3 = \begin{bmatrix} 0.677259 \\ 0.600216 \\ 0.26958 \\ 0.228426 \\ 0.237084 \end{bmatrix},$$

and it is not equal to (nor linearly dependent on) GM. Both vectors are equal or linearly dependent only for a consistent matrix for $n > 3$. For $n = 3$, eigenvector and geometric means are linearly dependent vectors.

♦

6. Conclusions

This study provided a proof that the limit for the inconsistency reduction process is the PC matrix induced by the vector of geometric means of the initial matrix. It also provided examples showing that the vector of geometric means and the right principal eigenvector are not linearly dependent for the size of a PC matrix $n > 3$ (they are for $n = 3$).

The sequence of matrices occurring during the inconsistency reduction process converges to the matrix induced by the vector of geometric means, but not by the right principal eigenvector. This is of considerable importance for practical applications. The inconsistency reduction approach is well aligned with the "GIGO" (garbage in, garbage out) computing principle, which comes to the following: *Improving the approximation accuracy for the inconsistent data makes very little sense*. The inconsistency reduction process must take place before such approximation is attempted. More research (e.g., by Monte Carlo simulations) is needed to investigate algorithms for inconsistency reduction for various indicators.

The inconsistency reduction process is based on the reduction of inconsistency in individual triads. It must be stressed that replacing the initial PC matrix by the consistent one generated by the geometric means of rows sometimes may be worse than applying just a few steps of the algorithm. Making a matrix perfectly consistent usually requires changing all its elements, while the reduction process allows detecting the worst assessments and trying to reduce inconsistency in future comparisons.

Evidently, geometric means can be computed even with a pocket calculator although Gnumerics and Excel are better tools for it. Computing the principal eigenvector is not as easy as computing geometric means. An eigenvalue perturbation problem exists, and finding the eigenvectors and eigenvalues of a system that is perturbed from known eigenvectors and eigenvalues is not an entirely trivial problem to solve. In addition, the Bauer–Fike theorem stipulates that the sensitivity of the eigenvalues is estimated by the condition number. In other words, computing eigenvalues and eigenvectors with high accuracy may be challenging, while computing geometric means with high accuracy is, in practice, a trivial task.

It is worth pointing out that the simplified version of pairwise comparisons, published by Koczkodaj and Szybowski (2015), does not have inconsistencies. PC matrix elements are generated from a set of principal generators, preserving the consistency condition. However, without inconsistency analysis, it is difficult to correct the input if input data are inaccurate.

The pairwise comparisons method has been implemented, as part of cloud computing support, for a group decision making process used by the software development team at Health Sciences North (a regional hospital in Sudbury, Ontario, Canada, with a service area comparable to Holland). The *SourceForge.net* repository was used to make the software available for downloading *pro publico bono*.

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Computations in our examples have been made with MS Excel and Wolfram|Alpha.

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