

## FRACTIONAL DESCRIPTOR CONTINUOUS–TIME LINEAR SYSTEMS DESCRIBED BY THE CAPUTO–FABRIZIO DERIVATIVE

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The Weierstrass–Kronecker theorem on the decomposition of the regular pencil is extended to fractional descriptor continuous-time linear systems described by the Caputo–Fabrizio derivative. A method for computing solutions of continuous-time systems is presented. Necessary and sufficient conditions for the positivity and stability of these systems are established. The discussion is illustrated with a numerical example.

**Keywords:** fractional system, descriptor system, continuous-time, linear system, Caputo–Fabrizio derivative.

### 1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial condition state remains forever in the positive orthant for all nonnegative inputs. An overview of the state of the art in positive systems theory is given in the monographs of Farina and Rinaldi (2000) as well as Kaczorek (2001), and in the papers by Kaczorek (1997; 1998a; 2011a; 2014a; 2014b; 2015b). Models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

Fractional systems were investigated by Kaczorek (2012) and Ostalczyk (2016). Descriptor (singular) linear systems were considered in many papers and books (Gantmacher, 1959; Campbell *et al.*, 1976; Van Dooren, 1979; Kucera and Zagalak, 1988; Dai, 1989; Fahmy and O’Reill, 1989; Kaczorek, 1997; 1998a; Bru *et al.*, 2000; 2003; Virnik, 2008; Dodig and Stosic, 2009; Duan, 2010). Positive standard and descriptor systems and their stability were analyzed by Kaczorek (1998a; 2001; 2011b; 2014b; 2015b) and Virnik (2008). Positive linear systems with different fractional orders were addressed by Kaczorek (2011b; 2012), along with descriptor positive discrete-time and continuous-time nonlinear systems (Kaczorek, 2014a), the positivity and linearization of nonlinear discrete-time systems by state feedbacks (Kaczorek, 2014b), or new

stability tests of positive standard and fractional linear systems (Kaczorek, 2011a). The stability and robust stabilization of discrete-time switched systems were analyzed by Zhang *et al.* (2014a; 2014b), while the controllability of dynamical systems was investigated by Klamka (2013).

Recently, a new definition of the fractional derivative without a singular kernel has been proposed (Caputo and Fabrizio, 2015; Losada and Nieto, 2015). Using this new definition, the fractional descriptor continuous-time linear systems will be investigated.

The paper is organized as follows. In Section 2, the Weierstrass–Kronecker decomposition theorem is applied and the solution of the state equation of fractional descriptor continuous-time linear systems is given. Necessary and sufficient conditions for positivity are established in Section 3 and for asymptotic stability in Section 4, where also tests for checking stability are given. A numerical example is presented in Section 5. Concluding remarks are given in Section 6.

The following notation will be used:  $\mathbb{R}$ , the set of real numbers;  $\mathbb{R}^{n \times m}$ , the set of  $n \times m$  matrices;  $\mathbb{R}_+^{n \times m}$ , the set of real  $n \times m$  matrices with nonnegative entries and  $\mathbb{R}_+^n = \mathbb{R}_+^{n \times 1}$ ;  $M_n$ , the set of  $n \times n$  Metzler matrices (with nonnegative off-diagonal entries);  $I_n$ , the  $n \times n$  identity matrix.

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## 2. Fractional descriptor continuous-time linear systems

Consider the fractional descriptor continuous-time linear system

$$E^{CF} D^\alpha x(t) = Ax(t) + Bu(t), \quad 0 < \alpha < 1, \quad (1a)$$

$$y(t) = Cx(t), \quad (1b)$$

where  $x(t) \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}^m$ ,  $y(t) \in \mathbb{R}^p$  are the state, input and output vectors,  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{p \times n}$ ,  $D \in \mathbb{R}^{p \times m}$ , and

$$\begin{aligned} {}^{CF}D^\alpha x(t) &= \frac{d^\alpha x(t)}{dt^\alpha} \\ &= \frac{1}{1-\alpha} \int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right) \dot{x}(\tau) d\tau, \quad (2) \\ \dot{x}(\tau) &= \frac{dx(\tau)}{d\tau}, \quad t \geq 0, \end{aligned}$$

is the Caputo–Fabrizio fractional derivative of order  $\alpha$  of the state vector  $x(t) \in \mathbb{R}^n$  (Caputo and Fabrizio, 2015; Losada and Nieto, 2015).

It is assumed that  $\det E = 0$  and

$$\det[E\lambda - A] \neq 0 \quad (3)$$

for some  $\lambda \in \mathbb{C}$ .

It is well-known (Kaczorek, 1998b) that if (3) holds then there exists a pair of nonsingular matrices  $P, Q \in \mathbb{R}^{n \times n}$  such that

$$P[E\lambda - A]Q = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} \lambda - \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \quad (4)$$

$$n_1 + n_2 = n,$$

where  $n_1 = \deg \det[E\lambda - A]$ ,  $A_1 \in \mathbb{R}^{n_1 \times n_1}$  and  $N \in \mathbb{R}^{n_2 \times n_2}$  is a nilpotent matrix with the index  $\mu$  (i.e.,  $N^\mu = 0$  and  $N^{\mu-1} \neq 0$ ). The matrices  $P, Q, A_1$  can be found by the use of elementary row and column operations (Kaczorek, 1998b).

Premultiplying (1a) by the matrix  $P \in \mathbb{R}^{n \times n}$ , introducing the new state vector

$$\begin{aligned} \bar{x}(t) = Q^{-1}x(t) &= \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix}, \\ \bar{x}_1(t) &= \begin{bmatrix} \bar{x}_{11}(t) \\ \bar{x}_{12}(t) \\ \vdots \\ \bar{x}_{1n_1}(t) \end{bmatrix}, \\ \bar{x}_2(t) &= \begin{bmatrix} \bar{x}_{21}(t) \\ \bar{x}_{22}(t) \\ \vdots \\ \bar{x}_{2n_2}(t) \end{bmatrix}, \end{aligned} \quad (5)$$

and using (4), we obtain

$$\frac{d^\alpha \bar{x}_1(t)}{dt^\alpha} = A_1 \bar{x}_1(t) + B_1 u(t), \quad (6a)$$

$$N \frac{d^\alpha \bar{x}_2(t)}{dt^\alpha} = \bar{x}_2(t) + B_2 u(t), \quad (6b)$$

where

$$PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad B_1 \in \mathbb{R}^{n_1 \times m}, \quad B_2 \in \mathbb{R}^{n_2 \times m}. \quad (7)$$

**Theorem 1.** The solution  $\bar{x}_1(t)$  of Eqn. (6a) for a given initial condition  $\bar{x}_1(0) = \bar{x}_{10} \in \mathbb{R}^{n_1}$  and an input  $u(t) \in \mathbb{R}^m$ ,  $t \geq 0$ , has the form

$$\begin{aligned} \bar{x}_1(t) &= e^{\hat{A}_1 t} (\hat{x}_{10} + \hat{B}_1 u_0) \\ &+ \int_0^t e^{\hat{A}_1(t-\tau)} \hat{B}_1 [\beta u(\tau) + \dot{u}(\tau)] d\tau, \end{aligned} \quad (8a)$$

where

$$\begin{aligned} \hat{A}_1 &= \alpha [I_{n_1} - (1-\alpha)A_1]^{-1} A_1, \\ \hat{B}_1 &= [I_{n_1} - (1-\alpha)A_1]^{-1} (1-\alpha)B_1, \\ \hat{x}_{10} &= [I_{n_1} - (1-\alpha)A_1]^{-1} \bar{x}_{10}, \\ e^{\hat{A}_1 t} &= \mathcal{L}^{-1} \{ [I_{n_1} s - \hat{A}_1]^{-1} \}, \end{aligned} \quad (8b)$$

$$\beta = \frac{\alpha}{1-\alpha}, \quad \dot{u}(\tau) = \frac{du(\tau)}{d\tau}, \quad u(0) = u_0.$$

The proof is given by Kaczorek (2015a).

**Theorem 2.** The solution  $\bar{x}_2(t)$  of Eqn. (6b) for a given initial condition  $\bar{x}_2(0) = \bar{x}_{20} \in \mathbb{R}^{n_2}$  and an input  $u(t) \in \mathbb{R}^m$ ,  $t \geq 0$ , has the form

$$\begin{aligned} \bar{x}_2(t) &= e^{\hat{N}t} (\tilde{N} \bar{x}_{20} + \tilde{B}_2 u_0) \\ &+ \int_0^t e^{\hat{N}(t-\tau)} \tilde{B}_2 [\beta u(\tau) + \dot{u}(\tau)] d\tau, \end{aligned} \quad (9a)$$

where

$$\begin{aligned} \hat{N} &= \alpha [N - I_{n_2}(1-\alpha)]^{-1}, \\ \tilde{N} &= [N - I_{n_2}(1-\alpha)]^{-1} N, \\ \tilde{B}_2 &= [N - I_{n_2}(1-\alpha)]^{-1} (1-\alpha)B_2, \\ \beta &= \frac{\alpha}{1-\alpha}. \end{aligned} \quad (9b)$$

*Proof.* Using Laplace transform ( $\mathcal{L}$ ) in (6b) as well as the convolution theorem, we obtain

$$\begin{aligned} N \mathcal{L} \left[ \frac{d^\alpha \bar{x}_2(t)}{dt^\alpha} \right] &= N \frac{1}{1-\alpha} \mathcal{L} \left[ \int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right) \dot{\bar{x}}_2(\tau) d\tau \right] \\ &= \mathcal{L}[\bar{x}_2(t)] + B_2 \mathcal{L}[u(t)] \end{aligned} \quad (10)$$

and

$$N \frac{1}{1-\alpha} \left\{ \frac{1}{s+\beta} [s\bar{X}_2(s) - \bar{x}_{20}] \right\} = \bar{X}_2(s) + B_2U(s), \tag{11}$$

where

$$\begin{aligned} \bar{X}_2(s) &= \mathcal{L}[\bar{x}_2(t)] = \int_0^\infty \bar{x}_2(t)e^{-st} dt, \\ U(s) &= \mathcal{L}[u(t)], \\ \beta &= \frac{\alpha}{1-\alpha}, \\ \mathcal{L}[e^{-\beta t}] &= \frac{1}{s+\beta}, \\ \mathcal{L} \left[ \int_0^t \exp \left( -\frac{\alpha}{1-\alpha}(t-\tau) \right) \dot{\bar{x}}_2(\tau) d\tau \right] & \\ &= \frac{1}{s+\beta} [s\bar{X}_2(s) - \bar{x}_{20}], \\ \mathcal{L}[\dot{\bar{x}}_2(t)] &= s\bar{X}_2(s) - \bar{x}_{20}. \end{aligned} \tag{12}$$

From (10), we have

$$\begin{aligned} \{s[N - I_{n_2}(1-\alpha)] - I_{n_2}\alpha\} \bar{X}_2(s) \\ = N\bar{x}_{20} + (s+\beta)\bar{B}_2U(s), \end{aligned} \tag{13}$$

where  $\bar{B}_2 = (1-\alpha)B_2$ .

Note that the matrix  $[N - I_{n_2}(1-\alpha)]$  is invertible. After premultiplication of (13) by  $[N - I_{n_2}(1-\alpha)]^{-1}$ , we obtain

$$\begin{aligned} [I_{n_2}s - \hat{N}]\bar{X}_2(s) &= [N - I_{n_2}(1-\alpha)]^{-1}N\bar{x}_{20} \\ &+ (s+\beta)[N - I_{n_2}(1-\alpha)]^{-1}\bar{B}_2U(s) \\ &= \tilde{N}\bar{x}_{20} + \beta\tilde{B}_2U(s) + \tilde{B}_2[sU(s) - u_0] + \tilde{B}_2u_0, \end{aligned} \tag{14a}$$

where

$$\begin{aligned} \hat{N} &= \alpha[N - I_{n_2}(1-\alpha)]^{-1}, \\ \tilde{N} &= [N - I_{n_2}(1-\alpha)]^{-1}N, \\ \tilde{B}_2 &= [N - I_{n_2}(1-\alpha)]^{-1}\bar{B}_2 \end{aligned} \tag{14b}$$

and

$$\begin{aligned} \bar{X}_2(s) &= [I_{n_2}s - \hat{N}]^{-1}\tilde{N}\bar{x}_{20} \\ &+ [I_{n_2}s - \hat{N}]^{-1}\tilde{B}_2u_0 \\ &+ \beta[I_{n_2}s - \hat{N}]^{-1}\tilde{B}_2U(s) \\ &+ [I_{n_2}s - \hat{N}]^{-1}\tilde{B}_2[sU(s) - u_0]. \end{aligned} \tag{15}$$

Taking into account that

$$\mathcal{L}^{-1}\{[I_{n_2}s - \hat{N}]^{-1}\} = e^{\hat{N}t} \tag{16}$$

and using the inverse Laplace transform as well as the convolution theorem, we obtain (9). ■

From (8) and (9), we can see that both solutions have a similar form, which is completely different from the standard Caputo derivative, where the subsystem (6b) has a strictly singular solution with Dirac impulses.

### 3. Positive fractional linear systems

In this section, the necessary and sufficient conditions for the positivity of fractional descriptor continuous-time linear systems described by Eqns. (1) will be established.

**Definition 1.** The fractional descriptor continuous-time linear system (1) is called (internally) *positive* if and only if  $x(t) \in \mathbb{R}_+^n$  and  $y(t) \in \mathbb{R}_+^p$ ,  $t \geq 0$ , for any consistent initial conditions  $x_0 \in \mathbb{R}_+^n$  and all admissible inputs  $u(t) \in \mathbb{R}_+^m$ ,  $\dot{u}(t) \in \mathbb{R}_+^m$ ,  $t \geq 0$ .

**Definition 2.** The matrix  $Q \in \mathbb{R}_+^{n \times n}$  is called *monomial* if in each row and column only one entry is positive and the remaining entries are zero.

It is well known (Kaczorek, 2001) that  $Q^{-1} \in \mathbb{R}_+^{n \times n}$  if and only if the matrix  $Q \in \mathbb{R}_+^{n \times n}$  is monomial. It is assumed that for the positive system (1) the decomposition (4) is possible for the monomial matrix  $Q$ . In this case,  $x(t) = Q\bar{x}(t) \in \mathbb{R}_+^n$  if and only if  $\bar{x}(t) \in \mathbb{R}_+^n$ ,  $t \geq 0$ . It is also well known that premultiplication of Eqn. (1) by the matrix  $P$  does not change its solution  $x(t)$ .

**Lemma 1.** Let  $\hat{A}_1 \in M_{n_1}$  and  $0 < \alpha < 1$ . Then

$$e^{\hat{A}_1 t} \in \mathbb{R}_+^{n_1 \times n_1} \text{ for } t \geq 0. \tag{17}$$

The proof is similar to the one given by Kaczorek (2001).

**Theorem 3.** Let the decomposition (4) of the system (1) be possible for a monomial matrix  $Q \in \mathbb{R}_+^{n \times n}$ . The subsystem (6a) for  $0 < \alpha < 1$  is positive if and only if

$$\hat{A}_1 \in M_{n_1}, \quad \hat{B}_1 \in \mathbb{R}_+^{n_1 \times m}. \tag{18}$$

*Proof.*

(Sufficiency) If  $\hat{A}_1 \in M_{n_1}$  and  $\hat{B}_1 \in \mathbb{R}_+^{n_1 \times m}$  then from (8) we have  $\bar{x}_1(t) \in \mathbb{R}_+^{n_1}$ ,  $t \geq 0$  since by Lemma 1  $e^{\hat{A}_1 t} \in \mathbb{R}_+^{n_1 \times n_1}$  and  $\bar{x}_{10} \in \mathbb{R}_+^{n_1}$ ,  $u(t) \in \mathbb{R}_+^m$ ,  $\dot{u}(t) \in \mathbb{R}_+^m$ ,  $t \geq 0$ .

(Necessity) Let  $u(t) = 0$ ,  $t \geq 0$  and  $\bar{x}_{10} = e_i$  (the  $i$ -th column of the identity matrix  $I_{n_1}$ ). The trajectory remains in the orthant  $\mathbb{R}_+^{n_1}$  only if  ${}^{CF}D^\alpha \bar{x}_1(0) = \hat{A}_1 e_i \geq 0$ , which implies  $\hat{A}_1 \in M_{n_1}$ . If  $\bar{x}_{10} = 0$ , then  ${}^{CF}D^\alpha \bar{x}_1(0) = \hat{B}_1 u(0) \geq 0$  and this implies  $\hat{B}_1 \in \mathbb{R}_+^{n_1 \times m}$  since  $u(0) \in \mathbb{R}_+^m$  is arbitrary. ■

**Lemma 2.** If  $\lambda_k$ ,  $k = 1, \dots, n_1$ , are the eigenvalues of the matrix  $A_1$ , then the eigenvalues of the matrix  $\hat{A}_1 = \alpha[I_{n_1} - (1-\alpha)A_1]^{-1}A_1$  are given by

$$\hat{\lambda}_k = \alpha[1 - (1-\alpha)\lambda_k]^{-1}\lambda_k. \tag{19}$$

*Proof.* It is well known (Gantmacher, 1959) that if  $f(\lambda_k)$  is well defined on the spectrum  $\lambda_k, k = 1, \dots, n_1$ , of the matrix  $A_1$  then the eigenvalues of the matrix  $f(A)$  are given by  $f(\lambda_k), k = 1, \dots, n_1$ . In this case,  $f(\hat{A}) = \alpha[I_{n_1} - (1 - \alpha)A_1]^{-1}A_1$ . ■

**Lemma 3.** *The matrix  $\bar{A}_1 = (1 - \alpha)A_1 \in \mathbb{R}^{n_1 \times n_1}$  for  $0 < \alpha < 1$  is asymptotically stable if and only if the matrix  $A_1$  is asymptotically stable.*

*Proof.* The eigenvalues  $\bar{\lambda}_k, k = 1, \dots, n_1$  of the matrix  $\bar{A}_1$  are related with the eigenvalues  $\lambda_k, k = 1, \dots, n_1$ , of the matrix  $A_1$  by

$$\bar{\lambda}_k = (1 - \alpha)\lambda_k, \quad k = 1, \dots, n_1, \quad (20)$$

since the characteristic polynomials of the matrices are related by the equality

$$\begin{aligned} \det[I_{n_1}\bar{\lambda}_k - \bar{A}_1] &= \det[I_{n_1}\bar{\lambda}_k - (1 - \alpha)A_1] \\ &= (1 - \alpha)^{n_1} \det \left[ I_{n_1} \frac{\bar{\lambda}_k}{1 - \alpha} - A_1 \right] \\ &= (1 - \alpha)^{n_1} \det[I_n\lambda_k - A_1]. \end{aligned} \quad (21)$$

Therefore, from (20) it follows that  $\text{Re } \bar{\lambda}_k < 0, k = 1, \dots, n_1$ , if and only if  $\text{Re } \lambda_k < 0, k = 1, \dots, n_1$ . ■

**Lemma 4.** *The matrix*

$$\hat{A}_1 = \alpha[I_{n_1} - (1 - \alpha)A_1]^{-1}A_1 \in M_{n_1} \quad (22)$$

*is asymptotically stable if and only if the eigenvalues  $\lambda_k = -\alpha_k + j\beta_k, k = 1, \dots, n_1$ , of the matrix  $A_1$  satisfy the condition*

$$[1 + (1 - \alpha)\alpha_k]\alpha_k + (1 - \alpha)\beta_k^2 = n(k) > 0.$$

*Proof.* From (22) for  $\hat{\lambda}_k = -\hat{\alpha}_k + j\hat{\beta}_k$  and  $\lambda_k = -\alpha_k + j\beta_k, k = 1, \dots, n_1$ , we have

$$\begin{aligned} \hat{\lambda}_k &= -\hat{\alpha}_k + j\hat{\beta}_k = \alpha[1 - (1 - \alpha)\lambda_k]^{-1}\lambda_k \\ &= \alpha[1 - (1 - \alpha)(-\alpha_k + j\beta_k)]^{-1}(-\alpha_k + j\beta_k) \\ &= \alpha \frac{1 + (1 - \alpha)\alpha_k + j(1 - \alpha)\beta_k}{[1 + (1 - \alpha)\alpha_k]^2 + [(1 - \alpha)\beta_k]^2} (-\alpha_k + j\beta_k) \\ &= \alpha \left( \frac{-[1 + (1 - \alpha)\alpha_k]\alpha_k - (1 - \alpha)\beta_k^2}{[1 + (1 - \alpha)\alpha_k]^2 + [(1 - \alpha)\beta_k]^2} \right. \\ &\quad \left. + j \frac{[1 + (1 - \alpha)\alpha_k]\beta_k - (1 - \alpha)\alpha_k\beta_k}{[1 + (1 - \alpha)\alpha_k]^2 + [(1 - \alpha)\beta_k]^2} \right) \end{aligned} \quad (23)$$

and

$$\begin{aligned} \hat{\alpha}_k &= \alpha \left( \frac{[1 + (1 - \alpha)\alpha_k]\alpha_k + (1 - \alpha)\beta_k^2}{[1 + (1 - \alpha)\alpha_k]^2 + [(1 - \alpha)\beta_k]^2} \right) \\ &= \alpha \frac{n(k)}{d(k)}, \quad k = 1, \dots, n. \end{aligned} \quad (24)$$

From (24) it follows that  $\hat{\alpha}_k > 0, k = 1, \dots, n_1$ , if and only if  $n(k) > 0, k = 1, \dots, n_1$ . ■

**Lemma 5.** *The matrices*

$$\begin{aligned} \hat{A}_1 &= \alpha[I_{n_1} - (1 - \alpha)A_1]^{-1}A_1 \in M_{n_1}, \\ \hat{B} &= [I_{n_1} - (1 - \alpha)A_1]^{-1}(1 - \alpha)B_1 \in \mathbb{R}_+^{n_1 \times m} \end{aligned} \quad (25)$$

*if  $A_1 \in M_{n_1}$  is asymptotically stable and  $B_1 \in \mathbb{R}_+^{n_1 \times m}$ .*

*Proof.* The matrix  $[I_{n_1} - (1 - \alpha)A_1]^{-1} \in \mathbb{R}_+^{n_1 \times n_1}$  if the matrix  $A_1 \in M_{n_1}$  is asymptotically stable (Kaczorek, 2001). Therefore, by Lemma 3 and  $(1 - \alpha)B_1 \in \mathbb{R}_+^{n_1 \times m}$  for  $0 < \alpha < 1$ , (24) holds if  $A_1 \in M_{n_1}$  is asymptotically stable. ■

From Lemma 5 and Theorem 3, we have the following.

**Theorem 4.** *Let the decomposition (4) of the system (1) be possible for a monomial matrix  $Q \in \mathbb{R}_+^{n \times n}$ . The subsystem (6a) for  $0 < \alpha < 1$  is positive if  $A_1 \in M_{n_1}$  is asymptotically stable and  $B_1 \in \mathbb{R}_+^{n_1 \times m}$ .*

From Theorem 4, we have stronger restrictions for the positivity of the subsystem (6a) described by the Caputo–Fabrizio derivative. In contrast to the standard Caputo derivative, the matrix  $A_1$  must be asymptotically stable. The subsystem (6b) also has an additional condition which is proved in the following theorem.

**Theorem 5.** *Let the decomposition (4) of the system (1) be possible for a monomial matrix  $Q \in \mathbb{R}_+^{n \times n}$ . The subsystem (6b) for  $0 < \alpha < 1$  is positive if and only if*

$$-B_2 \in \mathbb{R}_+^{n_2 \times m}, \quad v_{20} = \tilde{N}\bar{x}_{20} + \tilde{B}_2u_0 \in \mathbb{R}_+^{n_2}. \quad (26)$$

*Proof.*

(Sufficiency) Observe that the matrix  $[N - I_{n_2}(1 - \alpha)]$  is asymptotically stable and  $-\hat{N} \in \mathbb{R}_+^{n_2 \times n_2}$  (Kaczorek, 2001). If  $v_{20} \in \mathbb{R}_+^{n_2}$ , then  $e^{\hat{N}t}v_{20} \in \mathbb{R}_+^{n_2}$ . By assumption,  $u(\tau) + \dot{u}(\tau) \in \mathbb{R}_+^{n_2}, t \geq 0$  and  $\beta u(\tau) + \dot{u}(\tau) \in \mathbb{R}_+^{n_2}$  since  $\beta > 0$  and  $e^{\hat{N}(t-\tau)}\tilde{B}_2 \in \mathbb{R}_+^{n_2}, t \geq 0$ , since  $\tilde{B}_2 \in \mathbb{R}_+^{n_2}$ . Therefore,

$$\int_0^t e^{\hat{N}(t-\tau)}\tilde{B}_2[\beta u(\tau) + \dot{u}(\tau)] d\tau \in \mathbb{R}_+^{n_2}, \quad t \geq 0.$$

(Necessity) The proof of necessity is based on Eqn. (6b). To simplify the notation, it is assumed that the matrix  $N$  has the form

$$N = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{n_2 \times n_2}. \quad (27)$$

From (6b) and (27), we have

$$\begin{aligned} & \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \frac{d^\alpha}{dt^\alpha} \begin{bmatrix} \bar{x}_{21}(t) \\ \bar{x}_{22}(t) \\ \vdots \\ \bar{x}_{2n_2}(t) \end{bmatrix} \\ &= \begin{bmatrix} \bar{x}_{21}(t) \\ \bar{x}_{22}(t) \\ \vdots \\ \bar{x}_{2n_2}(t) \end{bmatrix} + \begin{bmatrix} B_{21} \\ B_{22} \\ \vdots \\ B_{2n_2} \end{bmatrix} u(t) \end{aligned} \quad (28)$$

and

$$\begin{aligned} \bar{x}_{2n_2}(t) &= -B_{2n_2}u(t), \\ \bar{x}_{2n_2-1}(t) &= \frac{d^\alpha \bar{x}_{2n_2}(t)}{dt^\alpha} - B_{2n_2-1}u(t), \\ &\vdots \\ \bar{x}_{21}(t) &= \frac{d^\alpha \bar{x}_{22}(t)}{dt^\alpha} - B_{21}u(t). \end{aligned} \quad (29)$$

Assuming

$$\frac{d^\alpha \bar{x}_{2n_2}}{dt^\alpha} \geq 0,$$

the subsystem (6b) is positive if and only if the conditions (26) are satisfied. ■

The considerations can be easily extended to the case when the matrix  $N$  in (6b) has the form

$$N = \text{blockdiag}[ N_1 \ \cdots \ N_q ], \quad q > 1 \quad (30)$$

and  $N_k$  for  $k = 1, 2, \dots, q$  has the form (27).

**Theorem 6.** *Let the decomposition (4) of the system (1) be possible for a monomial matrix  $Q \in \mathbb{R}_+^{n \times n}$ . The system (1) for  $0 < \alpha < 1$  is positive if and only if*

- (i) *the conditions of Theorem 4 and (26) are satisfied,*
- (ii)  *$C \in \mathbb{R}_+^{p \times n}$ .*

*Proof.* By Theorems 3–5 the solutions of Eqns. (6a) and (6b) are positive if and only if the conditions of Theorem 4 and (26) are met. From (1b) and (5), we have

$$y(t) = CQQ^{-1}x(t) = \bar{C}x(t), \quad (31)$$

where  $\bar{C} = CQ$ .

For a monomial matrix  $Q \in \mathbb{R}_+^{n \times n}$ , we have

$$\bar{C} \in \mathbb{R}_+^{p \times n} \text{ if and only if } C \in \mathbb{R}_+^{p \times n} \quad (32)$$

and

$$y(t) \in \mathbb{R}_+^p \text{ if and only if } C \in \mathbb{R}_+^{p \times n}. \quad (33)$$

### 4. Stability of positive systems

Consider the positive fractional descriptor continuous-time linear system (1) with  $u(t) = 0$ . Note that  $\bar{x}_2 = 0$  and the stability of the positive system (1) depends only on the stability of the subsystem (6a) described by the equation

$$\frac{d^\alpha \bar{x}_1(t)}{dt^\alpha} = A_1 \bar{x}_1(t), \quad \bar{x}_1(t) \in \mathbb{R}_+^{n_1}, \quad A_1 \in M_{n_1}. \quad (34)$$

**Definition 3.** The positive fractional descriptor continuous-time linear system (1) is called *asymptotically stable* if

$$\lim_{t \rightarrow \infty} \bar{x}_1(t) = 0 \quad \text{for all } \bar{x}_{10} \in \mathbb{R}_+^{n_1}. \quad (35)$$

**Theorem 7.** *The positive fractional system (34) is (asymptotically) stable if and only if one of the following equivalent conditions is satisfied (the matrix  $\hat{A}_1$  is defined by (8b)):*

- (i) *All coefficients of the polynomial*

$$\begin{aligned} & \det[I_{n_1}s - \hat{A}_1] \\ &= s^n + \hat{a}_{n-1}s^{n-1} + \cdots + \hat{a}_1s + \hat{a}_0 \end{aligned} \quad (36)$$

*are positive, i.e.,  $\hat{a}_k > 0$  for  $k = 0, 1, \dots, n_1 - 1$ .*

- (ii) *All principal minors  $M_k, k = 1, \dots, n_1$  of the matrix  $-\hat{A}_1$  are positive, i.e.,*

$$\begin{aligned} M_1 &= |-\hat{a}_{11}| > 0, \\ M_2 &= \begin{vmatrix} -\hat{a}_{11} & -\hat{a}_{12} \\ -\hat{a}_{21} & -\hat{a}_{22} \end{vmatrix} > 0, \\ &\vdots \\ M_n &= \det[-\hat{A}_1] > 0. \end{aligned} \quad (37)$$

- (iii) *The diagonal entries of the matrices*

$$\hat{A}_{1,n-k}^{(k)} \quad \text{for } k = 1, \dots, n_1 - 1 \quad (38a)$$

*are negative, where  $\hat{A}_{1,n-k}^{(k)}$  are defined as follows:*

$$\begin{aligned} \hat{A}_{1,n}^{(0)} &= \hat{A}_1 \\ &= \begin{bmatrix} \hat{a}_{11}^{(0)} & \cdots & \hat{a}_{1,n}^{(0)} \\ \vdots & \ddots & \vdots \\ \hat{a}_{n,1}^{(0)} & \cdots & \hat{a}_{n,n}^{(0)} \end{bmatrix} \\ &= \begin{bmatrix} \hat{a}_{11}^{(0)} & \hat{b}_{n-1}^{(0)} \\ \hat{c}_{n-1}^{(0)} & \hat{A}_{1,n-1}^{(0)} \end{bmatrix}, \end{aligned}$$

$$\hat{A}_{1,n-1}^{(0)} = \begin{bmatrix} \hat{a}_{22}^{(0)} & \cdots & \hat{a}_{2,n}^{(0)} \\ \vdots & \ddots & \vdots \\ \hat{a}_{n,2}^{(0)} & \cdots & \hat{a}_{n,n}^{(0)} \end{bmatrix}, \quad (38b)$$

$$\hat{b}_{n-1}^{(0)} = [\hat{a}_{12}^{(0)} \quad \cdots \quad \hat{a}_{1,n}^{(0)}],$$

$$\hat{c}_{n-1}^{(0)} = \begin{bmatrix} \hat{a}_{21}^{(0)} \\ \vdots \\ \hat{a}_{n,1}^{(0)} \end{bmatrix}$$

and

$$\hat{A}_{1,n-k}^{(k)} = \hat{A}_{1,n-k}^{(k-1)} - \frac{\hat{c}_{n-k}^{(k-1)} \hat{b}_{n-k}^{(k-1)}}{\hat{a}_{k+1,k+1}^{(k-1)}}$$

$$= \begin{bmatrix} \hat{a}_{k+1,k+1}^{(k)} & \cdots & \hat{a}_{k+1,n}^{(0)} \\ \vdots & \ddots & \vdots \\ \hat{a}_{n,k+1}^{(0)} & \cdots & \hat{a}_{n,n}^{(0)} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{a}_{k+1,k+1}^{(k)} & \hat{b}_{n-k-1}^{(k)} \\ \hat{c}_{n-k-1}^{(k)} & \hat{A}_{n-k-1}^{(k)} \end{bmatrix},$$

$$\hat{A}_{1,n-k-1}^{(k)} = \begin{bmatrix} \hat{a}_{k+2,k+2}^{(k)} & \cdots & \hat{a}_{k+2,n}^{(0)} \\ \vdots & \ddots & \vdots \\ \hat{a}_{n,k+2}^{(0)} & \cdots & \hat{a}_{n,n}^{(0)} \end{bmatrix},$$

$$\hat{b}_{n-k-1}^{(0)} = [\hat{a}_{k+1,k+2}^{(k)} \quad \cdots \quad \hat{a}_{k+1,n}^{(k)}],$$

$$\hat{c}_{n-k-1}^{(k)} = \begin{bmatrix} \hat{a}_{k+2,k+1}^{(k)} \\ \vdots \\ \hat{a}_{n,k+1}^{(k)} \end{bmatrix}, \quad (38c)$$

for  $k = 1, \dots, n_1 - 1$ .

(iv) All diagonal entries of the upper (lower) triangular matrix

$$\tilde{A}_{1,u} = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1,n} \\ 0 & \tilde{a}_{22} & \cdots & \tilde{a}_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{a}_{n,n} \end{bmatrix}, \quad (39)$$

$$\tilde{A}_{1,l} = \begin{bmatrix} \tilde{a}_{11} & 0 & \cdots & 0 \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n,1} & \tilde{a}_{n,2} & \cdots & \tilde{a}_{n,n} \end{bmatrix}$$

are negative, i.e.,

$$\tilde{a}_{kk} < 0 \quad \text{for } k = 1, \dots, n_1,$$

and matrices  $\tilde{A}_1$  have been obtained from the matrix  $\hat{A}_1$  with the use of an elementary row operation.

(v) There exists a strictly positive vector

$$\lambda = [\lambda_1 \quad \cdots \quad \lambda_n]^T,$$

$$\lambda_k > 0, \quad k = 1, \dots, n_1$$

such that

$$\hat{A}_1 \lambda < 0. \quad (40)$$

Proof. Substituting in (8)  $u(t) = 0, t \geq 0$ , we obtain the solution of Eqn. (34) in the form

$$\bar{x}_1(t) = e^{\hat{A}_1 t} \bar{x}_{10}. \quad (41)$$

The system (34) is stable if and only if

$$\lim_{t \rightarrow \infty} e^{\hat{A}_1 t} = 0 \quad \text{for all } \bar{x}_{10} \in \mathbb{R}_+^n. \quad (42)$$

The condition (42) is satisfied if and only if  $\hat{A}_1 \in M_{n_1}$ . In the work of Kaczorek (2001), it is shown that the system (34) with  $\hat{A}_1 \in M_{n_1}$  is asymptotically stable if and only if one of the conditions (1)–(4) is satisfied. If the system is asymptotically stable then from the condition one we have  $\hat{a}_0 = \det[-\hat{A}_1] > 0$  and  $-\hat{A}_1^{-1} \in \mathbb{R}_+^{n \times n}$  (Kaczorek, 2001). Then, using (40), we obtain  $(-\hat{A}_1^{-1})(-\hat{A}_1)\lambda > 0$  and  $\lambda > 0$  if and only if the system is asymptotically stable. ■

### 5. Numerical example

Consider the fractional descriptor continuous-time system described by Eqn. (1a) for  $\alpha = 0.5$  and

$$E = \begin{bmatrix} -0.4 & 0 & -0.5 & 0 \\ -0.2 & 0 & 0 & 0 \\ 0.4 & 1 & 0.5 & 0 \\ 0.2 & 0 & 0 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} -0.2 & 1.8 & 0.5 & 0 \\ 0.4 & 0.4 & 0 & 0 \\ 0.2 & -1.8 & -0.5 & 0.5 \\ -0.4 & 0.6 & 0 & 0 \end{bmatrix}, \quad (43)$$

$$B = \begin{bmatrix} -1 & -3.6 \\ 0 & -0.8 \\ -1 & 2.6 \\ 0 & -0.2 \end{bmatrix},$$

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} 1(t) \\ \sin(t) + 1(t) \end{bmatrix},$$

$$1(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geq 0. \end{cases}$$

The pencil is regular since

$$\det[E\lambda - A] = -0.05(\lambda + 1)(\lambda + 2) \neq 0. \quad (44)$$

In this case,

$$P = \begin{bmatrix} -1 & 3 & 0 & 1 \\ 0 & -3 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad (45)$$

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} = PEQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} = PAQ = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (46)$$

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ -2 & -1 \\ 0 & -1 \end{bmatrix}.$$

The matrix

$$\begin{aligned} \hat{A}_1 &= \alpha[I_2 - (1 - \alpha)A_1]^{-1}A_1 \\ &= \begin{bmatrix} 1.5 & -0.5 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -0.5 & 0.5 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -0.3333 & 0.1667 \\ 0 & -0.5 \end{bmatrix} \end{aligned} \quad (47)$$

is an asymptotically stable Metzler matrix since its eigenvalues are  $\lambda_1 = -0.3333, \lambda_2 = -0.5$ .

Note that the matrix

$$\begin{aligned} \hat{B}_1 &= [I_1 - (1 - \alpha)A_1]^{-1}(1 - \alpha)B_1 \\ &= \begin{bmatrix} 1.5 & -0.5 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 & 0.5 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.3333 & 0.5 \\ 0 & 0.5 \end{bmatrix} \end{aligned} \quad (48)$$

has positive entries.

Using the Sylvester theorem, we may find the matrix

$$\begin{aligned} e^{\hat{A}_1 t} &= Z_1 e^{\lambda_1 t} + Z_2 e^{\lambda_2 t}, \\ Z_1 &= \frac{\hat{A} - I_2 \lambda_2}{\lambda_1 - \lambda_2}, \quad Z_2 = \frac{\hat{A} - I_2 \lambda_1}{\lambda_2 - \lambda_1}. \end{aligned} \quad (49)$$

From (47) and (49), we obtain

$$\begin{aligned} e^{\hat{A}_1 t} &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} e^{-0.3333t} + \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} e^{-0.5t} \\ &= \begin{bmatrix} e^{-0.3333t} & e^{-0.3333t} - e^{-0.5t} \\ 0 & e^{-0.5t} \end{bmatrix}. \end{aligned} \quad (50)$$

Using (8a) with (43), (47)–(50) and

$$\bar{x}_{10} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

we can find the desired solution of the subsystem (6a):

$$\bar{x}_1(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix}, \quad (51)$$

where

$$\begin{aligned} \xi_1(t) &= -0.7333e^{-0.3333t} - 0.2e^{-0.5t} - 0.4 \cos(t) \\ &\quad + 0.6 \sin(t) + 3, \\ \xi_2(t) &= 0.2e^{-0.5t} - 0.2 \cos(t) + 0.6 \sin(t) + 1. \end{aligned}$$

The solution (51) of the subsystem (6a) is shown in Fig. 1.

The matrices  $\tilde{B}_2, \hat{N}$  and  $\tilde{N}$  have the form

$$\begin{aligned} \tilde{B}_2 &= [N - I_2(1 - \alpha)]^{-1}(1 - \alpha)B_2 \\ &= \begin{bmatrix} -0.5 & 1 \\ 0 & -0.5 \end{bmatrix}^{-1} \begin{bmatrix} -1 & -0.5 \\ 0 & -0.5 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}, \\ \hat{N} &= \alpha[N - I_2(1 - \alpha)]^{-1} = 0.5 \begin{bmatrix} -0.5 & 1 \\ 0 & -0.5 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}, \\ \tilde{N} &= [N - I_2(1 - \alpha)]^{-1}N \\ &= \begin{bmatrix} -0.5 & 1 \\ 0 & -0.5 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}. \end{aligned} \quad (52)$$

From (52), we have  $\tilde{B}_2 \in \mathbb{R}_+^{n_2 \times m}$  since  $-B_2 \in \mathbb{R}_+^{n_2 \times m}$  and  $-\hat{N} \in \mathbb{R}_+^{n_2 \times n_2}$ . We can find the matrix  $e^{\hat{N}t}$  using the inverse Laplace transform,

$$\begin{aligned} e^{\hat{N}t} &= \mathcal{L}^{-1}\{[I_{n_2}s - \hat{N}]^{-1}\} \\ &= \mathcal{L}^{-1} \begin{bmatrix} \frac{1}{s+1} & \frac{-2}{(s+1)^2} \\ 0 & \frac{1}{s+1} \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} & -2te^{-t} \\ 0 & e^{-t} \end{bmatrix}. \end{aligned} \quad (53)$$

Using (9a) with (43), (52), (53) and

$$\bar{x}_{20} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

we can find the desired solution of the subsystem (6b):

$$\bar{x}_2(t) = \begin{bmatrix} -e^{-t} + \cos(t) + 2 \sin(t) + 3 \\ \sin(t) + 1 \end{bmatrix}. \quad (54)$$

The solution (54) of the subsystem (6b) is shown in Fig. 2.

The fractional descriptor system (43) is positive since the matrix  $Q$  defined by (45) is monomial and the conditions of Theorem 6 are satisfied. The system (43) is also asymptotically stable since the matrix  $\hat{A}_1$  is a Metzler matrix with eigenvalues  $\lambda_1 = -0.3333, \lambda_2 = -0.5$ .

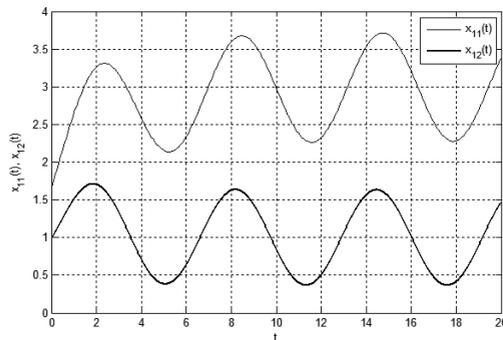


Fig. 1. State vector of the subsystem (6a).

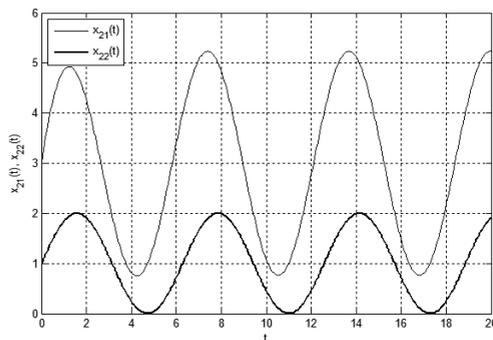


Fig. 2. State vector of the subsystem (6b).

## 6. Concluding remarks

The Weierstrass–Kronecker theorem on the decomposition of the regular pencil was extended to fractional descriptor continuous-time linear systems described by the Caputo–Fabrizio derivative. The solution to the state equation was given (Theorems 1 and 2). Necessary and sufficient conditions for the positivity (Theorems 3, 4 and 5) and stability of the systems were established. Tests for checking the asymptotic stability of the systems (Theorem 7) were also presented. The discussion was illustrated with a numerical example.

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