Int. J. Appl. Math. Comput. Sci., 2021, Vol. 31, No. 2, 219–232
DOI: 10.34768/amcs-2021-0015

MITTAG-LEFFLER STABILITY FOR A TIMOSHENKO PROBLEM

NASSER-EDDINE TATAR [a]

[a] Department of Mathematics and Statistics
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia
e-mail: tatarn@kfupm.edu.sa

A Timoshenko system of a fractional order between zero and one is investigated here. Using a fractional version of resolvents, we establish an existence and uniqueness theorem in an appropriate space. Moreover, it is proved that lower order fractional terms (in the rotation component) are capable of stabilizing the system in a Mittag-Leffler fashion. Therefore, they deserve to be called damping terms. This is shown through the introduction of some new functionals and some fractional inequalities, and the establishment of some properties, involving fractional derivatives. In the case of different wave speeds of propagation we obtain convergence to zero.

Keywords: Caputo fractional derivative, Mittag-Leffler stability, multiplier technique, resolvent operator.

1. Introduction

Random diffusion of microscopic particles has been observed in a large number of processes. It varies from localized diffusion (normal diffusion) all the way to ballistic diffusion passing by sub-diffusion and super-diffusion. The classification depends on an exponent in the expression of the mean square displacement. The exponent one corresponds to ordinary diffusion, whereas the exponent two is for ballistic diffusion. The intermediary exponents correspond to fractional derivatives. Therefore, in complex media, it is appropriate to be in the fractional calculus context.

The wave equation with a frictional damping

$$\rho\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}=a\frac{\partial^2 u}{\partial x^2},$$

commonly known as the telegraph equation, has been generalized to the fractional case

$$\rho\frac{\partial^\alpha u}{\partial t^\alpha}+\frac{\partial^\beta u}{\partial t^\beta}=a\frac{\partial^2 u}{\partial x^2}$$

by Atanackovic *et al*. (2007). The authors treated the Cauchy problem and the signalling problem for unbounded and bounded spatial domains. The problems for $0<\beta\leq\alpha\leq 2$, $\alpha>1$ (as well as $0\leq\beta\leq\alpha\leq 1$), are solved and their solutions are expressed in the forms of series and integral representations. To this end, the authors used separation of variables and Laplace transforms. We refer the readers to the references there in for the problem without the lower-order fractional term.

Orsingher and Beghin (2004), as well as Beghin and Orsingher (2003), obtained the Fourier transforms of the fundamental solutions. The Adomian decomposition is used by Momani (2005) to derive analytic and approximate solutions. In the work of Chen *et al*. (2008), the separation-of-variables method is utilized to find analytical solutions for Dirichlet, Neumann and Robin type boundary conditions. We refer also to Anderson *et al*. (2018). In the survey paper by Klamka *et al*. (2020), the authors presented the most important results on the controllability and stability of some fractional differential equations of an order between zero and one (see also the work of Kaczorek (2020) and the references therein). The importance of these questions and the difficulties encountered in their treatments are highlighted. Moreover, the main tools and techniques are mentioned. The authors concluded that, while the linear case is clear, the nonlinear case is still open. In the presence of delays, the situation is even more involved.

In this paper we are concerned with the fractional problem

$$\begin{cases}\rho_1 D^\alpha\left(D^\alpha\varphi\right) - k(\varphi_x+\psi)_x = 0,\ 0<x<1,\ t>0,\\ \rho_2 D^\alpha\left(D^\alpha\psi + a\psi\right) - b\psi_{xx} + k(\varphi_x+\psi) = 0\end{cases} \tag{1}$$

with the boundary and initial conditions

$$\begin{cases}\varphi(0,t)=\varphi(1,t)=0,\ \psi(0,t)=\psi(1,t)=0,\ t>0,\\ \varphi(x,0)=\varphi_0(x),\ \psi(x,0)=\psi_0(x),\ x\in(0,1)\end{cases} \tag{2}$$

where $1/2 < \alpha < 1$ and D^α is the (time) Caputo fractional derivative defined below.

This is the generalization of the well-known Timoshenko system

$$\begin{cases}\rho_1\varphi_{tt} - k(\varphi_x+\psi)_x = 0,\ 0<x<1,\ t>0,\\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x+\psi) = 0,\ 0<x<1,\ t>0,\end{cases}$$

where φ denotes the transverse displacement from its equilibrium and ψ is the rotation of the beam. The constants ρ_1, ρ_2, k and b are known positive parameters. This system enjoys the property of dealing with the cases where the transverse shear strain is important.

By its nature, this system is conservative. Raposo *et al*. (2005) or Rivera and Racke (2008) stabilized the system by adding two velocity feedback terms, one in each equation. Then, in the work of Soufyane and Whebe (2003), it was realized that only one damping in the second equation (ψ_t) is enough to stabilize the system uniformly in the case of equal speeds of propagation and asymptotically otherwise (see also Ammar-Khodja *et al*., 2007). These two results were improved by Rivera *et al*. (2002) for exponential stability and polynomial stability, respectively. Nonlinear damping was also treated in several papers (Alabau-Boussouira, 2007; Cavalcanti *et al*., 2014; Mustafa and Messaoudi, 2010). The case of frictional damping in the first component (with Neumann boundary conditions) was discussed by Almeida Júnior *et al*. (2013). Rotating beams were also studied by many authors. For instance, Sklyar and Szkibiel (2013) discussed the (integer-order) Timoshenko beam clamped to a rotating disk. They determined an appropriate control with the aid of the torque using the angular acceleration of the disk to stabilize the system.

To keep the length of the paper reasonable, we do not report here on other ways of stabilizing the system such as viscoelastic, thermal and boundary controls. Fractional models are found to be suitable for the description of propagation in complex environments such as viscoelastic media (Mainardi, 2010; Sandev and Tomovski, 2019). They also describe adequately the vibrations of beams and cables manufactured of smart materials. Moreover, they are often used to model diffusion in anomalous media, having in mind that the classical Fick law is unable to deal with such cases.

The mean square displacement is not linear but rather nonlinear. This results in a non-local stress-strain constitutive relationship. Incidentally, this approach provides better models which avoid the unpleasant infinite speed of propagation in conventional diffusion. To the best of the author's knowledge, such a system has not been explored in the literature yet. In this work, we first discuss how we can prove the existence and uniqueness of a solution in a suitable underlying space. In addition, we investigate the question of stability of the above system. From our results, it will appear that the lower-order fractional term $a\rho_2 D^\alpha\psi$ by itself is capable of stabilizing the system. The rate of stability is shown to be of the Mittag-Leffler type in the case of equal speeds of propagation. That is, the energy of the system decays to equilibrium as a Mittag-Leffler function (of a negative argument). This rate is the equivalent to the exponential rate in the (second) integer-order case. This is established by replacing first order derivatives in the standard functionals by fractional ones and with the help of some properties of fractional derivatives ('fractional chain rule' and fractional inequalities). Otherwise, if the wave equations do not propagate at the same speed, we prove a kind of Lyapunov asymptotic stability by employing a generalized Barbalat theorem.

In the next section, we provide some preliminaries. Section 3 is devoted to the existence and uniqueness issue. Then we introduce our functionals and state (with proofs) some lemmas. Stability is treated in Section 4.

2. Definitions and propositions

This section contains some definitions and useful propositions.

Definition 1. (*Gorenflo* et al.*, 2014; Podlubny, 1999*) The Riemann–Liouville fractional integral of order $\gamma > 0$ is given by

$$I^\gamma\chi(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\chi(s)\,\mathrm{d}s, \quad \gamma>0$$

for any measurable function χ provided that the right-hand side exists. Here $\Gamma(\gamma)$ denotes the Gamma function.

Definition 2. (*Gorenflo* et al.*, 2014; Podlubny, 1999*) The fractional derivative of order γ in the sense of Caputo is given by

$$D^\gamma\chi(t) = \frac{1}{\Gamma(1-\gamma)}\int_0^t (t-s)^{-\gamma}\chi'(s)\,\mathrm{d}s,$$

$$0<\gamma<1,$$

$$D^{\gamma}\chi(t) = \frac{1}{\Gamma(2-\gamma)}\int_0^t (t-s)^{1-\gamma}\chi''(s)\,\mathrm{d}s, \qquad 1<\gamma<2,$$

whereas the one in the sense of Riemann–Liouville is defined by

$$^{RL}D^{\gamma}\chi(t) = \frac{1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-\gamma}\chi(s)\,\mathrm{d}s, \qquad 0<\gamma<1$$

provided that the integrals exist.

The Riemann–Liouville fractional derivative is defined by

$$D^{\gamma}\chi(t) = \frac{1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-\gamma}\chi(s)\,\mathrm{d}s, \qquad 0<\gamma<1$$

and is related to the Caputo fractional derivative by the formula

$$^{RL}D^{\gamma}\chi(t) = \frac{\chi(0)t^{-\gamma}}{\Gamma(1-\gamma)} + D^{\gamma}\chi(t), \qquad 0<\gamma<1, \quad t>0.$$

Next, we give the definitions of one-parametric and two-parametric Mittag-Leffler functions (Gorenflo *et al.*, 2014),

$$E_{\alpha}(z) := \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, \qquad \mathbf{Re}(\alpha)>0,$$

and

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}, \qquad \mathbf{Re}(\alpha)>0, \quad \mathbf{Re}(\beta)>0,$$

respectively. Note that $E_{\alpha,1}(z) \equiv E_{\alpha}(z)$.

Proposition 1. (Li *et al.*, 2010) *If $\chi(t)$ is a differentiable function satisfying*

$$D^{\alpha}\chi(t) \le -\gamma\chi(t), \quad 0<\alpha<1$$

for some $\gamma>0$, then $\chi(t)\le\chi(0)E_{\alpha}(-\gamma t^{\alpha})$, $t\ge 0$. If the derivative is of Riemann–Liouville type, then the decay is of the form $t^{\alpha-1}E_{\alpha,\alpha}(-\gamma t^{\alpha})$.

Proposition 2. (Gorenflo *et al.*, 2014) *For $\alpha,\beta>0$, we have the identity*

$$\lambda z^{\alpha}E_{\alpha,\alpha+\beta}(\lambda z^{\alpha}) = E_{\alpha,\beta}(\lambda z^{\alpha}) - \frac{1}{\Gamma(\beta)}.$$

Proposition 3. (Gorenflo *et al.*, 2014, p. 61) *For $\mu,\alpha,\beta>0$, we have the relation*

$$\frac{1}{\Gamma(\mu)}\int_0^z (z-t)^{\mu-1}E_{\alpha,\beta}(\lambda t^{\alpha})t^{\beta-1}\,\mathrm{d}t = z^{\mu+\beta-1}E_{\alpha,\mu+\beta}(\lambda z^{\alpha}), \quad z>0.$$

Unfortunately, the chain rule does not hold in the fractional case. We shall manage to work instead with the following rule.

Proposition 4. (Alikhanov, 2012) *Let $f(t)$ and $g(t)$ be absolutely continuous functions. Then, for $t\ge 0$ and $0<\alpha<1$, we have*

$$\begin{aligned} &f(t)D^{\alpha}g(t) + g(t)D^{\alpha}f(t) \\ &= D^{\alpha}(fg(t)) \\ &+ \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}}\int_0^{\xi}\frac{f'(\eta)\,\mathrm{d}\eta}{(t-\eta)^{\alpha}}\int_0^{\xi}\frac{g'(s)\,\mathrm{d}s}{(t-s)^{\alpha}}. \end{aligned}$$

If $f=g$, we recover the well-known inequality

$$D^{\alpha}(f^2(t)) \le 2f(t)D^{\alpha}f(t).$$

A vector version is also valid.

3. Existence and uniqueness

Without loss of generality, we may assume that $a\rho_2=1$. Let $\mathcal{U}=(\varphi,\tilde{\varphi},\psi,\tilde{\psi})^T$, $\mathcal{U}_0=(\varphi_0,\tilde{\varphi}_0,\psi_0,\tilde{\psi}_0)^T$. Then the system (1) can be written as

$$D^{\alpha}\mathcal{U} = M\mathcal{U}, \quad \mathcal{U}(t)=\mathcal{U}_0, \tag{3}$$

where

$$M = \begin{pmatrix} 0 & I_d & 0 & 0 \\ \dfrac{k\partial_x^2}{\rho_1} & 0 & \dfrac{k\partial_x}{\rho_1} & 0 \\ 0 & 0 & 0 & I_d \\ -\dfrac{k\partial_x}{\rho_2} & 0 & \dfrac{b\partial_x^2}{\rho_2} - \dfrac{kI_d}{\rho_2} & -\dfrac{I_d}{\rho_2} \end{pmatrix}.$$

Let

$$\mathcal{H} = H_0^1(0,1)\times L^2(0,1)\times H_0^1(0,1)\times L^2(0,1)$$

and

$$D(M) = \left\{\mathcal{U} = \left(\varphi,\tilde{\varphi},\psi,\tilde{\psi}\right)^T \in \mathcal{H} : \varphi,\psi\in H^2(0,1)\right\}.$$

Instead of semigroups, in the fractional case we employ the notion of resolvents. Let $\mathcal{L}(X) := \mathcal{L}(X;X)$

(the space of bounded linear operators from X into X) and

$$\delta_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0, \quad \alpha \geq 0 \tag{4}$$

be the Gelfand–Shilov function. For $\alpha = 0$, this function corresponds to the well-known Dirac delta function.

Consider the problem

$$D^\gamma u = \mathcal{P}u, \quad u(0) = u_0. \tag{5}$$

Definition 3. (*Bajlekova, 2001*) The family of bounded linear operators $(R_\gamma(t))_{t\geq 0}$ determines a solution operator for (5) if

(a) $R_\gamma : [0,\infty) \to \mathcal{L}(X)$ is strongly continuous and $R_\gamma(0) = I$;

(b) $\forall w \in D(\mathcal{P})$, $R_\gamma(t)w \in D(\mathcal{P})$, and $\mathcal{P}R_\gamma(t)w = R_\gamma(t)\mathcal{P}w$, $t \geq 0$;

(c) $\forall w \in D(\mathcal{P})$, $R_\gamma(t)w$ is a solution of

$$u(t) = w + \int_0^t \delta_\gamma(t-s)\mathcal{P}u(s)\,\mathrm{d}s, \quad t \geq 0.$$

It is shown by Bajlekova (2001, Thm. 3.1 and Cor. 3.2) and Prüss (2013, Thm. 2.1) that, if $\mathcal{P}$ (a linear closed densely defined operator in a Banach space X) is an infinitesimal generator of a C_0-semigroup, then it is also the generator of a unique 'subordinated resolvent' $\{R_\gamma(t)\}_{t\geq 0}$ given by

$$R_\gamma(t) := \frac{1}{2\pi i}\int_\Gamma \mu^{\gamma-1}e^{\mu t}R(\mu^\gamma,\mathcal{P})\,\mathrm{d}\mu, \quad t \geq 0,$$

where $R(\mu,\mathcal{P}) := (\mu I - \mathcal{P})^{-1}$ is the resolvent operator of $\mathcal{P}$ and

$$\begin{aligned}\Gamma* = &\{te^{i\theta} : t \geq r\} \cup \{re^{i\zeta} : -\theta \leq \zeta \leq \theta\} \\ &\cup \{te^{-i\theta} : t \geq r\}\end{aligned}$$

is oriented counterclockwise, where $\pi/2 < \theta < \phi$ and r is determined in the proof. Therefore, the problem (5), is well-posed in the space of functions $u \in C([0,\infty);X)$ satisfying $\delta_{1-\gamma} * (u - u_0) \in C^1([0,\infty);X)$ (the mild solution being also a classical one) and for $u_0 \in D(\mathcal{P})$,

$$u(t) = R_\gamma(t)u_0, \quad t \geq 0.$$

In our case, as M is the generator of a C_0-semigroup, for $\mathcal{U}_0 \in D(M)$, our problem has a unique solution

$$\mathcal{U}(t) = R_\alpha(t)\mathcal{U}_0, \quad t \geq 0 \tag{6}$$

with

$$\begin{aligned}\varphi,\psi \in C(R^+, H^2(0,1) \cap H_0^1(0,1)) \\ \cap C^1(R^+, H_0^1(0,1))\end{aligned}$$

such that $\delta_{2-2\alpha} * (\varphi - \varphi_0) \in C^2\left([0,\infty), L^2(0,1)\right)$ and

$$\delta_{2-2\alpha} * (\psi - \psi_0) \in C^2\left([0,\infty), L^2(0,1)\right).$$

It is worth noting that, by the continuity of φ_t and ψ_t nearby zero, we have $D^\alpha\varphi(0) = 0$ and $D^\alpha\psi(0) = 0$.

4. Stability results

In this section, we shall proceed in a direct manner with the adaptation of the arguments employed in the integer-order case to the present fractional one. The new functionals will be validated and, in particular, we will clarify how to deal with the undesirable terms that arise from the application of the 'fractional chain rule' property (Proposition 4 above).

Definition 4. The solution (φ,ψ) of system (1) is said to be Mittag-Leffler stable if

$$\|(\varphi,\psi)\| \leq \left[M(\varphi_0,\psi_0)E_\alpha(-\lambda(t-t_0)^\alpha)\right]^\kappa,$$

where $\kappa > 0$, $\alpha \in (0,1)$, $\lambda > 0$, $M(\varphi_0,\psi_0) \geq 0$ and $M(0) = 0$.

In the above definition the norm will be that of the underlying space defined in Section 3 or the energy functional, for $t \geq 0$,

$$\begin{aligned}E(t) = \frac{1}{2}\Big[&\rho_1\|D^\alpha\varphi\|^2 + \rho_2\|D^\alpha\psi\|^2 \\ &+ b\|\psi_x\|^2 + k\|\varphi_x + \psi\|^2\Big],\end{aligned}$$

where $\|\cdot\|$ is the L^2-norm. To shorten the notation, we will write

$$\begin{aligned}\mathfrak{T}_\chi(t) = &\frac{\alpha}{2\Gamma(1-\alpha)} \\ &\times \int_0^1\int_0^t \frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^\xi \frac{\chi'(\eta)\,\mathrm{d}\eta}{(t-\eta)^\alpha}\right)^2 \mathrm{d}x.\end{aligned}$$

Assuming that our initial data satisfy $\mathcal{U}_0 \in D(M)$, it is easy to see that all the terms $\mathfrak{T}_\varphi(t)$, $\mathfrak{T}_\psi(t)$, $\mathfrak{T}_{D^\alpha\varphi}(t)$, $\mathfrak{T}_{D^\alpha\psi}(t)$, $\mathfrak{T}_{\varphi_x+\psi}(t)$ and $\mathfrak{T}_{\psi_x}(t)$ are well-defined in the above underlying space. A straightforward fractional differentiation (using Proposition 4) shows that

$$\begin{aligned}&D^\alpha E(t) \\ &= \rho_1\int_0^1 D^\alpha\varphi D^\alpha\left(D^\alpha\varphi\right)\mathrm{d}x - \rho_1\mathfrak{T}_{D^\alpha\varphi}(t)\end{aligned}$$

$$
\begin{aligned}
&+\rho_2 \int_0^1 D^\alpha \psi D^\alpha \left(D^\alpha \psi\right) \mathrm{d}x - \rho_2 \mathfrak{T}_{D^\alpha \psi}(t) \\
&+ b \int_0^1 \psi_x \left(D^\alpha \psi_x\right) \mathrm{d}x - b \mathfrak{T}_{\psi_x}(t) \\
&+ k \int_0^1 \left(\varphi_x + \psi\right) D^\alpha \left(\varphi_x + \psi\right) \mathrm{d}x \\
&- k \mathfrak{T}_{\varphi_x + \psi}(t)
\end{aligned} \tag{7}
$$

and, along solutions of (1) and (2), we have

$$
\begin{aligned}
&\rho_1 \int_0^1 D^\alpha \varphi D^\alpha \left(D^\alpha \varphi\right) \mathrm{d}x \\
&+ \rho_2 \int_0^1 D^\alpha \psi D^\alpha \left(D^\alpha \psi\right) \mathrm{d}x \\
&= k \int_0^1 \left(D^\alpha \varphi\right) \left(\varphi_x + \psi\right)_x \mathrm{d}x + b \int_0^1 \left(D^\alpha \psi\right) \psi_{xx} \mathrm{d}x \\
&\quad - \left\|D^\alpha \psi\right\|^2 - k \int_0^1 \left(D^\alpha \psi\right) \left(\varphi_x + \psi\right) \mathrm{d}x \\
&= -k \int_0^1 (\varphi_x + \psi) D^\alpha (\varphi_x + \psi) \mathrm{d}x \\
&\quad - b \int_0^1 \left(D^\alpha \psi_x\right) \psi_x \mathrm{d}x - \left\|D^\alpha \psi\right\|^2 .
\end{aligned} \tag{8}
$$

Substitution of (8) into (7) gives

$$
\begin{aligned}
D^\alpha E(t) = &- \left\|D^\alpha \psi\right\|^2 - \rho_1 \mathfrak{T}_{D^\alpha \varphi}(t) - \rho_2 \mathfrak{T}_{D^\alpha \psi}(t) \\
&- b \mathfrak{T}_{\psi_x}(t) - k \mathfrak{T}_{\varphi_x + \psi}(t), \quad t \geq 0.
\end{aligned} \tag{9}
$$

Clearly, $D^\alpha E(t) \leq 0, t \geq 0$.

Remark 1. Unlike the integer order case, the relation $D^\alpha E(t) \leq 0$, $t \geq 0$ does not mean that the energy is non-increasing. However, we may infer a mere Lyapunov stability with the help of the next proposition.

Consider the equation

$$
D^\alpha x(t) = f(x, t)
$$

with zero as the equilibrium.

Proposition 5. (Duarte-Mermoud *et al.*, 2015) *If there exist a continuous Lyapunov function $Z(x(t), t)$ and a function $\vartheta_1(.)$ in $\mathcal{K}$ (the class of strictly increasing continuous functions $h : [0, +\infty) \to [0, +\infty)$ satisfying $h(0) = 0$) such that, for all $x \neq 0$, $\vartheta_1(\|x(t)\|) \leq Z(x(t), t)$ and $D^\alpha Z(x(t), t) \leq 0$, $0 < \alpha \leq 1$, then the equilibrium is Lyapunov stable.*

To go further, we continue with the introduction of appropriate functionals (by replacing integer-order derivatives with fractional ones in the standard functionals).

Lemma 1. *The functional*

$$
U_1(t) := -\int_0^1 \left(\rho_1 \varphi D^\alpha \varphi + \rho_2 \psi D^\alpha \psi\right) \mathrm{d}x, \quad t \geq 0,
$$

satisfies

$$
\begin{aligned}
D^\alpha U_1(t) \leq &-\rho_1 \left\|D^\alpha \varphi\right\|^2 + \left(\frac{1}{4\varepsilon_0} - \rho_2\right) \left\|D^\alpha \psi\right\|^2 \\
&+ k \left\|\varphi_x + \psi\right\|^2 + (b + \varepsilon_0) \left\|\psi_x\right\|^2 \\
&+ 2\rho_1 \mathfrak{T}_{\varphi_x + \psi}(t) + (2\rho_1 + \rho_2) \mathfrak{T}_{\psi_x}(t) \\
&+ \rho_1 \mathfrak{T}_{D^\alpha \varphi}(t) + \rho_2 \mathfrak{T}_{D^\alpha \psi}(t), \quad \varepsilon_0 > 0.
\end{aligned}
$$

Proof. In view of Proposition 4 and the equations of the system (1), we can write

$$
\begin{aligned}
&D^\alpha U_1(t) \\
&\quad = -\rho_1 \left\|D^\alpha \varphi\right\|^2 \\
&\qquad + \frac{\alpha \rho_1}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{\varphi'(\eta)\, \mathrm{d}\eta}{(t-\eta)^\alpha}\right) \\
&\qquad \times \left(\int_0^\xi \frac{\left[D^\alpha \varphi(s)\right]' \mathrm{d}s}{(t-s)^\alpha}\right) \mathrm{d}x \\
&\qquad + \frac{\alpha \rho_2}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{\psi'(\eta)\, \mathrm{d}\eta}{(t-\eta)^\alpha}\right) \\
&\qquad \times \left(\int_0^\xi \frac{\left[D^\alpha \psi(s)\right]' \mathrm{d}s}{(t-s)^\alpha}\right) \mathrm{d}x \\
&\qquad - \rho_2 \left\|D^\alpha \psi\right\|^2 + k \left\|\varphi_x + \psi\right\|^2 \\
&\qquad + \int_0^1 \psi D^\alpha \psi \, \mathrm{d}x + b \left\|\psi_x\right\|^2 .
\end{aligned}
$$

It is also easy to see that

$$
\begin{aligned}
\int_0^1 \varphi D^\alpha \varphi \, \mathrm{d}x &\leq \varepsilon_0 \left\|\varphi\right\|_2^2 + \frac{1}{4\varepsilon_0} \left\|D^\alpha \varphi\right\|^2 \\
&\leq \varepsilon_0 \left\|\varphi_x\right\|_2^2 + \frac{1}{4\varepsilon_0} \left\|D^\alpha \varphi\right\|^2 \\
&\leq 2\varepsilon_0 \left\|\varphi_x + \psi\right\|^2 + 2\varepsilon_0 \left\|\psi_x\right\|^2 \\
&\quad + \frac{1}{4\varepsilon_0} \left\|D^\alpha \varphi\right\|^2 ,
\end{aligned}
$$

$$
\int_0^1 \psi D^\alpha \psi \, \mathrm{d}x \leq \varepsilon_0 \left\|\psi_x\right\|^2 + \frac{1}{4\varepsilon_0} \left\|D^\alpha \psi\right\|^2 ,
$$

$$\frac{\alpha}{\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^\xi \frac{\varphi'(\eta)\,\mathrm{d}\eta}{(t-\eta)^\alpha}\right) \times\left(\int_0^\xi \frac{[D^\alpha\varphi(s)]'\,\mathrm{d}s}{(t-s)^\alpha}\right)\mathrm{d}x \le \mathfrak{T}_\varphi(t)+\mathfrak{T}_{D^\alpha\varphi}(t) \le 2\mathfrak{T}_{\varphi_x+\psi}(t)+2\mathfrak{T}_{\psi_x}(t)+\mathfrak{T}_{D^\alpha\varphi}(t)$$

and

$$\frac{\alpha}{\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^\xi \frac{\psi'(\eta)\,\mathrm{d}\eta}{(t-\eta)^\alpha}\right) \times\left(\int_0^\xi \frac{[D^\alpha\psi(s)]'\,\mathrm{d}s}{(t-s)^\alpha}\right)\mathrm{d}x \le \mathfrak{T}_{\psi_x}(t)+\mathfrak{T}_{D^\alpha\psi}(t).$$

These estimates imply

$$D^\alpha U_1(t) \le -\rho_1\|D^\alpha\varphi\|^2+\left(\frac{1}{4\varepsilon_0}-\rho_2\right)\|D^\alpha\psi\|^2 + k\|\varphi_x+\psi\|^2+(b+\varepsilon_0)\|\psi_x\|^2 + 2\rho_1\mathfrak{T}_{\varphi_x+\psi}(t)+(2\rho_1+\rho_2)\mathfrak{T}_{\psi_x}(t) + \rho_1\mathfrak{T}_{D^\alpha\varphi}(t)+\rho_2\mathfrak{T}_{D^\alpha\psi}(t).$$

■

Our second functional is the following.

Lemma 2. *The fractional derivative of*

$$U_2(t)=\rho_2\int_0^1 D^\alpha\psi(\varphi_x+\psi)\,\mathrm{d}x + \rho_2\int_0^1 \psi_x D^\alpha\varphi\,\mathrm{d}x,\quad t\ge 0$$

fulfills, for $t\ge 0$ and $\varepsilon_0,\varepsilon_1>0$,

$$D^\alpha U_2(t) \le (\varepsilon_0-k)\|\varphi_x+\psi\|^2+\left(\frac{1}{4\varepsilon_0}+\rho_2\right)\|D^\alpha\psi\|^2 + \varepsilon_1\left[\varphi_x^2(1)+\varphi_x^2(0)\right]+\frac{b^2}{4\varepsilon_1}\left[\psi_x^2(1)+\psi_x^2(0)\right] + \left(\frac{k\rho_2}{\rho_1}-b\right)\int_0^1(\varphi_x+\psi)_x\psi_x\,\mathrm{d}x + \rho_2\mathfrak{T}_{D^\alpha\psi}(t)+\rho_2\mathfrak{T}_{\varphi_x+\psi}(t) + \rho_2\mathfrak{T}_{\psi_x}(t)+\rho_2\mathfrak{T}_{D^\alpha\varphi}(t).$$

Proof. Proposition 4 and the equations of the system imply

$$\begin{aligned} D^\alpha U_2(t) &= \rho_2\int_0^1 D^\alpha\left(D^\alpha\psi\right)(\varphi_x+\psi)\,\mathrm{d}x + \rho_2\int_0^1 D^\alpha\psi D^\alpha(\varphi_x+\psi)\,\mathrm{d}x \\ &\quad + \rho_2\int_0^1 D^\alpha\psi_x D^\alpha\varphi\,\mathrm{d}x + \rho_2\int_0^1\psi_x D^\alpha\left(D^\alpha\varphi\right)\mathrm{d}x \\ &\quad - \frac{\alpha\rho_2}{\Gamma(1-\alpha)}\int_0^1\int_0^t\frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^\xi\frac{[D^\alpha\psi(\eta)]'\,\mathrm{d}\eta}{(t-\eta)^\alpha}\right) \times\left(\int_0^\xi\frac{(\varphi_x+\psi)'(s)\,\mathrm{d}s}{(t-s)^\alpha}\right)\mathrm{d}x \\ &\quad - \frac{\alpha\rho_2}{\Gamma(1-\alpha)}\int_0^1\int_0^t\frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^\xi\frac{[\psi_x(\eta)]'\,\mathrm{d}\eta}{(t-\eta)^\alpha}\right) \times\left(\int_0^\xi\frac{[D^\alpha\varphi(s)]'\,\mathrm{d}s}{(t-s)^\alpha}\right)\mathrm{d}x \end{aligned}$$

or

$$\begin{aligned} D^\alpha U_2(t) &= \int_0^1(\varphi_x+\psi)\Big[-D^\alpha\psi + b\psi_{xx}-k(\varphi_x+\psi)\Big]\,\mathrm{d}x \\ &\quad + \rho_2\int_0^1 D^\alpha\psi D^\alpha(\varphi_x+\psi)\,\mathrm{d}x + \rho_2\int_0^1 D^\alpha\psi_x D^\alpha\varphi\,\mathrm{d}x + \frac{k\rho_2}{\rho_1}\int_0^1\psi_x(\varphi_x+\psi)_x\,\mathrm{d}x \\ &\quad - \frac{\alpha\rho_2}{\Gamma(1-\alpha)}\int_0^1\int_0^t\frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}} \times\left(\int_0^\xi\frac{[D^\alpha\psi(\eta)]'\,\mathrm{d}\eta}{(t-\eta)^\alpha}\right)\left(\int_0^\xi\frac{(\varphi_x+\psi)'(s)\,\mathrm{d}s}{(t-s)^\alpha}\right)\mathrm{d}x \end{aligned}$$

$$- \frac{\alpha\rho_2}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \times \left(\int_0^\xi \frac{[\psi_x(\eta)]' \, d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{[D^\alpha \varphi(s)]' \, ds}{(t-s)^\alpha} \right) dx$$

and, integrating by parts, we find

$$\begin{aligned} &D^\alpha U_2(t) \\ &= -\int_0^1 (\varphi_x + \psi) D^\alpha \psi \, dx + b \left[\varphi_x \psi_x\right]_0^1 \\ &\quad - b \int_0^1 (\varphi_x + \psi)_x \psi_x \, dx \\ &\quad - k \left\|\varphi_x + \psi\right\|^2 + \rho_2 \left\|D^\alpha \psi\right\|^2 \\ &\quad - \rho_2 \int_0^1 D^\alpha \psi_x D^\alpha \varphi \, dx + \rho_2 \int_0^1 D^\alpha \psi_x D^\alpha \varphi \, dx \\ &\quad + \frac{k\rho_2}{\rho_1} \int_0^1 \psi_x (\varphi_x + \psi)_x \, dx \\ &\quad - \frac{\alpha\rho_2}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\ &\quad \times \left(\int_0^\xi \frac{[D^\alpha \psi(\eta)]' \, d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{(\varphi_x + \psi)'(s) \, ds}{(t-s)^\alpha} \right) dx \\ &\quad - \frac{\alpha\rho_2}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\ &\quad \times \left(\int_0^\xi \frac{[\psi_x(\eta)]' \, d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{[D^\alpha \varphi(s)]' \, ds}{(t-s)^\alpha} \right) dx. \end{aligned}$$

After cancelling some terms and applying the Young inequality, we get

$$\begin{aligned} D^\alpha U_2(t) \le{}& \varepsilon_0 \left\|\varphi_x + \psi\right\|^2 + \frac{1}{4\varepsilon_0} \left\|D^\alpha \psi\right\|^2 \\ &+ \varepsilon_1 \left[\varphi_x^2(1) + \varphi_x^2(0)\right] \\ &+ \frac{b^2}{4\varepsilon_1} \left[\psi_x^2(1) + \psi_x^2(0)\right] \\ &+ \left(\frac{k\rho_2}{\rho_1} - b \right) \int_0^1 (\varphi_x + \psi)_x \psi_x \, dx \\ &- k \left\|\varphi_x + \psi\right\|^2 + \rho_2 \left\|D^\alpha \psi\right\|^2 \\ &+ \rho_2 \mathfrak{T}_{D^\alpha \psi}(t) + \rho_2 \mathfrak{T}_{\varphi_x + \psi}(t) \\ &+ \rho_2 \mathfrak{T}_{\psi_x}(t) + \rho_2 \mathfrak{T}_{D^\alpha \varphi}(t) \end{aligned}$$

or simply

$$\begin{aligned} &D^\alpha U_2(t) \\ &\le (\varepsilon_0 - k) \left\|\varphi_x + \psi\right\|^2 + \left(\frac{1}{4\varepsilon_0} + \rho_2 \right) \left\|D^\alpha \psi\right\|^2 \\ &\quad + \varepsilon_1 \left[\varphi_x^2(1) + \varphi_x^2(0)\right] + \frac{b^2}{4\varepsilon_1} \left[\psi_x^2(1) + \psi_x^2(0)\right] \\ &\quad + \left(\frac{k\rho_2}{\rho_1} - b \right) \int_0^1 (\varphi_x + \psi)_x \psi_x \, dx + \rho_2 \mathfrak{T}_{D^\alpha \psi}(t) \\ &\quad + \rho_2 \mathfrak{T}_{\varphi_x + \psi}(t) + \rho_2 \mathfrak{T}_{\psi_x}(t) + \rho_2 \mathfrak{T}_{D^\alpha \varphi}(t). \end{aligned} \tag{10}$$

The proof is complete. ■

The next functional,

$$U_3(t) = \rho_2 \int_0^1 \vartheta(x) D^\alpha \psi \left(b\psi_x(t)\right) dx, \quad t \ge 0,$$

where $\vartheta(x) = 2 - 4x$, so that $\vartheta(0) = -\vartheta(1) = 2$, will incorporate the boundary terms in (10).

Lemma 3. *Along the solutions of (1) and (2), we have*

$$\begin{aligned} D^\alpha U_3(t) \le{}& -b^2 \psi_x^2(1) - b^2 \psi_x^2(0) \\ &+ \left[2b^2 + \frac{kb}{\varepsilon_2} + \frac{\varepsilon_0}{2} \right] \left\|\psi_x\right\|^2 \\ &+ \varepsilon_2 kb \left\|\varphi_x + \psi\right\|^2 + 2 \left(\frac{1}{\varepsilon_0} + b\rho_2 \right) \left\|D^\alpha \psi\right\|^2 \\ &+ 2b\rho_2 \mathfrak{T}_{\psi_x}(t) + 2b\rho_2 \mathfrak{T}_{D^\alpha \psi}(t), \\ &\varepsilon_2 > 0, \quad t \ge 0. \end{aligned}$$

Proof. Again, Proposition 4 yields

$$\begin{aligned} D^\alpha U_3(t) ={}& b\rho_2 \int_0^1 \vartheta(x) \psi_x D^\alpha \left(D^\alpha \psi\right) dx \\ &+ b\rho_2 \int_0^1 \vartheta(x) D^\alpha \psi D^\alpha \psi_x \, dx \\ &- \frac{\alpha b\rho_2}{\Gamma(1-\alpha)} \int_0^1 \vartheta(x) \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\ &\times \left(\int_0^\xi \frac{[D^\alpha \psi(\eta)]' \, d\eta}{(t-\eta)^\alpha} \right) \\ &\times \left(\int_0^\xi (t-s)^{-\alpha} \psi_x'(s) \, ds \right) dx \end{aligned}$$

and therefore

$$\begin{aligned} D^\alpha U_3(t) \le{}& b^2 \int_0^1 \vartheta(x)\psi_{xx}\psi_x \,\mathrm{d}x \\ &- kb \int_0^1 \vartheta(x)(\varphi_x+\psi)\psi_x \,\mathrm{d}x \\ &- \int_0^1 \vartheta(x)\psi_x D^\alpha \psi \,\mathrm{d}x \\ &+ b\rho_2 \int_0^1 \vartheta(x) D^\alpha\psi D^\alpha \psi_x \,\mathrm{d}x \\ &+ 2b\rho_2 \mathfrak{T}_{\psi_x}(t) + 2b\rho_2 \mathfrak{T}_{D^\alpha\psi}(t), \quad t \ge 0. \end{aligned} \tag{11}$$

Using the Young inequality, it is clear that

$$\int_0^1 \vartheta(x)\psi_x D^\alpha\psi \,\mathrm{d}x \le \frac{2}{\varepsilon_0}\left\|D^\alpha\psi\right\|^2 + \frac{\varepsilon_0}{2}\left\|\psi_x\right\|^2, \ \varepsilon_0>0 \tag{12}$$

and, for $\varepsilon_1 > 0$,

$$kb\int_0^1 \vartheta(x)(\varphi_x+\psi)\psi_x \,\mathrm{d}x \le \varepsilon_2 kb\left\|\varphi_x+\psi\right\|^2 + \frac{kb}{\varepsilon_2}\left\|\psi_x\right\|^2. \tag{13}$$

Gathering all the previous relations (11)–(13), we get

$$\begin{aligned} &D^\alpha U_3(t) \\ &\le \frac{b^2}{2}\left[\vartheta(x)\psi_x^2\right]_0^1 - \frac{b^2}{2}\int_0^1 \vartheta'(x)\psi_x^2 \,\mathrm{d}x \\ &\quad + \varepsilon_2 kb\left\|\varphi_x+\psi\right\|^2 + \left(\frac{kb}{\varepsilon_2}+\frac{\varepsilon_0}{2}\right)\left\|\psi_x\right\|^2 \\ &\quad + \frac{2}{\varepsilon_0}\left\|D^\alpha\psi\right\|^2 - \frac{b\rho_2}{2}\int_0^1 \vartheta'(x)\left(D^\alpha\psi\right)^2 \,\mathrm{d}x \\ &\quad + 2b\rho_2\mathfrak{T}_{\psi_x}(t) + 2b\rho_2\mathfrak{T}_{D^\alpha\psi}(t), \end{aligned}$$

that is,

$$\begin{aligned} &D^\alpha U_3(t) \\ &\le -b^2\psi_x^2(1) - b^2\psi_x^2(0) \\ &\quad + \left[2b^2 + \frac{kb}{\varepsilon_2} + \frac{\varepsilon_0}{2}\right]\left\|\psi_x\right\|^2 \\ &\quad + \varepsilon_2 kb\left\|\varphi_x+\psi\right\|^2 + 2\left(\frac{1}{\varepsilon_0}+b\rho_2\right)\left\|D^\alpha\psi\right\|^2 \\ &\quad + 2b\rho_2\mathfrak{T}_{\psi_x}(t) + 2b\rho_2\mathfrak{T}_{D^\alpha\psi}(t), \quad t\ge 0. \end{aligned}$$

This ends the proof. ■

Now we turn to the boundary terms $\varphi_x^2(1)$ and $\varphi_x^2(0)$. They will be dealt with through the functional

$$U_4(t) = \rho_1 \int_0^1 \vartheta(x)\varphi_x D^\alpha\varphi \,\mathrm{d}x, \quad t\ge 0.$$

Lemma 4. *For the functional $U_4(t)$, we have*

$$\begin{aligned} D^\alpha U_4(t) \le{}& 2\rho_1\left\|D^\alpha\varphi\right\|^2 + k\left\|\psi_x\right\|^2 \\ &- k\left[\varphi_x^2(1)+\varphi_x^2(0)\right] + 3k\left\|\varphi_x\right\|^2 \\ &+ 2\rho_1\mathfrak{T}_{\varphi_x}(t) + 2\rho_1\mathfrak{T}_{D^\alpha\varphi}(t), \quad t\ge 0. \end{aligned}$$

Proof. Applying the operator D^α to the functional $U_4(t)$ gives

$$\begin{aligned} &D^\alpha U_4(t) \\ &= \rho_1\int_0^1 \vartheta(x) D^\alpha\varphi_x D^\alpha\varphi \,\mathrm{d}x \\ &\quad + \rho_1\int_0^1 \vartheta(x)\varphi_x D^\alpha\left(D^\alpha\varphi\right) \,\mathrm{d}x \\ &\quad - \frac{\alpha\rho_1}{\Gamma(1-\alpha)}\int_0^1 \vartheta(x)\int_0^t \frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}} \\ &\quad \times \left(\int_0^\xi \frac{\left[\varphi_x(\eta)\right]' \,\mathrm{d}\eta}{(t-\eta)^\alpha}\right)\left(\int_0^\xi \frac{\left[D^\alpha\varphi(\eta)\right]' \,\mathrm{d}\eta}{(t-\eta)^\alpha}\right)\mathrm{d}x \\ &= \frac{\rho_1}{2}\left[\vartheta(x)\left(D^\alpha\varphi\right)^2\right]_0^1 \\ &\quad - \frac{\rho_1}{2}\int_0^1 \vartheta'(x)\left(D^\alpha\varphi\right)^2 \,\mathrm{d}x \\ &\quad + k\int_0^1 \vartheta(x)\varphi_x(\varphi_x+\psi)_x \,\mathrm{d}x + 2\rho_1\mathfrak{T}_{\varphi_x}(t) \\ &\quad + 2\rho_1\mathfrak{T}_{D^\alpha\varphi}(t), \end{aligned}$$

and hence

$$\begin{aligned} &D^\alpha U_4(t) \\ &\le 2\rho_1\left\|D^\alpha\varphi\right\|^2 + \frac{k}{2}\left[\vartheta(x)\varphi_x^2\right]_0^1 + k\left\|\varphi_x\right\|^2 \\ &\quad - \frac{k}{2}\int_0^1 \vartheta'(x)\varphi_x^2 \,\mathrm{d}x + k\left\|\psi_x\right\|^2 \\ &\quad + 2\rho_1\mathfrak{T}_{\varphi_x}(t) + 2\rho_1\mathfrak{T}_{D^\alpha\varphi}(t) \\ &\le 2\rho_1\left\|D^\alpha\varphi\right\|^2 - k\left[\varphi_x^2(1)+\varphi_x^2(0)\right] \\ &\quad + 3k\left\|\varphi_x\right\|^2 + k\left\|\psi_x\right\|^2 \\ &\quad + 2\rho_1\mathfrak{T}_{\varphi_x}(t) + 2\rho_1\mathfrak{T}_{D^\alpha\varphi}(t), \quad t\ge 0. \end{aligned}$$

The proof is complete. ■

Let w be the solution of the problem

$$\begin{cases} -w_{xx} = \psi_x, \ x \in (0,1), \\ w(0) = w(1) = 0. \end{cases} \tag{14}$$

Our last functional is

$$U_5(t) := \int_0^1 (\rho_1 w D^\alpha \varphi + \rho_2 \psi D^\alpha \psi)\, \mathrm{d}x.$$

Lemma 5. *The above functional $U_5(t)$ satisfies*

$$\begin{aligned} D^\alpha U_5(t) \le{}& \left(\frac{\rho_1}{\varepsilon_0} + \frac{1}{4\varepsilon_0} + \rho_2\right) \|D^\alpha \psi\|^2 \\ &+ \frac{\varepsilon_0 \rho_1}{4} \|D^\alpha \varphi\|^2 \\ &+ (\varepsilon_0 - b) \|\psi_x\|^2 \\ &+ \rho_1 \mathfrak{T}_w(t) + \rho_1 \mathfrak{T}_{D^\alpha \varphi}(t) \\ &+ \rho_2 \mathfrak{T}_\psi(t) + \rho_2 \mathfrak{T}_{D^\alpha \psi}(t), \quad t \ge 0. \end{aligned}$$

Proof. Application of Proposition 4 one more time leads to

$$\begin{aligned} &D^\alpha U_5(t) \\ &= \rho_1 \int_0^1 D^\alpha w D^\alpha \varphi \, \mathrm{d}x + \rho_1 \int_0^1 w D^\alpha \left(D^\alpha \varphi\right) \mathrm{d}x \\ &\quad + \rho_2 \int_0^1 \psi D^\alpha (D^\alpha \psi)\, \mathrm{d}x + \rho_2 \|D^\alpha \psi\|^2 \\ &\quad - \frac{\alpha \rho_1}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}} \\ &\quad \times \left(\int_0^\xi \frac{[w(\eta)]' \, \mathrm{d}\eta}{(t-\eta)^\alpha}\right) \left(\int_0^\xi \frac{[D^\alpha \varphi(\eta)]' \, \mathrm{d}\eta}{(t-\eta)^\alpha}\right) \mathrm{d}x \\ &\quad - \frac{\alpha \rho_2}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[\psi(\eta)]' \, \mathrm{d}\eta}{(t-\eta)^\alpha}\right) \\ &\quad \times \left(\int_0^\xi \frac{[D^\alpha \psi(\eta)]' \, \mathrm{d}\eta}{(t-\eta)^\alpha}\right) \mathrm{d}x \end{aligned}$$

and, along the solutions of (1), it is easy to see that

$$\begin{aligned} &D^\alpha U_5(t) \\ &\le \frac{\rho_1}{\varepsilon_0} \|D^\alpha w\|^2 + \frac{\varepsilon_0 \rho_1}{4} \|D^\alpha \varphi\|^2 \\ &\quad + k \int_0^1 w(\varphi_x + \psi)_x \, \mathrm{d}x \\ &\quad + \int_0^1 \psi \left[b \psi_{xx} - D^\alpha \psi - k(\varphi_x + \psi)\right] \mathrm{d}x \\ &\quad + \rho_2 \|D^\alpha \psi\|^2 + \rho_1 \mathfrak{T}_w(t) + \rho_1 \mathfrak{T}_{D^\alpha \varphi}(t) \\ &\quad + \rho_2 \mathfrak{T}_\psi(t) + \rho_2 \mathfrak{T}_{D^\alpha \psi}(t). \end{aligned} \tag{15}$$

Having in mind that $\|D^\alpha w\|^2 \le \|D^\alpha \psi\|^2$, from the last relation (15), we deduce, that

$$\begin{aligned} &D^\alpha U_5(t) \\ &\le \left(\frac{\rho_1}{\varepsilon_0} + \frac{1}{4\varepsilon_0} + \rho_2\right) \|D^\alpha \psi\|^2 \\ &\quad + \frac{\varepsilon_0 \rho_1}{4} \|D^\alpha \varphi\|^2 + (\varepsilon_0 - b) \|\psi_x\|^2 \\ &\quad + \rho_1 \mathfrak{T}_w(t) + \rho_1 \mathfrak{T}_{D^\alpha \varphi}(t) + \rho_2 \mathfrak{T}_\psi(t) \\ &\quad + \rho_2 \mathfrak{T}_{D^\alpha \psi}(t), \quad t \ge 0. \end{aligned}$$

■

4.1. Equal speeds of propagation.

Here, we will prove our stability result in the case of equal speeds of propagation. For this purpose, we consider the functional

$$U(t) := N E(t) + \sum_{i=1}^5 M_i U_i(t), \qquad N, M_i \ge 0, \quad t \ge 0$$

with N and M_i to be determined inside the proof, and make use of the lemmas proven in the previous section. First of all, note that this functional $U(t)$ is equivalent to $E(t)$.

Theorem 1. *Assume that the initial data satisfy $\mathcal{U}_0 \in D(M)$ and $\rho_1/k = \rho_2/b$, then the solution of (1) and (2) approaches zero as time goes to infinity in a Mittag-Leffler fashion. That is, there exit two positive constants η and B (depending on $E(0)$) such that*

$$E(t) \le B E_\alpha(-\eta t^\alpha), \quad t \ge 0.$$

Proof. Our objective is to reach an inequality of the form $D^\alpha U(t) \le -C U(t)$ for some positive constant C. Thanks to Lemmas 1–5 and the relation (9), after choosing $M_2 = 5M_1 = 20 M_4$, $\varepsilon_1 = k/20$, $\varepsilon_2 = 1$ and $M_3 < 10 M_4 \min\{10/k, 1/b\}$, we infer that

$$\begin{aligned} &D^\alpha U(t) \\ &\le \left[M_5 \left(\frac{4\rho_1 + 1}{4\varepsilon_0} + \rho_2\right) + 2M_3 \left(\frac{1}{\varepsilon_0} + b\rho_2\right)\right. \end{aligned}$$

$$
\begin{aligned}
&-N + 4M_4\left(\frac{3}{2\varepsilon_0} + 4\rho_2\right)\Big]\,\|D^\alpha\psi\|^2\\
&+\rho_1\left(M_5\frac{\varepsilon_0}{4} - 2M_4\right)\|D^\alpha\varphi\|^2\\
&+\Big[10M_4k + 20M_4(\varepsilon_0 - k)\\
&+M_3kb\Big]\,\|\varphi_x+\psi\|^2 + \Big[M_4(4b+7k+4\varepsilon_0)\\
&+M_3\left(2b^2 + kb + \frac{\varepsilon_0}{2}\right) + M_5(\varepsilon_0 - b)\Big]\,\|\psi_x\|^2\\
&+\rho_2\left(24M_4 + 2M_3b + M_5 - N\right)\mathfrak{T}_{D^\alpha\psi}(t)\\
&+\rho_1\left(6M_4 + 20M_4\frac{\rho_2}{\rho_1} + M_5 - N\right)\mathfrak{T}_{D^\alpha\varphi}(t)\\
&+\Big[12M_4(\rho_1+2\rho_2) + 2M_3b\rho_2\\
&+M_5(\rho_2+\rho_1) - Nb\Big]\mathfrak{T}_{\psi_x}(t)\\
&+\left[12M_4\rho_1 + 20M_4\rho_2 - Nk\right]\mathfrak{T}_{\varphi_x+\psi}(t), \quad t \ge 0.
\end{aligned}
$$

For N large enough, several coefficients can already be made negative. Namely,

$$
\begin{cases}
M_5\left(\frac{4\rho_1+1}{4\varepsilon_0} + \rho_2\right) + 2M_3\left(\frac{1}{\varepsilon_0} + b\rho_2\right)\\
+4M_4\left(\frac{3}{2\varepsilon_0} + 4\rho_2\right) < N,\\
24M_4 + 2M_3b + M_5 < N,\\
6M_4 + 20M_4\frac{\rho_2}{\rho_1} + M_5 < N,\\
12M_4(\rho_1+2\rho_2) + 2M_3b\rho_2 + M_5(\rho_2+\rho_1) < Nb,\\
12M_4\rho_1 + 20M_4\rho_2 < Nk.
\end{cases}
$$

We can therefore neglect ε_0 in the first step of the selection of the parameters. There remains only to choose M_5 so that

$$bM_5 > M_4(4b+7k) + bM_3(2b+k).$$

Next, we go backwards and select ε_0 and N. Hence, we have $D^\alpha U(t) \le -C_1E(t)$, for a $C_1 > 0$. By the equivalence of $U(t)$ and $E(t)$, we get $D^\alpha U(t) \le -\eta U(t)$, $t \ge 0$ for some $\eta > 0$. This yields the Mittag-Leffler decay $U(t) \le C_2E_\alpha(-\eta t^\alpha)$, $t \ge 0$, for a $C_2 > 0$. The statement of the theorem follows from another application of the equivalence. The proof is complete. ■

4.2. Non-equal speed of propagation. This is a more often encountered practical situation ($k\rho_2/\rho_1 \ne b$). Passing to a higher-order 'fractional' energy functional, we obtain a relation ($D^\alpha V(t) \le -CE(t)$) similar to the one in the integer-order case ($\alpha = 1$). However, opposite to the integer-order case where a stability of order t^{-1} can be derived, the derivation of a corresponding stability for the fractional case is not clear. We could, nevertheless, prove that $\liminf_{t\to\infty} E(t) = 0$ by applying the following result generalizing Barbalat's theorem.

Theorem 2. (Gallegos *et al.*, 2015) *Let g be a nonnegative bounded function such that $I^\alpha g(t)$ is uniformly bounded. Then, we have*

$$\liminf_{t\to\infty} g(t) = 0.$$

Theorem 3. *Assume that $\mathcal{U}_0 \in D(M)$ and the speeds of propagation are not necessarily equal. Then,*

$$\liminf_{t\to\infty} E(t) = 0.$$

We need to determine a functional whose fractional derivative is non-positive. We recall that the difficulty here is in how to deal with the higher-order term $\int_0^1 (\varphi_x+\psi)_x\psi_x\,\mathrm{d}x$. It cannot be controlled by terms in the (first-order) energy. Thereupon, we pass to a higher-order energy functional. Rewriting $E(t)$ in the form $E(t) = E(t,\varphi,\psi)$ to account for the dependence on φ and ψ, we define the '2α-order' energy by

$$E_s(t) = E(t, D^\alpha\varphi, D^\alpha\psi), \quad t \ge 0.$$

Bearing in mind the equation $D^\alpha\mathcal{U} = M\mathcal{U}$, this functional is well-defined for $\mathcal{U}_0 \in D(M)$.

Applying D^α to

$$
\begin{cases}
\rho_1 D^\alpha\left[D^\alpha\left(D^\alpha\varphi\right)\right] - kD^\alpha(\varphi_x+\psi)_x = 0,\\
\rho_2 D^\alpha\left[D^\alpha\left(D^\alpha\psi\right)\right] + D^\alpha\left(D^\alpha\psi\right) - bD^\alpha\psi_{xx}\\
+kD^\alpha(\varphi_x+\psi) = 0
\end{cases}
$$

and multiplying by $D^\alpha(D^\alpha\varphi)$ and $D^\alpha(D^\alpha\psi)$ the two equations, respectively, we find

$$
\begin{cases}
\rho_1 D^\alpha\left(D^\alpha\varphi\right)D^\alpha\left[D^\alpha\left(D^\alpha\varphi\right)\right]\\
-kD^\alpha\left(D^\alpha\varphi\right)D^\alpha(\varphi_x+\psi)_x = 0,\\
\rho_2 D^\alpha\left(D^\alpha\psi\right)D^\alpha\left[D^\alpha\left(D^\alpha\psi\right)\right] + \left[D^\alpha\left(D^\alpha\psi\right)\right]^2\\
-bD^\alpha\left(D^\alpha\psi\right)D^\alpha\psi_{xx}\\
+kD^\alpha\left(D^\alpha\psi\right)D^\alpha(\varphi_x+\psi) = 0.
\end{cases}
$$

As, for $t \ge 0$,

$$
\begin{aligned}
E_s(t) = \frac{1}{2}\Big[&\rho_1\left\|D^\alpha\left(D^\alpha\varphi\right)\right\|^2 + \rho_2\left\|D^\alpha\left(D^\alpha\psi\right)\right\|^2\\
&+b\left\|D^\alpha\psi_x\right\|^2 + k\left\|D^\alpha\left(\varphi_x+\psi\right)\right\|^2\Big],
\end{aligned}
$$

it appears that

$$D^\alpha E_s(t) \le -\left\|D^\alpha\left(D^\alpha\psi\right)\right\|^2 \le 0, \quad t \ge 0. \tag{16}$$

Next, we modify $U_2(t)$ by

$$
\begin{aligned}
\tilde{U}_2(t) = \rho_2\int_0^1 D^\alpha\psi(\varphi_x+\psi)\,\mathrm{d}x - \rho_2\int_0^1 D^\alpha\psi\varphi_x\,\mathrm{d}x\\
+\tfrac{b\rho_1}{k}\int_0^1\left[\psi_xD^\alpha\varphi + \varphi_xD^\alpha\psi\right]\mathrm{d}x.
\end{aligned}
$$

Lemma 6. *For the above functional $\tilde{U}_2(t)$, we have*

$$\begin{aligned}
&D^\alpha \tilde{U}_2(t)\\
&\le \varepsilon_3\left[\varphi_x^2(1)+\varphi_x^2(0)\right]+\frac{b^2}{4\varepsilon_3}\left[\psi_x^2(1)+\psi_x^2(0)\right]\\
&\quad+(3\varepsilon_0-k)\left\|\varphi_x+\psi\right\|^2+\left(\frac{1}{4\varepsilon_0}+\rho_2\right)\left\|D^\alpha\psi\right\|^2\\
&\quad+2\varepsilon_0\left\|\psi_x\right\|^2\\
&\quad+\frac{1}{4\varepsilon_0}\left(\frac{b\rho_1}{k}-\rho_2\right)^2\left\|D^\alpha\left(D^\alpha\psi\right)\right\|^2\\
&\quad+\mathfrak{T}_{D^\alpha\varphi}(t)+\left(2\rho_2+\frac{b\rho_1}{k}\right)\mathfrak{T}_{D^\alpha\psi}(t)\\
&\quad+\left(3\rho_2+\frac{2b\rho_1}{k}\right)\mathfrak{T}_{\varphi_x+\psi}(t)\\
&\quad+\left[2\left(\rho_2+\frac{b\rho_1}{k}\right)+1\right]\mathfrak{T}_{\psi_x}(t),
\end{aligned}$$

for $\varepsilon_0,\varepsilon_3>0,\ t\ge 0.$

Proof. First, by Proposition 4,

$$\begin{aligned}
&D^\alpha \tilde{U}_2(t)\\
&=\rho_2\int_0^1 D^\alpha\left(D^\alpha\psi\right)\left(\varphi_x+\psi\right)\mathrm{d}x\\
&\quad+\rho_2\int_0^1 D^\alpha\psi D^\alpha(\varphi_x+\psi)\,\mathrm{d}x\\
&\quad-\rho_2\int_0^1 D^\alpha\psi D^\alpha\varphi_x\,\mathrm{d}x-\rho_2\int_0^1 D^\alpha\left(D^\alpha\psi\right)\varphi_x\mathrm{d}x\\
&\quad-\frac{\alpha\rho_2}{\Gamma(1-\alpha)}\int_0^1\int_0^t\frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}}\\
&\quad\times\left(\int_0^\xi\frac{\left[D^\alpha\psi(\eta)\right]'\,\mathrm{d}\eta}{(t-\eta)^\alpha}\right)\left(\int_0^\xi\frac{(\varphi_x+\psi)'(s)\mathrm{d}s}{(t-s)^\alpha}\right)\mathrm{d}x\\
&\quad-\frac{\alpha\rho_2}{\Gamma(1-\alpha)}\int_0^1\int_0^t\frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}}\\
&\quad\times\left(\int_0^\xi\frac{\left[D^\alpha\psi(\eta)\right]'\,\mathrm{d}\eta}{(t-\eta)^\alpha}\right)\left(\int_0^\xi\frac{\left[\varphi_x(s)\right]'\,\mathrm{d}s}{(t-s)^\alpha}\right)\mathrm{d}x\\
&\quad+\frac{b\rho_1}{k}D^\alpha\int_0^1\left[\psi_x D^\alpha\varphi+\varphi_x D^\alpha\psi\right]\,\mathrm{d}x,
\end{aligned}$$

and the second equation of the system is

$$\begin{aligned}
&D^\alpha \tilde{U}_2(t)\\
&=-\rho_2\int_0^1 D^\alpha\left(D^\alpha\psi\right)\varphi_x\mathrm{d}x\\
&\quad+\int_0^1(\varphi_x+\psi)\Big[-D^\alpha\psi+b\psi_{xx}\\
&\quad-k(\varphi_x+\psi)\Big]\,\mathrm{d}x+\rho_2\left\|D^\alpha\psi\right\|^2\\
&\quad-\frac{\alpha\rho_2}{\Gamma(1-\alpha)}\int_0^1\int_0^t\frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^\xi\frac{\left[D^\alpha\psi(\eta)\right]'\mathrm{d}\eta}{(t-\eta)^\alpha}\right)\\
&\quad\times\left(\int_0^\xi\frac{(\varphi_x+\psi)'(s)\,\mathrm{d}s}{(t-s)^\alpha}\right)\mathrm{d}x\\
&\quad-\frac{\alpha\rho_2}{\Gamma(1-\alpha)}\int_0^1\int_0^t\frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^\xi\frac{\left[D^\alpha\psi(\eta)\right]'\,\mathrm{d}\eta}{(t-\eta)^\alpha}\right)\\
&\quad\times\left(\int_0^\xi\frac{\left[\varphi_x(s)\right]'\,\mathrm{d}s}{(t-s)^\alpha}\right)\mathrm{d}x\\
&\quad+\frac{b\rho_1}{k}\left[D^\alpha\int_0^1\left[\psi_x D^\alpha\varphi+\varphi_x D^\alpha\psi\right]\mathrm{d}x\right]
\end{aligned}$$

or

$$\begin{aligned}
&D^\alpha \tilde{U}_2(t)\\
&=-\int_0^1 D^\alpha\psi(\varphi_x+\psi)\,\mathrm{d}x+b\left[\varphi_x\psi_x\right]_0^1\\
&\quad-b\int_0^1(\varphi_x+\psi)_x\psi_x\,\mathrm{d}x-k\left\|\varphi_x+\psi\right\|^2\\
&\quad+\rho_2\left\|D^\alpha\psi\right\|^2-\rho_2\int_0^1 D^\alpha\left(D^\alpha\psi\right)\varphi_x\,\mathrm{d}x\\
&\quad-\frac{\alpha\rho_2}{\Gamma(1-\alpha)}\int_0^1\int_0^t\frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^\xi\frac{\left[D^\alpha\psi(\eta)\right]'\,\mathrm{d}\eta}{(t-\eta)^\alpha}\right)\\
&\quad\times\left(\int_0^\xi\frac{(\varphi_x+\psi)'(s)\mathrm{d}s}{(t-s)^\alpha}\right)\mathrm{d}x\\
&\quad-\frac{\alpha\rho_2}{\Gamma(1-\alpha)}\int_0^1\int_0^t\frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^\xi\frac{\left[D^\alpha\psi(\eta)\right]'\mathrm{d}\eta}{(t-\eta)^\alpha}\right)
\end{aligned}$$

$$\times\left(\int_0^{\xi}\frac{[\varphi_x(s)]'\,\mathrm{d}s}{(t-s)^{\alpha}}\right)\mathrm{d}x$$

$$+\frac{b\rho_1}{k}D^{\alpha}\int_0^1[\psi_x D^{\alpha}\varphi+\varphi_x D^{\alpha}\psi]\,\mathrm{d}x.$$

Using the first equation in the system, applying the Young inequality and the fact that

$$\begin{aligned}
&D^{\alpha}\int_0^1[\psi_x D^{\alpha}\varphi+\varphi_x D^{\alpha}\psi]\,\mathrm{d}x\\
&=D^{\alpha}\int_0^1\psi_x D^{\alpha}\varphi\,\mathrm{d}x-D^{\alpha}\int_0^1\varphi D^{\alpha}\psi_x\,\mathrm{d}x\\
&=\int_0^1 D^{\alpha}\psi_x D^{\alpha}\varphi\mathrm{d}x+\int_0^1\psi_x D^{\alpha}\left(D^{\alpha}\varphi\right)\,\mathrm{d}x\\
&\quad-\frac{\alpha}{\Gamma(1-\alpha)}\int_0^1\int_0^t\frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{\psi_{xt}(\eta)\,\mathrm{d}\eta}{(t-\eta)^{\alpha}}\right)\\
&\quad\times\left(\int_0^{\xi}\frac{[D^{\alpha}\varphi(s)]'\,\mathrm{d}s}{(t-s)^{\alpha}}\right)\mathrm{d}x\\
&\quad+\int_0^1 D^{\alpha}\varphi_x D^{\alpha}\psi\mathrm{d}x+\int_0^1\varphi_x D^{\alpha}\left(D^{\alpha}\psi\right)\,\mathrm{d}x\\
&\quad-\frac{\alpha}{\Gamma(1-\alpha)}\int_0^1\int_0^t\frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{\varphi_{xt}(\eta)\,\mathrm{d}\eta}{(t-\eta)^{\alpha}}\right)\\
&\quad\times\left(\int_0^{\xi}\frac{[D^{\alpha}\psi(s)]'\,\mathrm{d}s}{(t-s)^{\alpha}}\right)\mathrm{d}x
\end{aligned}$$

we get

$$\begin{aligned}
&D^{\alpha}\tilde{U}_2(t)\\
&\leq\varepsilon_0\left\|\varphi_x+\psi\right\|^2+\frac{1}{4\varepsilon_0}\left\|D^{\alpha}\psi\right\|^2\\
&\quad+\varepsilon_3\left[\varphi_x^2(1)+\varphi_x^2(0)\right]+\frac{b^2}{4\varepsilon_3}\left[\psi_x^2(1)+\psi_x^2(0)\right]\\
&\quad-k\left\|\varphi_x+\psi\right\|^2+\rho_2\left\|D^{\alpha}\psi\right\|^2\\
&\quad+\left(\frac{b\rho_1}{k}-\rho_2\right)\int_0^1 D^{\alpha}\left(D^{\alpha}\psi\right)\varphi_x\,\mathrm{d}x\\
&\quad-\frac{\alpha\rho_2}{\Gamma(1-\alpha)}\int_0^1\int_0^t\frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{[D^{\alpha}\psi(\eta)]'\,\mathrm{d}\eta}{(t-\eta)^{\alpha}}\right)\\
&\quad\times\left(\int_0^{\xi}\frac{(\varphi_x+\psi)'(s)\,\mathrm{d}s}{(t-s)^{\alpha}}\right)\mathrm{d}x\\
&\quad-\frac{\alpha}{\Gamma(1-\alpha)}\left(\rho_2+\frac{b\rho_1}{k}\right)\int_0^1\int_0^t\frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}}\\
&\quad\times\left(\int_0^{\xi}\frac{[D^{\alpha}\psi(\eta)]'\,\mathrm{d}\eta}{(t-\eta)^{\alpha}}\right)\left(\int_0^{\xi}\frac{[\varphi_x(s)]'\,\mathrm{d}s}{(t-s)^{\alpha}}\right)\mathrm{d}x\\
&\quad-\frac{\alpha}{\Gamma(1-\alpha)}\int_0^1\int_0^t\frac{\mathrm{d}\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{\psi_{xt}(\eta)\,\mathrm{d}\eta}{(t-\eta)^{\alpha}}\right)\\
&\quad\times\left(\int_0^{\xi}\frac{[D^{\alpha}\varphi(s)]'\,\mathrm{d}s}{(t-s)^{\alpha}}\right)\mathrm{d}x.
\end{aligned}$$

Further applications of the Young inequality yield

$$\begin{aligned}
&D^{\alpha}\tilde{U}_2(t)\\
&\leq\lambda_1\left[\varphi_x^2(1)+\varphi_x^2(0)\right]+\frac{b^2}{4\lambda_1}\left[\psi_x^2(1)+\psi_x^2(0)\right]\\
&\quad+(\varepsilon_0-k)\left\|\varphi_x+\psi\right\|^2\\
&\quad+\left(\frac{1}{4\varepsilon_0}+\rho_2\right)\left\|D^{\alpha}\psi\right\|^2+\varepsilon_0\left\|\varphi_x\right\|^2\\
&\quad+\frac{1}{4\varepsilon_0}\left(\frac{b\rho_1}{k}-\rho_2\right)^2\left\|D^{\alpha}\left(D^{\alpha}\psi\right)\right\|^2\\
&\quad+\rho_2\mathfrak{T}_{D^{\alpha}\psi}(t)+\rho_2\mathfrak{T}_{\varphi_x+\psi}(t)\\
&\quad+\left(\rho_2+\frac{b\rho_1}{k}\right)\mathfrak{T}_{D^{\alpha}\psi}(t)+\left(\rho_2+\frac{b\rho_1}{k}\right)\mathfrak{T}_{\varphi_x}(t)\\
&\quad+\mathfrak{T}_{\psi_x}(t)+\mathfrak{T}_{D^{\alpha}\varphi}(t)
\end{aligned}$$

and

$$\begin{aligned}
&D^{\alpha}\tilde{U}_2(t)\\
&\leq\varepsilon_3\left[\varphi_x^2(1)+\varphi_x^2(0)\right]+\frac{b^2}{4\varepsilon_3}\left[\psi_x^2(1)+\psi_x^2(0)\right]\\
&\quad+(3\varepsilon_0-k)\left\|\varphi_x+\psi\right\|^2\\
&\quad+\left(\frac{1}{4\varepsilon_0}+\rho_2\right)\left\|D^{\alpha}\psi\right\|^2+2\varepsilon_0\left\|\psi_x\right\|^2\\
&\quad+\frac{1}{4\varepsilon_0}\left(\frac{b\rho_1}{k}-\rho_2\right)^2\left\|D^{\alpha}\left(D^{\alpha}\psi\right)\right\|^2\\
&\quad+\mathfrak{T}_{D^{\alpha}\varphi}(t)+\left(2\rho_2+\frac{b\rho_1}{k}\right)\mathfrak{T}_{D^{\alpha}\psi}(t)
\end{aligned}$$

$$+\left(3\rho_2+\frac{2b\rho_1}{k}\right)\mathfrak{T}_{\varphi_x+\psi}(t)$$
$$+\left[2\left(\rho_2+\frac{b\rho_1}{k}\right)+1\right]\mathfrak{T}_{\psi_x}(t).$$

This completes the proof. ■

Let, for $t \geq 0$,

$$V(t) := \tilde{N}E(t)+\tilde{M}_6E_s(t) + \sum_{i=1}^{5}\tilde{M}_i\tilde{U}_i(t), \quad \tilde{N},\tilde{M}_i \geq 0,$$

with $\tilde{M}_1 = 4\tilde{M}_4$, $\tilde{M}_2 = k\tilde{M}_3/3 = 11\tilde{M}_4$ and $\varepsilon_3 = k/12$, where $\tilde{U}_i(t) = U_i(t)$, $i = 1, 3, 4, 5$.

Proof. (Theorem 3) Taking into account the relations (9), (16) and Lemmas 1, 3–6, we obtain

$$D^\alpha V(t)$$
$$\leq \left[\tilde{M}_4\left(7\rho_2+\frac{66}{k}b\rho_2+\frac{66}{k\varepsilon_0}+\frac{15}{4\varepsilon_0}\right)-\tilde{N}\right.$$
$$\left.\times +\tilde{M}_5\left(\frac{\rho_1}{\varepsilon_0}+\frac{1}{4\varepsilon_0}+\rho_2\right)\right]\|D^\alpha\psi\|^2$$
$$+\rho_1\left(\frac{\varepsilon_0\tilde{M}_5}{4}-2\tilde{M}_4\right)\|D^\alpha\varphi\|^2$$
$$+\left[\frac{11\tilde{M}_4}{4\varepsilon_0}\left(\frac{b\rho_1}{k}-\rho_2\right)^2-\tilde{M}_6\right]\|D^\alpha(D^\alpha\psi)\|^2$$
$$\times\left\{\tilde{M}_4\left[4b+26\varepsilon_0+\frac{33}{k}\left(2b^2+\frac{kb}{\varepsilon_2}+\frac{\varepsilon_0}{2}\right)+7k\right]\right.$$
$$\left.+\tilde{M}_5(\varepsilon_0-b)\right\}\times\|\psi_x\|^2+\tilde{M}_4\Big[4k+11(3\varepsilon_0-k)$$
$$+\frac{33}{k}\varepsilon_2kb+6k\Big]\|\varphi_x+\psi\|^2$$
$$+\left[6\tilde{M}_4\rho_1+11\tilde{M}_4+\tilde{M}_5\rho_1-\tilde{N}\rho_1\right]\mathfrak{T}_{D^\alpha\varphi}(t)$$

$$+\rho_2\left\{\tilde{M}_4\left[26+\frac{11b\rho_1}{k\rho_2}+\frac{66}{k}b\right]+\tilde{M}_5-\tilde{N}\right\}\mathfrak{T}_{D^\alpha\psi}(t)$$
$$+\left\{\tilde{M}_4\left[12\rho_1+26\rho_2+\frac{22b\rho_1}{k}+\frac{66}{k}b\rho_2+11\right]\right.$$
$$\left.-\tilde{N}b+\tilde{M}_5(\rho_2+\rho_1)\right\}\mathfrak{T}_{\psi_x}(t)$$
$$+\left[\tilde{M}_4\left(12\rho_1+33\rho_2+\frac{22b\rho_1}{k}\right)-\tilde{N}k\right]\mathfrak{T}_{\varphi_x+\psi}(t).$$

For $\tilde{N}$ large enough (and consequently small ε_0) and

$$\tilde{M}_6 \geq \frac{11\tilde{M}_4}{4\varepsilon_0}\left(\frac{b\rho_1}{k}-\rho_2\right)^2,$$

we require

$$\begin{cases}\tilde{M}_4\left[4b+\frac{33}{k}\left(2b^2+\frac{kb}{\varepsilon_2}\right)+7k\right]<b\tilde{M}_5,\\ \frac{33}{k}\varepsilon_2 b<1.\end{cases}$$

To this end, it suffices that $\varepsilon_2 = k/34b$ and $\tilde{M}_5$ be large enough. Proceeding backwards, we select ε_0 and $\tilde{N}$. We end up with

$$D^\alpha V(t) \leq -CE(t), \quad t \geq 0.$$

Therefore,

$$I^\alpha E(t) \leq -C^{-1}I^\alpha\left(D^\alpha V(t)\right), \quad t \geq 0,$$

or

$$I^\alpha E(t) \leq -C^{-1}\left[V(t)-V(0)\right] \leq C^{-1}V(0), \quad t \geq 0.$$

Having fulfilled the hypotheses of Theorem 2, we deduce that $\liminf_{t\to\infty} E(t) = 0$.

■

Acknowledgment

The author is very grateful for the financial support and the facilities provided by the King Fahd University of Petroleum and Minerals through the project number IN171006 (Interdisciplinary Research Center for Intelligent Manufacturing and Robotics).

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Nasser-eddine Tatar is currently a full professor in the Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals. He is mainly interested in partial differential equations of hyperbolic and mixed types, integro-differential equations, fractional differential equations, inequalities and neural network theory. He is a recipient of the Shoman Prize and some other research awards from the KFUPM. He is the author of more than 160 publications in international journals. He is a member of several mathematics associations and an associate editor for a dozen of international journals.

Received: 20 February 2021
Revised: 16 March 2021
Accepted: 30 March 2021