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SZEGŐ'S FIRST LIMIT THEOREM IN TERMS OF A REALIZATION OF A CONTINUOUS-TIME TIME-VARYING SYSTEM

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It is shown that the limit in an abstract version of Szegő's limit theorem can be expressed in terms of the antistable dynamics of the system. When the system dynamics are regular, it is shown that the limit equals the difference between the antistable Lyapunov exponents of the system and those of its inverse. In the general case, the elements of the dichotomy spectrum give lower and upper bounds.

Keywords: time-varying systems, exponential dichotomies, coprime, inner/outer factorizations

1. Introduction

In the early part of the last century, Szegő proved two formulae regarding the limits of certain Toeplitz matrices in terms of their symbols. Since then, these two limits have received considerable attention, see (Böttcher and Silbermann, 1999) and the references therein. In the case where the symbol can be expressed as a rational matrix function, Gohberg *et al.* (1987) showed a relationship between these limits and the realization of the symbol.

Consider a continuous matrix function Φ on the unit circle. Furthermore, assume that the symbol Φ is a rational matrix function with realization

$$\Phi(z) = I + C(zI - A)^{-1}B, \quad |z| = 1.$$

Assume that $\det \Phi(z) \neq 0$, for $|z| = 1$, and that the winding number $(1/2\pi) \arg\det \Phi(\mathrm{e}^{i\omega})|_{\omega=-\pi}^{\pi}$ is equal to zero. Moreover, assume that $A^{\times} := A - BC$ has no eigenvalues on the unit circle. As shown in (Gohberg *et al.*, 1987), Szegő's first limit theorem can then be stated as follows:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\det \Phi(\mathrm{e}^{i\omega})\, \mathrm{d}\omega = \sum_i \log|z_i| - \sum_i \log|p_i|,$$

where z_i (resp. p_i) are the eigenvalues of $A^{\times}$ (resp. A) with norm greater than one.

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For symbols arising from realizations of continuous-time systems, analogues of the original Szegő formulae were obtained in (Ahiezer, 1964; Kac, 1954). Their connection with the symbol can be found in (Gohberg *et al.*, 1987). In this case, the Toeplitz matrix is replaced by the Wiener-Hopf operator

$$(Fw)(t) = w(t) + \int_0^\infty f(t-\tau)w(\tau)\,\mathrm{d}\tau, \quad t \geq 0, \tag{1}$$

where f is an $m \times m$ matrix function with entries in $\mathcal{L}_1(-\infty,\infty)$ and symbol

$$F(\omega) = I + C(i\omega I - A)^{-1}B, \quad -\infty < \omega < \infty,$$

A and $A^\times$ having no eigenvalues on the imaginary axis. If the function $f(t)$ is continuous, we can consider the operator

$$[F_T w](t) = \int_0^T f(t-\tau)w(\tau)\,\mathrm{d}\tau$$

as the truncation and look at the limit

$$\lim_{T\to\infty} \frac{1}{T} \log\det(I + F_T) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \log\det F(i\omega)\,\mathrm{d}\omega = \sum_i \mathrm{Re}\, z_i - \sum_i \mathrm{Re}\, p_i, \tag{2}$$

where z_i (resp. p_i) are the eigenvalues of $A^\times$ (resp. A) in the right-hand plane.

A second generalization of Szegő's limit was provided in (Dym and Ta'assan, 1981). While they consider a very general situation, their results can be used where the function (1) is replaced by

$$(Fw)(t) = w(t) + \int_0^\infty f(t,\tau)w(\tau)\,\mathrm{d}\tau, \quad t \geq 0.$$

In this paper we show that, for this case, the limit obtained in (Dym and Ta'assan, 1981) can also be expressed in terms of the antistable dynamics of a realization for F. The results considered here are continuous-time versions of those in (Iglesias, 2001). We note that the use of Szegő's limits has a long history in the engineering community, see (Grenander and Szegő, 1958; Iglesias, 2002).

The remainder of the paper is organized as follows: Section 2 provides some necessary preliminaries on Lyapunov exponents, exponential dichotomies, inner/outer and coprime factorizations. In Section 3 we present the relevant results from (Dym and Ta'assan, 1981). We then present our main results in Section 4.

2. Preliminaries

We consider linear time-varying systems Σ_G admitting a state-space representation

$$\Sigma_G := \begin{cases} \dot{x}(t) &= A(t)x(t) + B(t)w(t) \\ y(t) &= C(t)x(t) + D(t)w(t) \end{cases} = \left[\begin{array}{c|c} A(t) & B(t) \\ \hline C(t) & D(t) \end{array}\right]. \tag{3}$$

We will assume that all the matrix functions $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, $C(t) \in \mathbb{R}^{m\times n}$, $D(t) \in \mathbb{R}^{m\times m}$ are continuous, bounded functions of t defined for $t \geq 0$. With this system we associate an operator $\boldsymbol{G}$ mapping the input $\boldsymbol{w}$ to output $\boldsymbol{y}$. This operator has an integral representation

$$y(t) = D(t)w(t) + \int_{-\infty}^{t} G(t,\tau)w(\tau)\,\mathrm{d}t,$$

where the kernel $G(t,\tau)$ equals

$$G(t,\tau) = C(t)\Phi_A(t,\tau)B(\tau), \quad \text{if } t \geq \tau,$$

and zero otherwise. The matrix function $\Phi_A(t,\tau)$ is the *transition matrix* which equals

$$\Phi_A(t,\tau) = X(t)X^{-1}(\tau),$$

where $X(t)$ is the *fundamental* solution to the matrix differential equation

$$\dot{X}(t) = A(t)X(t), \quad X(0) = X_0$$

and X_0 is invertible. The following result is standard, see, e.g., (Rugh, 1996), and will be needed in the sequel.

Lemma 1. (Liouville's formula) *The transition matrix for $A(t)$ satisfies*

$$\log\det\Phi_A(t,\tau) = \int_{\tau}^{t} \operatorname{trace}\big[A(\sigma)\big]\,\mathrm{d}\sigma$$

for every t and τ.

2.1. Lyapunov Exponents and Exponential Dichotomies

For time-varying systems, the *Lyapunov exponents* or *characteristic numbers* play the same role as the real parts of the eigenvalues of the time-invariant matrix $A(t) \equiv A$. We now present, following (Dieci *et al.*, 1997), some basic results on Lyapunov exponents.

2.1.1. Lyapunov exponents

Consider the n-dimensional homogeneous system

$$\dot{x}(t) = A(t)x(t). \tag{4}$$

Suppose that $X(t)$ is the fundamental solution with an orthogonal initial condition X_0, and let $\{p_i\}_{i=1}^{n}$ be an orthonormal basis for $\mathbb{R}^n$. Then the characteristic numbers

$$\lambda_i(p_i) = \limsup_{T\to\infty} \frac{1}{T}\log\|X(T)p_i\| \tag{5}$$

are well defined.

Suppose that the orthonormal basis $\{p_i\}_{i=1}^n$ is chosen so as to minimize $\sum_{i=1}^n \lambda_i(p_i)$. The basis is then said to be *normal* and the corresponding λ_i are called the *Lyapunov exponents*. For now we will denote by λ_i the Lyapunov exponents associated with a normal basis. It is well-known that, in this case,

$$\sum_{i=1}^{n} \lambda_i \geq \limsup_{T\to\infty} \frac{1}{T}\int_0^T \operatorname{trace}\big[A(s)\big]\,\mathrm{d}s. \tag{6}$$

It is important to differentiate between the stable poles and zeros of the symbol of the integral operator (1), which do not contribute to the right-hand side of (2), and the unstable poles and zeros, which do. Time-varying systems which can be decomposed into stable and antistable components are said to possess an *exponential dichotomy.*

2.1.2. Exponential Dichotomy

The linear system (4) is said to possess an exponential dichotomy if there exists a projection P, and real constant $\gamma > 0,\ \lambda > 0$, such that

$$\begin{aligned} \big\|X(t)PX^{-1}(\tau)\big\| &\leq \gamma \exp\big(-\lambda(t-\tau)\big), \quad \text{for } t \geq \tau, \\ \big\|X(t)(I-P)X^{-1}(\tau)\big\| &\leq \gamma \exp\big(-\lambda(\tau-t)\big), \quad \text{for } \tau \geq t. \end{aligned}$$

Note that, if $\operatorname{rank}(P) = n_{\mathrm{s}}$, an exponential dichotomy implies that n_{s} fundamental solutions are uniformly exponentially stable, whereas $n_{\mathrm{u}} = n - n_{\mathrm{s}}$ are uniformly exponentially antistable. In this case we say that the exponential dichotomy is of rank n_{s}.

The existence of an exponential dichotomy allows us to define a stability preserving state space transformation (a Lyapunov transformation) that separates the stable and antistable parts of $A(t)$.

Lemma 2. *(Coppel, 1978, Ch. 5) If the function $A(t)$ in the realization (3) admits an exponential dichotomy, then there exists a bounded matrix function $V(t)$ with bounded inverses such that*

$$\left[\begin{array}{c|c} \big(\dot V(t) + V(t)A(t)\big)V^{-1}(t) & V(t)B(t) \\ \hline C(t)V^{-1}(t) & D(t) \end{array}\right] =: \left[\begin{array}{cc|c} A_{\mathrm{s}}(t) & 0 & B_{\mathrm{s}}(t) \\ 0 & A_{\mathrm{u}}(t) & B_{\mathrm{u}}(t) \\ \hline C_{\mathrm{s}}(t) & C_{\mathrm{u}}(t) & D(t) \end{array}\right],$$

where $A_{\mathrm{s}}(t)$ is stable and $A_{\mathrm{u}}(t)$ is antistable.

Whenever two matrices $A_1(t)$ and $A_2(t)$ are related by a Lyapunov transformation, we say that they are *kinematically similar*, and this will be denoted by $A_1 \simeq A_2$.

2.1.3. Dichotomy Spectrum

Exponential dichotomies permit us to present another form of spectral representation for linear time-varying systems related to Lyapunov exponents. The *dichotomy*, or *Sacker-Sell spectrum*, $\mathcal{S}_{\text{dich}}$, of the system (4) is the set of real values λ for which the translated systems

$$\dot{x}(t) = \big(A(t) - \lambda I\big)x(t)$$

fail to have an exponential dichotomy (Sacker and Sell, 1978). In general, the spectrum is a collection of compact non-overlapping intervals:

$$\mathcal{S}_{\text{dich}} = \bigcup_{i=1}^{m} [\underline{\lambda}_i, \overline{\lambda}_i],$$

where $m \le n$ and $\underline{\lambda}_1 \le \overline{\lambda}_1 < \underline{\lambda}_2 \le \overline{\lambda}_2 < \cdots < \underline{\lambda}_m \le \overline{\lambda}_m$, and n is the dimension of the $A(t)$ matrix.

Suppose that real-valued $\lambda_0, \ \lambda_1, \ldots, \lambda_m$ are chosen in the complement of $\mathcal{S}_{\text{dich}}$ so that

$$\lambda_0 < \underline{\lambda}_1 \le \overline{\lambda}_1 < \lambda_1 < \underline{\lambda}_2 \le \cdots \le \overline{\lambda}_{m-1} < \lambda_{m-1} < \underline{\lambda}_m \le \overline{\lambda}_m < \lambda_m. \tag{7}$$

It is straightforward to check that for λ_0 all the trajectories of (3) are antistable. Similarly, for λ_m all the trajectories of (4) are stable.

Now, the matrix $A(t) - \lambda_1 I$ admits an exponential dichotomy and thus, from Lemma 2, is kinematically similar to a block-diagonal matrix. Equivalently,

$$A(t) \simeq \begin{bmatrix} A_1(t) & 0 \\ 0 & \bar{A}_1(t) \end{bmatrix},$$

where $A_1(t)$ is a square matrix of order n_1. Repeating this process with λ_2 leads to

$$\bar{A}_1(t) \simeq \begin{bmatrix} A_2(t) & 0 \\ 0 & \bar{A}_2(t) \end{bmatrix},$$

where $A_2(t)$ is a square matrix of order n_2. Continuing this procedure will lead to a sequence of matrices $A_k(t)$ of size $n_k \times n_k$, for $k = 1, \ldots, m$, so that $A(t) \simeq \operatorname{diag}\{A_1(t), \ldots, A_n(t)\}$ and $n_1 + \cdots + n_m = n$. It should be stated that the resulting matrices $A_k(t)$ do *not* depend on the particular choice of λ_k provided that (7) holds.

Since $A(t) - \lambda_o I$ is antistable, there exist an $\epsilon > 0$ and a $K \ge 0$, both depending on λ_0 such that, for $t \ge s$,[1]

$$\Phi'_{A-\lambda_0 I}(t,s)\Phi_{A-\lambda_0 I}(t,s) \ge K^2 \exp\big(-2\epsilon(s-t)\big).$$

However, we have

$$\Phi_{A-\lambda_0 I}(t,s) = \Phi_A(t,s)e^{-\lambda_0(t-s)},$$

[1] We note that the prime ($'$) denotes the transpose and not a derivative.

so that

$$\Phi'_A(t,s)\Phi_A(t,s) \geq K^2 \exp\big(2(\lambda_0+\epsilon)(t-s)\big),$$

and thus

$$\log\det\Phi_A(t,s) \geq n\log K + n(\lambda_0+\epsilon)(t-s).$$

Let $t > s$. Dividing by $\tau := t-s$, taking the limit as $\tau \to \infty$ and using Lemma 1, we obtain

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_s^{s+\tau} \operatorname{trace}\big[A(\sigma)\big]\,\mathrm{d}\sigma \geq n(\lambda_0+\epsilon).$$

The above holds for any $\lambda_0 < \underline{\lambda}_1$. Passing to the limit $\lambda_0 \to \underline{\lambda}_1$, we have, for $t = T$ and $s = 0$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^{T} \operatorname{trace}\big[A(\sigma)\big]\,\mathrm{d}\sigma \geq n\underline{\lambda}_1.$$

For λ_m, since $A(t) - \lambda_m I$ is stable, we have

$$\Phi'_{A-\lambda_m I}(t,s)\Phi_{A-\lambda_m I}(t,s) \leq K^2 \exp\big(-2\epsilon(t-s)\big)$$

for some $K \geq 0$ and $\epsilon > 0$. Proceeding as above, we obtain

$$\lim_{T\to\infty} \frac{1}{T}\int_0^{T} \operatorname{trace}\big[A(\sigma)\big]\,\mathrm{d}\sigma \leq n\overline{\lambda}_m.$$

This procedure can be repeated for all $\lambda_k \in (\overline{\lambda}_k, \underline{\lambda}_{k+1})$, $k = 1,\dots,m-1$ to yield the following:

Lemma 3. *Suppose that the matrix function $A(t)$ has a dichotomy spectrum $\mathcal{S}_{\mathrm{dich}} = \bigcup_{k=1}^{m}[\underline{\lambda}_k, \overline{\lambda}_k]$ satisfying (7), and suppose that the corresponding $A_k(t)$ have orders n_k, $k = 1,\dots,m$. Then*

$$\sum_{k=1}^{m} n_k\underline{\lambda}_k \leq \lim_{T\to\infty} \frac{1}{T}\int_0^{T} \operatorname{trace}\big[A(\sigma)\big]\,\mathrm{d}\sigma \leq \sum_{k=1}^{m} n_k\overline{\lambda}_k.$$

2.1.4. Regular Systems

In the special case where each of these intervals is a point (not necessarily unique), the spectrum is known as a *point spectrum.* Here, each λ_i in the point spectrum equals a Lyapunov exponent, and the system is said to be *regular.* Moreover, in this case the lim sups in (5) and (6) can be replaced by limits, cf. (Dieci *et al.*, 1997).

Clearly, time-invariant systems are regular and the elements of the point spectrum are the real part of the eigenvalues of A. Similarly, if the matrix function $A(t)$ is periodic, then the Floquet theory (see (Rugh, 1996)) for a description) states that there exists a change of variables so that the resulting equation is time-invariant. The resulting point spectrum coincides with the Floquet one.

In general, however, systems will not have point spectra. An example of a 2×2 real matrix with almost periodic coefficients is given in (Millionščikov, 1969). Unfortunately, regularity is hard to verify for any particular system, though all time-invariant and periodic systems are regular. In these two cases, the spectral values are the magnitudes of the eigenvalues and Floquet characteristic exponents of the system. For systems involving a flow with an invariant probability measure, Oseledeč's multiplicative ergodic theory states that regularity occurs with probability one (Arnold, 1998).

2.2. Assumptions on the System

We will consider integral operators of the form $\boldsymbol{F} := \boldsymbol{G}^*\boldsymbol{G}$, where $\boldsymbol{G}$ admits a stabilizable and detectable state-space realization[2] given by (3) with $D(t) = I$. We will need the following additional assumptions. First of all, we need to differentiate between the stable and antistable dynamics of Σ_G. To this end, we make the following standing assumption:

Assumption 1. *The matrix $A(t)$ admits an exponential dichotomy of rank $n_{\mathrm{p_d}}$. Moreover, the antistable component has dichotomy spectrum*

$$\Lambda_{\mathrm{p}} = [\underline{p}_1, \overline{p}_1] \cup [\underline{p}_2, \overline{p}_2] \cup \cdots \cup [\underline{p}_m, \overline{p}_m]$$

with dimensions $n_1, \ldots, n_m$ and $\sum_{k=1}^{m} n_k = n_{\mathrm{p}}$, where $n_{\mathrm{p_d}} + n_{\mathrm{p}} = n$.

We will also need to differentiate between the stable and antistable zero dynamics of Σ_F. Thus, we put the following restriction:

Assumption 2. *The matrix $A^{\times}(t) = A(t) - B(t)C(t)$ admits an exponential dichotomy of rank $n_{\mathrm{z_d}}$. Moreover, the antistable component has dichotomy spectrum*

$$\Lambda_{\mathrm{z}} = [\underline{z}_1, \overline{z}_1] \cup [\underline{z}_2, \overline{z}_2] \cup \cdots \cup [\underline{z}_{m'}, \overline{z}_{m'}]$$

with dimensions $n'_1, \ldots, n'_{m'}$ and $\sum_{k=1}^{m'} n'_k = n_{\mathrm{z}}$, where $n_{\mathrm{z_d}} + n_{\mathrm{z}} = n$.

In the time-invariant case, the continuity of $f(t)$ in (2) requires that $CB = 0$. In our case, we impose the following condition:

Assumption 3. *The operator $\boldsymbol{G}$ has a relative degree of at least two, i.e. $C(t)B(t) = 0$ for all t.*

Finally, the following assumption is made for technical reasons:

Assumption 4. *The derivatives $\dot{B}(t)$ and $\dot{C}(t)$ are bounded.*

[2] A realization (3) is stabilizable if there exists a bounded $F(t)$ such that $A(t) + B(t)F(t)$ is uniformly exponentially stable. Similarly, it is detectable if there exists a bounded $L(t)$ such that $A(t) + L(t)C(t)$ is uniformly exponentially stable.

2.3. Factorizations

2.3.1. Inner/Outer Factorization

In the following, we will need to compute an inner/outer factorization of a bounded operator $\boldsymbol{N}$. That is, we seek two systems with associated input/output operators $\boldsymbol{N}_{\rm i}$ and $\boldsymbol{N}_{\rm o}$ such that $\boldsymbol{N} = \boldsymbol{N}_{\rm i}\boldsymbol{N}_{\rm o}$, where $\boldsymbol{N}_{\rm i}$, $\boldsymbol{N}_{\rm o}$ and $\boldsymbol{N}_{\rm o}^{-1}$ are all causal and bounded, and $\|\boldsymbol{N}_{\rm i}w\|_2 = \|w\|_2$ for any $w \in \mathcal{L}_2$, where $\mathcal{L}_2$ is the space of square Lebesgue-integrable, measurable vector-valued functions defined over the half-line $[0, \infty)$.

For a bounded operator which has the state representation

$$\Sigma_N = \left[\begin{array}{c|c} A(t) & B(t) \\ \hline C(t) & I \end{array}\right] \tag{8}$$

we can derive state-space representations of its inner and outer factors in terms of the solution of a related Riccati differential equation. In the following result, we say that $(A(t), B(t))$ is *uniformly completely controllable* if there exist positive constants σ, α and β such that

$$\alpha I \le W(t, t+\sigma) \le \beta I,$$

where

$$W(t, t+\sigma) := \int_t^{t+\sigma} \Phi_A(t,\tau)B(\tau)B'(\tau)\Phi_A'(t,\tau)\,\mathrm{d}\tau \tag{9}$$

is the controllability Gramian of $(A(t), B(t))$.

The Riccati differential equation

$$-\dot{X}(t) = A'(t)X(t) + X(t)A(t) - X(t)B(t)B'(t)X(t) \tag{10}$$

has a *stabilizing* solution $X(t)$ if $X(t) = X'(t) \ge 0$, $X(t)$ is bounded and $A(t) - B(t)B'(t)X(t)$ is stable.

Lemma 4. *Suppose that $A(t)$ admits an exponential dichotomy and that the pair $(A(t), B(t))$ is stabilizable. Then (10) has a stabilizing solution $X(t)$ and:*

(i) If $A(t)$ is stable, then $X(t) \equiv 0$ for all t.

(ii) If $A(t)$ is antistable, then $\exists\, \epsilon > 0$ such that $\epsilon I \le X(t)$ for all t.

(iii) If $(A(t), B(t)) = \left(\begin{bmatrix} A_{\rm s}(t) & 0 \\ 0 & A_{\rm u}(t)\end{bmatrix}, \begin{bmatrix} B_{\rm s}(t) \\ B_{\rm u}(t)\end{bmatrix}\right)$ admits an exponential dichotomy as in Lemma 2, then $X(t) = \begin{bmatrix} 0 & 0 \\ 0 & X_{\rm u}(t)\end{bmatrix}$ and $\exists\, \epsilon > 0$ such that $\epsilon I \le X_{\rm u}(t)$ for all t.

Proof. Note that the solution to the general equation (10) and that of item (iii) are related by $X(t) \mapsto T'(t)X(t)T(t)$ where $T(t)$ is the Lyapunov transformation of Lemma 2. Thus, to prove the existence of a general solution it is enough to prove (i)—which is trivial—and (ii). Note that the existence of the solution follows from the general result of (Vojtenko, 1987) that the Riccati differential equation has a solution iff the corresponding Hamiltonian has an exponential dichotomy. In this specific case, the Hamiltonian is block-diagonal and hence has an exponential dichotomy iff $A(t)$ has one. However, as we need to show the boundedness (from both above and below) of the solution, we will first show its existence and this will lead to the required bounds.

To prove (ii), we use the result of (Ilchmann and Kern, 1987), which states that $(A(t), B(t))$ is uniformly stabilizable iff the pair $(A_{\mathrm{u}}(t), B_{\mathrm{u}}(t))$ is uniformly completely controllable. Note that in (ii), $A_{\mathrm{u}} \equiv A$ and $B_{\mathrm{u}} \equiv B$.

To prove item (ii), our approach follows that of (Kalman, 1960); see also (Ravi *et al.*, 1991). Consider the finite-horizon matrix equation

$$\dot{Q}_T(t) = A(t)Q_T(t) + Q_T(t)A'(t) - B(t)B'(t), \quad Q_T(T) = 0.$$

The solution is $Q_T(t) = W(t,T)$, where $W(t,T)$ is given by (9). Note that

$$\alpha I \leq Q_T(t) \leq \beta I \quad \text{for } t < T - \sigma.$$

For two terminal times $T_2 > T_1$ it is straightforward to check that $Q_{T_2}(t) \geq Q_{T_1}(t)$ for all $t < T_1 - \sigma$. Thus $\{Q_T(\cdot)\}$, indexed by T, is a nondecreasing sequence of continuous functions that is bounded from below. Consequently, there exists a (unique) bounded function $Q(t)$ defined as $\lim_{T\to\infty} Q_T(t) = Q(t)$. As in (Kalman, 1960), it follows that $Q(t)$ satisfies

$$\dot{Q}(t) = A(t)Q(t) + Q(t)A'(t) - B(t)B'(t)$$

and $\alpha I \leq Q(t) \leq \beta I$. Thus $Q(t)$ is invertible. It is straightforward to check that $X(t) = Q^{-1}(t)$ is the solution to (10) and that it is bounded from both above and below. It remains to show that $A(t) - B(t)B'(t)X(t)$ is stable. This follows from the fact that due to the bounds on $Q(t)$ it is a Lyapunov transformation and thus $A(t) - B(t)B'(t)X(t)$ is kinematically similar to $-A'(t)$:

$$\dot{Q}(t) = \big(A(t) - B(t)B'(t)X(t)\big)Q(t) - Q(t)\big(-A'(t)\big)$$

and $-A'(t)$ is stable since $A(t)$ is anti-stable. ■

Lemma 5. *Consider the operator $\boldsymbol{N}$ with stabilizable state-space representation (8). Furthermore, assume that $A^{\times}(t)$ has an exponential dichotomy. Then $\boldsymbol{N}$ has an inner/outer factorization $\boldsymbol{N} = \boldsymbol{N}_{\mathrm{i}}\boldsymbol{N}_{\mathrm{o}}$. State-space representations for the two factors are*

$$\Sigma_{N_{\mathrm{i}}} = \left[\begin{array}{c|c} A(t) - B(t)B'(t)X(t) - B(t)C(t) & B(t) \\ \hline -B'(t)X(t) & I \end{array}\right] \tag{11}$$

and

$$\Sigma_{N_{\mathrm{o}}} = \left[\begin{array}{c|c} A(t) & B(t) \\ \hline B'(t)X(t) + C(t) & I \end{array}\right], \tag{12}$$

where $X(t)$ is the solution of the Riccati differential equation

$$-\dot{X}(t) = A^{\times'}(t)X(t) + X(t)A^{\times}(t) - X(t)B(t)B'(t)X(t). \tag{13}$$

Proof. The pair $(A^{\times}(t), B(t))$ is stabilizable iff $(A(t), B(t))$ is so. Thus, since $A^{\times}(t)$ has exponential dichotomy, the Riccati differential equation (13) has a stabilizing solution. That rest of the proof is a straightforward extension of the time-invariant result which can be obtained by the general formulae for inner/outer factorizations that are found on pp. 367 and 368 in (Zhou *et al.*, 1996). ■

2.3.2. Coprime Factorization

In order to associate Szegő's limit theorem with the appropriate state space representation, we will need to compute a left coprime factorization of an operator $\boldsymbol{G}$. The pair of systems with associated operators $\boldsymbol{N}$ and $\boldsymbol{M}$ form a *left coprime factorization* of $\boldsymbol{G}$ if $\boldsymbol{M}$ and $\boldsymbol{N}$ are both bounded operators, $\boldsymbol{M}^{-1}$ exists as a causal integral operator, $\boldsymbol{G} = \boldsymbol{M}^{-1}\boldsymbol{N}$, and $\boldsymbol{N}$ and $\boldsymbol{M}$ are left coprime; that is, there exist two bounded integral operators $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $\boldsymbol{MY} + \boldsymbol{NX} = \boldsymbol{I}$.

For the operator which has the state-space representation

$$\Sigma_G = \left[\begin{array}{c|c} A(t) & B(t) \\ \hline C(t) & I \end{array}\right] \tag{14}$$

we can derive state-space representations of its left coprime factors in terms of the solution of a related Riccati differential equation.

Lemma 6. *Suppose that $A(t)$ admits an exponential dichotomy and that the pair $(C(t), A(t))$ is detectable. Then the Riccati differential equation*

$$\dot{Y}(t) = A(t)Y(t) + Y(t)A'(t) - Y(t)C'(t)C(t)Y(t) \tag{15}$$

has a stabilizing solution $Y(t)$ and:

(i) If $A(t)$ is stable, then $Y(t) \equiv 0$ for all t.

(ii) If $A(t)$ is antistable, then $\exists\, \epsilon > 0$ such that $\epsilon I \leq Y(t)$ for all t.

(iii) If $(C(t), A(t)) = \left(\begin{bmatrix} C_{\mathrm{s}}(t) & C_{\mathrm{u}}(t) \end{bmatrix}, \begin{bmatrix} A_{\mathrm{s}}(t) & 0 \\ 0 & A_{\mathrm{u}}(t) \end{bmatrix}\right)$ admits an exponential dichotomy as in Lemma 2, then $Y(t) = \begin{bmatrix} 0 & 0 \\ 0 & Y_{\mathrm{u}}(t) \end{bmatrix}$ and $\exists\, \epsilon > 0$ such that $\epsilon I \leq Y_{\mathrm{u}}(t)$ for all t.

Proof. It is similar to that of Lemma 4 and therefore is omitted. ■

Lemma 7. *Suppose that the operator $\boldsymbol{G}$ admits a detectable state-space representation (14) and that $A(t)$ admits an exponential dichotomy. Then $\boldsymbol{G}$ admits a left coprime factorization $\boldsymbol{G} = \boldsymbol{M}^{-1}\boldsymbol{N}$, where $\boldsymbol{M}$ is co-inner, both of the two factors are bounded and have state-space representations*

$$\Sigma_M = \left[\begin{array}{c|c} A(t) - Y(t)C'(t)C(t) & -Y(t)C'(t) \\ \hline C(t) & I \end{array}\right]$$

and

$$\Sigma_N = \left[\begin{array}{c|c} A(t) - Y(t)C'(t)C(t) & B(t) - Y(t)C'(t) \\ \hline C(t) & I \end{array}\right]. \tag{16}$$

Proof. The proof follows the time-invariant result which can be obtained by the general formulae for left coprime factorizations that are found on p. 370 in (Zhou *et al.*, 1996). ■

We note that for the class of operators considered in Section 2.2, the operator $\boldsymbol{G}$ satisfies the assumptions of Lemma 7. Thus we can write

$$\boldsymbol{F} = \boldsymbol{G}^*\boldsymbol{G} = [\boldsymbol{M}^{-1}\boldsymbol{N}]^*[\boldsymbol{M}^{-1}\boldsymbol{N}] = \boldsymbol{N}^*\boldsymbol{N}$$

since $\boldsymbol{M}$ is co-inner.

We now show that the operator $\boldsymbol{N}$ satisfies the assumptions of Lemma 5. In particular, we write the state-space representation (8) as

$$\Sigma_N = \left[\begin{array}{c|c} A_N(t) & B_N(t) \\ \hline C_N(t) & I \end{array}\right] := \left[\begin{array}{c|c} A(t) - Y(t)C'(t)C(t) & B(t) - Y(t)C'(t) \\ \hline C(t) & I \end{array}\right].$$

Then

$$\begin{aligned} A_N^\times(t) &:= A_N(t) - B_N(t)C_N(t) \\ &= A(t) - Y(t)C'(t)C(t) - B(t)C(t) + Y(t)C'(t)C(t) = A^\times(t), \end{aligned} \tag{17}$$

so that $A_N^\times(t)$ admits an exponential dichotomy since $A^\times(t)$ does. Similarly, since $A_N(t)$ is stable, the pair $(A_N(t), B_N(t))$ is stabilizable.

This allows us to compute an inner/outer factorization of $\boldsymbol{N}$ as in Lemma 5. In particular, $\boldsymbol{F} = \boldsymbol{N}^*\boldsymbol{N} = \boldsymbol{N}_\mathrm{o}^*\boldsymbol{N}_\mathrm{o}$, where

$$\Sigma_{N_\mathrm{o}} = \left[\begin{array}{c|c} A_N(t) & B_N(t) \\ \hline B_N'(t)X_N(t) + C_N(t) & I \end{array}\right] \tag{18}$$

and $X_N(t)$ is the stabilizing solution of the Riccati differential equation

$$-\dot{X}_N(t) = A_N^{\times\prime}(t)X_N(t) + X_N(t)A_N^\times(t) - X_N(t)B_N(t)B_N'(t)X_N(t)$$

with $A_N^\times(t) = A_N(t) - B_N(t)C_N(t)$. Finally, using (17), we can write this as

$$-\dot{X}_N(t) = A^{\times\prime}(t)X_N(t) + X_N(t)A^\times(t) - X_N(t)B_N(t)B_N'(t)X_N(t). \tag{19}$$

3. Abstract Szegő Theorem

For continuous-time systems, (Dym and Ta'assan, 1981) provided an abstract version of Szegő's limit theorem. In particular, let $\boldsymbol{F}$ be a bounded integral operator with an $m \times m$ matrix-valued kernel $F(t,s)$ acting on $\mathcal{L}_2$, and let P_T be the projection defined as

$$P_T x(s) = \begin{cases} x(s) & \text{if } 0 \le s \le T, \\ 0 & \text{otherwise} \end{cases}$$

for $0 \le T < \infty$.

In the sequel, we define $\hat{\boldsymbol{F}} = \boldsymbol{F} - I$. It can be shown that $\log\det(I + P_T\hat{\boldsymbol{F}}P_T)$ serves as a natural continuous analogue of Szegő's limit. For our purposes, the main result of (Dym and Ta'assan, 1981) is the following.

Theorem 8. *(Dym and Ta'assan, 1981) Let* $\hat{\boldsymbol{F}}$ *be an integral operator with continuous kernel. Assume that* $P_T\hat{\boldsymbol{F}}P_T$ *is trace class and that* $I + P_T\hat{\boldsymbol{F}}P_T$ *is invertible for all* $0 \le t \le T$*. Then*

$$\log\det(I + P_T\hat{\boldsymbol{F}}P_T) = \operatorname{trace}\big[P_T\boldsymbol{U}_-P_T + P_T\boldsymbol{U}_+P_T\big], \tag{20}$$

where $\boldsymbol{F} = (I + \boldsymbol{U}_-)(I + \boldsymbol{U}_+)$ *is a right-factorization.*

To use this result on the operator $\boldsymbol{F}$ considered in Section 2.2, we first need to show that $\hat{\boldsymbol{F}} := \boldsymbol{F} - I$ is a trace class.

Lemma 9. *(Iglesias, 2002) Let* $\hat{\boldsymbol{F}} = \boldsymbol{F} - I$*, where* $\boldsymbol{F}$ *is the integral operator associated with the state-space representation (3). Furthermore, assume that the derivatives* $\dot{B}(t)$ *and* $\dot{C}(t)$ *are bounded. Then* $P_T\hat{\boldsymbol{F}}P_T$ *is a trace-class operator.*

Theorem 10. *Let* $\boldsymbol{F}$ *be the operator defined as in Section 2.2. Then*

$$\lim_{T\to\infty}\frac{1}{4T}\log\det(I + P_T\hat{\boldsymbol{F}}P_T) = \lim_{T\to\infty}\frac{1}{2T}\int_0^T \operatorname{trace}\big[\hat{N}_{\mathrm{o}}(t,t)\big]\,\mathrm{d}t. \tag{21}$$

Proof. The system associated with the input-ouput operator $\boldsymbol{F}$ has a state-space representation given by

$$\Sigma_F = \left[\begin{array}{c|c} \tilde{A}(t) & \tilde{B}(t) \\ \hline \tilde{C}(t) & 0 \end{array}\right] := \left[\begin{array}{cc|c} A(t) - B(t)C(t) & 0 & B(t) \\ C'(t)C(t) & -(A(t) - B(t)C(t))' & -C'(t) \\ \hline -C(t) & -B'(t) & 0 \end{array}\right].$$

The boundedness of $\tilde{A}$, $\tilde{B}$, $\dot{\tilde{B}}$, $\tilde{C}$ and $\dot{\tilde{C}}$ follows from that of the corresponding constituent matrices. Similarly, the continuity of $\tilde{B}$ and $\tilde{C}$ follows from that of B and C. Thus, from Lemma 9, $\hat{\boldsymbol{F}} := \boldsymbol{F} - I$ is a trace class. That $I + P_T\hat{\boldsymbol{F}}P_T$ is invertible follows from the state-space description. We can now apply Theorem 8. In our particular case, we get

$$\boldsymbol{U}_- = \boldsymbol{N}_{\mathrm{o}}^* - I \quad \text{and} \quad \boldsymbol{U}_+ = \boldsymbol{N}_{\mathrm{o}} - I.$$

Furthermore, we can evaluate the right-hand side of (20), as in the proof of Theorem 3.1 in (Dym and Ta'assan, 1981), see also (Gohberg and Kreĭn, 1969, Ch. III.11), as follows:

$$\operatorname{trace}\big[P_T\hat{\boldsymbol{N}}_{\mathrm{o}}^*P_T + P_T\hat{\boldsymbol{N}}_{\mathrm{o}}P_T\big] = 2\int_0^T \operatorname{trace}[\hat{N}_{\mathrm{o}}(t,t)]\,\mathrm{d}t.$$

■

In the next section we will show how the integral defined by (21) is tied to the antistable pole and zero dynamics of the system as in the time-invariant case.

4. Main Results

Now we can provide a connection between the general Szegő-type limit considered in the previous section and the antistable dynamics of the given system.

Theorem 11. *For the integral operator $\boldsymbol{F} = \boldsymbol{G}^*\boldsymbol{G}$, where $\boldsymbol{G}$ is the integral operator admitting the state-space realization (3) which satisfies Assumptions 1–4, define $\hat{\boldsymbol{F}} = \boldsymbol{F} - I$. Then*

$$\sum_{i=1}^{m} n_i \underline{p}_i - \sum_{i=1}^{m'} n_i \overline{z}_i \leq \lim_{T\to\infty} \frac{1}{2T} \log\det(I + P_T \hat{\boldsymbol{F}} P_T) \leq \sum_{i=1}^{m} n_i \overline{p}_i - \sum_{i=1}^{m'} n_i \underline{z}_i, \tag{22}$$

where $\underline{p}_i$ and $\overline{p}_i$, $i = 1,\dots,m$ are the spectral values defined in Assumption 1, and $\underline{z}_i$ and $\overline{z}_i$, $i = 1,\dots,m'$ are the spectral values defined in Assumption 2.

Proof. By Theorem 10 we need to compute the kernel of $\hat{\boldsymbol{N}}_{\mathrm{o}} = \boldsymbol{N}_{\mathrm{o}} - I$. From (18) we have

$$\hat{N}_{\mathrm{o}}(t,\tau) = \big[B_N'(t)X_N(t) + C_N(t)\big]\Phi_{A_N}(t,\tau)B_N(\tau), \quad t \geq \tau,$$

and thus

$$\begin{aligned}\hat{N}_{\mathrm{o}}(t,t) &= \big[B_N'(t)X_N(t) + C_N(t)\big]B_N(t)\\ &= B_N'(t)X_N(t)B_N(t) + C(t)\big[B(t) - Y(t)C(t)'\big]\\ &= B_N'(t)X_N(t)B_N(t) - C(t)Y(t)C'(t), \end{aligned} \tag{23}$$

where (23) follows from Assumption 3.

Moreover, from Lemma 4 we obtain

$$\mathrm{trace}\big[B_N'(t)X_N(t)B_N(t)\big] = \mathrm{trace}\big[B_{N_{\mathrm{u}}}(t)B_{N_{\mathrm{u}}}'(t)X_{N_{\mathrm{u}}}(t)\big].$$

Similarly, Lemma 6 gives

$$\mathrm{trace}\big[C(t)Y(t)C'(t)\big] = \mathrm{trace}\big[C_{\mathrm{u}}'(t)C_{\mathrm{u}}(t)Y_{\mathrm{u}}'(t)\big].$$

Now, note that the Riccati differential equation for $X_{N\mathrm{u}}$, given by (19), can be rewritten as

$$\dot{X}_{N\mathrm{u}}(t) = \Big(-\big[A_{\mathrm{u}}^{\times}(t) - B_{N_{\mathrm{u}}}(t)B_{N_{\mathrm{u}}}'(t)X_{N_{\mathrm{u}}}(t)\big]'\Big)X_{N\mathrm{u}}(t) + X_{N\mathrm{u}}(t)\big(-A_{\mathrm{u}}^{\times}(t)\big).$$

For any t and τ the solution of this equation is

$$X_{N_{\mathrm{u}}}(t) = \Phi_{-[A_{\mathrm{u}}^{\times} - B_{N_{\mathrm{u}}}B_{N_{\mathrm{u}}}'X_{N_{\mathrm{u}}}']'}(t,\tau)X_{N_{\mathrm{u}}}(\tau)\Phi_{A_{\mathrm{u}}^{\times}}(\tau,t).$$

In particular, let $t = T$ and $\tau = 0$. Now, taking the logarithm of the determinant of both sides of this equation leads to

$$\begin{aligned}\log\det X_{N_{\mathrm{u}}}(T) = {}& \log\det X_{N_{\mathrm{u}}}(0) + \log\det \Phi_{-[A_{\mathrm{u}}^{\times} - B_{N_{\mathrm{u}}}B_{N_{\mathrm{u}}}'X_{N_{\mathrm{u}}}']'}(T,0)\\ &+ \log\det \Phi_{A_{\mathrm{u}}^{\times}}(0,T).\end{aligned}$$

Furthermore,

$$\lim_{T\to\infty} \frac{1}{T}\Big(\log\det X_{N_\mathrm{u}}(T) - \log\det X_{N_\mathrm{u}}(0)\Big) = 0,$$

since $X_{N\mathrm{u}}(t)$ is bounded from both above and below. Thus

$$\lim_{T\to\infty} \frac{1}{T}\left(\log\det \Phi_{-[A_\mathrm{u}^\times - B_{N_\mathrm{u}}B'_{N_\mathrm{u}}X_{N_\mathrm{u}}]}(T,0) + \log\det\Phi_{A_\mathrm{u}^\times}(0,T)\right) = 0.$$

Applying Lemma 1 to both the transition matrices yields

$$\begin{aligned}
&\log\det \Phi_{-[A_\mathrm{u}^\times - B_{N_\mathrm{u}}B'_{N_\mathrm{u}}X_{N_\mathrm{u}}]'}(T,0)\\
&\quad = \int_0^T \mathrm{trace}[B_{N_\mathrm{u}}(t)B'_{N_\mathrm{u}}(t)X'_{N_\mathrm{u}}(t) - A_\mathrm{u}^\times(t)]\,\mathrm{d}t\\
&\quad = \int_0^T \mathrm{trace}[B_{N_\mathrm{u}}(t)B'_{N_\mathrm{u}}(t)X'_{N_\mathrm{u}}(t)]\,\mathrm{d}t - \int_0^T \mathrm{trace}[A_\mathrm{u}^\times(t)]\,\mathrm{d}t
\end{aligned}$$

and

$$\log\det\Phi_{A_\mathrm{u}^\times}(0,T) = -\int_0^T \mathrm{trace}\left[A_\mathrm{u}^\times(t)\right]\mathrm{d}t.$$

Thus

$$\lim_{T\to\infty}\frac{1}{2T}\int_0^T \mathrm{trace}[B_{N_\mathrm{u}}(t)B'_{N_\mathrm{u}}(t)X'_{N_\mathrm{u}}(t)]\,\mathrm{d}t = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{trace}[A_\mathrm{u}^\times(t)]\,\mathrm{d}t.$$

Similarly, working with the Riccati differential equation (15), we can show that

$$\lim_{T\to\infty}\frac{1}{2T}\int_0^T \mathrm{trace}\left[C'_\mathrm{u}(t)C_\mathrm{u}(t)Y_\mathrm{u}(t)\right]\mathrm{d}t = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{trace}\left[A_\mathrm{u}(t)\right]\mathrm{d}t.$$

By Theorem 10 and (21), we have

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{4T}\log\det(I + P_T\hat{\boldsymbol{F}}P_T)\\
&\quad = \lim_{T\to\infty}\frac{1}{2T}\int_0^T \mathrm{trace}\left[\hat{N}_o(t,t)\right]\mathrm{d}t\\
&\quad = \lim_{T\to\infty}\frac{1}{2T}\int_0^T \mathrm{trace}\left[B_{N_\mathrm{u}}(t)B'_{N_\mathrm{u}}(t)X'_{N_\mathrm{u}}(t) - C'_\mathrm{u}(t)C_\mathrm{u}(t)Y_\mathrm{u}(t)\right]\mathrm{d}t\\
&\quad = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{trace}\left[A_\mathrm{u}^\times(t) - A_\mathrm{u}(t)\right]\mathrm{d}t.
\end{aligned}$$

Finally, by Lemma 3 we obtain (22). ■

Corollary 12. *If, in addition to the assumptions of Theorem 11, the antistable component of the system dynamics* A_{u} *and* $A_{\mathrm{u}}^{\times}$ *are regular, then*

$$\lim_{T\to\infty}\frac{1}{2T}\log\det(I+P_T\hat{\boldsymbol{F}}P_T)=\sum_{i=1}^{m}n_ip_i-\sum_{i=1}^{m'}n_i'z_i,$$

where p_i *and* z_i *are the Lyapunov exponents of the antistable component* A_{u} *and* $A_{\mathrm{u}}^{\times}$*, respectively.*

Proof. It follows directly from the definition of the regularity and Theorem 11. ■

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