Int. J. Appl. Math. Comput. Sci., 2016, Vol. 26, No. 4, 841–854
DOI: 10.1515/amcs-2016-0060

PERFORMANCE EVALUATION OF AN $M/G/n$-TYPE QUEUE WITH BOUNDED CAPACITY AND PACKET DROPPING

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A queueing system of the $M/G/n$-type, $n \geq 1$, with a bounded total volume is considered. It is assumed that the volumes of the arriving packets are generally distributed random variables. Moreover, the AQM-type mechanism is used to control the actual buffer state: each of the arriving packets is dropped with a probability depending on its volume and the occupied volume of the system at the pre-arrival epoch. The explicit formulae for the stationary queue-size distribution and the loss probability are found. Numerical examples illustrating theoretical formulae are given as well.

Keywords: AQM algorithms, finite buffer, loss probability, total packet volume, queue-size distribution.

1. Introduction

Queueing systems with bounded buffer capacities are natural models for in-depth investigation of various types of engineering, economic and transport phenomena. They are especially widely used in the analysis of processes occurring in nodes of packet telecommunication networks, and, in consequence, are one of the main analytic tools in network performance evaluation. Theoretically, single and batch arrivals are usually applied as models of the input packet flow. A kind of generalization of classical batch arrivals is an arriving process in which the incoming packets occur individually but have different volumes randomly distributed. In this model the notion of a “finite buffer” stands for a certain nonrandom maximal buffer capacity V, not for the maximal number of packets being allowed for waiting for service in the waiting room (Tikhonenko, 1991; 2005; Tikhonenko and Kempa, 2012; 2013; 2015). It seems such a model can be better adjusted to real-life packet-oriented networks modeling, in which the volume of the buffer measured in bytes (not in packets) is deterministic.

Due to finite resources of network switches (like, e.g., Internet routers), buffer overflows may occur, and some packets can be lost and must be retransmitted by the source host. Because of a complex nature of the Internet traffic, where the phenomena of, e.g., burstiness and self-similarity can be observed (Klemm *et al.*, 2003), packets can be lost in series. In practice, to reduce the risk of overflows in routers' input/output buffers, active queue management (AQM) mechanisms are used. The main idea of AQM is preventive probabilistic packet dropping even when the buffer is not saturated. In consequence, the queue of waiting packets is reduced and, in a long-term perspective, the arrival intensity is decreased and adjusted to transmission possibilities, as a reaction of the TCP protocol to packet losses. In the work of Floyd and Jacobson (1993), the first AQM-type algorithm, called random early detection (RED), was introduced. The RED approach defines a dropping function “filtering” the input flow and rejecting an incoming packet with a probability depending usually on the average or instantaneous queue size at the pre-arrival epoch. In the literature, different analytic forms of dropping functions were proposed: linear (Bonald *et al.*, 2000), doubly linear (GRED algorithm) (Floyd, 2000), exponential (REM algorithm) (Athuraliya *et al.*, 2001; Liu

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et al., 2005) and quadratic (Zhou *et al.*, 2006). One can find different results devoted to theory and applications of active queue management in the works of Chydziński (2010), Chydziński and Chróst (2011), Domańska *et al.* (2014), Floyd (2001), Hao and Wei (2005), Rosolen *et al.* (1999), Sun and Wang (2007), Suresh and Gol (2005), and Xiong *et al.* (2005). In particular, Chydziński and Chróst (2011) obtained a compact-form representation for the steady-state queue-size distribution in the $M/G/1$ system with a finite waiting room and an input flow controlled by a general-type dropping function, and Chydziński (2010) discussed the problem of stability of AQM-type models. New results on transient and equilibrium stochastic characteristics of queueing models with finite waiting rooms can also be found, e.g., in the work of Rusek *et al.* (2014), where the *MAP*-type arrival process is assumed, and in the paper by Woźniak *et al.* (2014), where, additionally, a cost-optimization problem of the $GI/M/1/N$-type model with single server vacations was solved via an evolutionary approach.

Kempa (2011) considered a Markovian model with a finite waiting room and a general-type dropping function, separately for the case of single and batch arrivals. The representations for different steady-state stochastic characteristics were found there, namely, the queue-size distribution, the number of packets (batches of packets) lost consecutively, and the time between two accepted arrivals. In the work of Tikhonenko and Kempa (2012), the formula for the queue-size distribution was obtained for a stationary $M/M/1/N$-type system with the above-mentioned generalized arrival process, in which the incoming packets have generally distributed volumes and the total system capacity is bounded. An extension of results obtained by Tikhonenko and Kempa (2012) for the case of a multi-channel model is included in their further work (Tikhonenko and Kempa, 2013). New results for time-dependent queue-size distributions in finite AQM-type models were obtained by Kempa (2013a; 2013b; 2013c).

In the article we generalize results of Tikhonenko and Kempa (2012; 2013) to the case of a multi-server system with Poisson arrivals but generally distributed service times. We replace the classical dropping function by an "accepting" function that qualifies the arriving packet with a probability that depends on the actual occupied volume of the system at the pre-arrival instant, and on the volume of the arriving packet.

Thus, the remaining part of the paper is organized as follows. In Section 2 we give the mathematical description of the queueing model considered, present the Markov process describing its evolution, and introduce the necessary notation. In Section 3 we present results for the "classical" $M/G/n/m$-type system without packet dropping. Section 4 contains the main result. In this section we build the system of Kolmogorov-type equations for the stationary queue-size distribution and find its solution. Section 5 contains numerical examples illustrating theoretical results, and in the last section some concluding remarks can be found.

2. Model and auxiliary results

Let us consider a multi-server queueing system in which successive packets arrive according to a Poisson process with intensity a, and are characterized by their volumes which are generally distributed positive random variables with a distribution function $L(\cdot)$. Packets are served individually with a general-type distribution function $B(\cdot)$, independently of their volumes. Sequences of successive inter-arrival and service times as well as volumes of the arriving packets are supposed to be totally independent. The total volume of the system, i.e., the sum of the volumes of all packets present in the system at an arbitrary time instant, is bounded by a non-random value V. The system contains n identical servers working independently and one waiting room with m places. Therefore, the total number of packets present in the system is bounded by $m+n$. In some cases we can assume that $m = \infty$; then, it is possible that the number of packets can be unlimited, while their total volume remains bounded by V.

Let $\eta(t)$ be the number of packets present in the system at a fixed time instant t. Of course, $\nu(t) = \min\big(\eta(t), n\big)$ denotes the number of packets being served at this instant. Assume additionally that the packets being on service at time t are numbered randomly, i.e., if exactly k packets are being served, then we have $k!$ possibilities of numbering from 1 to k, each one with the probability $1/k!$. This assumption helps us to simplify some future formulae and has no impact on the approach. Let $\xi_j^*(t)$ be the residual service time of the j-th packet being on service at time t.

Observe that the well-known "classical" $M/G/n/m$-type system is, in fact, a special case of the system described above when $L(x) = 0$ for $x \le 1$, $L(x) = 1$ for $x > 1$ and $V = m + n$. The evolution of the "classical" system can be described by the following Markov process:

$$\big(\eta(t);\, \xi_j^*(t), j = 1, \ldots, \nu(t)\big). \tag{1}$$

For the system considered, with a finite buffer and a bounded total volume, we need to supplement this process with additional characterizations. Let $\zeta_i(t)$ be the volume of the i-th packet present in the system at time t. Then $\sigma(t) = \sum_{i=1}^{\eta(t)} \zeta_i(t)$ is the "transient" volume of the system at time t, i.e., the sum of the volumes of all packets present in the system at this instant. Now, the system considered, denoted by $M/G/n/(m,V)$ (Tikhonenko, 1991; 2005; 2006; Tikhonenko and Kempa, 2012; 2013), can be described by the following Markov process:

$$\big(\eta(t);\, \zeta_i(t), i = 1,\dots,\eta(t); \xi_j^*(t), j = 1,\dots,\nu(t)\big). \tag{2}$$

Here we make the assumption that the waiting packets (in the case of $\eta(t) > n$) are numbered from $n+1$ to $\eta(t)$ successively as they occur. Of course, if $\eta(t) = 0$, then also $\sigma(t) = 0$.

Define the following vectors:

$$\begin{aligned} Y_k &= (y_1,\dots,y_k),\\ Y_k^j &= (y_1,\dots,y_{j-1},y_{j+1},\dots,y_k),\\ (Y_k, z) &= (y_1,\dots,y_k,z). \end{aligned} \tag{3}$$

Assume that there exists a stationary state of the system, i.e., the following limits exist in the sense of weak convergence:

$$\eta(t) \Rightarrow \eta, \quad \sigma(t) \Rightarrow \sigma, \quad \nu(t) \Rightarrow \nu,$$

$$\xi_j^*(t) \Rightarrow \xi_j^*, \quad \zeta_i(t) \Rightarrow \zeta_i. \tag{4}$$

In the stationary state the stochastic process (1) can be characterized by the functions

$$\widehat{w}_k(Y_l) = \mathbf{P}\{\eta = k;\, \xi_j^* < y_j, j = 1,\dots,l\}, \tag{5}$$

where $l = \min(k,n)$, $k = 1,\dots,n+m$. Denote by $\widehat{p}_0 = \mathbf{P}\{\eta = 0\}$ the stationary probability that the system is empty. It is easy to note that other stationary probabilities for the states of the "classical" $M/G/n/m$ system are equal to $\widehat{p}_k = \mathbf{P}\{\eta = k\} = \widehat{w}_k(\infty_l)$, where

$$\infty_l = (\underbrace{\infty,\dots,\infty}_{l}).$$

Similarly, one can describe the stochastic process (2) in the stationary state by the following functions:

$$\begin{aligned} g_k(x, Y_l)\,\mathrm{d}x =& \mathbf{P}\{\eta = k;\, \sigma \in [x, x+\mathrm{d}x);\\ & \xi_j^* < y_j, j = 1,\dots,l\}, \end{aligned} \tag{6}$$

where l is defined as previously.

Let us note that for the process (2) we can define the functions

$$w_k(Y_l) = \int_0^V g_k(x, Y_l)\,\mathrm{d}x, \tag{7}$$

having for this process the same probabilistic sense as the functions $\widehat{w}_k(Y_l)$ for the process (1).

Define also for the process (2) the following stationary probabilities:

$$p_0 = \mathbf{P}\{\eta = 0\}, \quad p_k = \mathbf{P}\{\eta = k\} = w_k(\infty_l), \tag{8}$$

where $k = 1,\dots,m+n$, and η stands for the number of packets present in the system in the stationary state.

Note that the functions $g_k(x, Y_l)$, $\widehat{w}_k(Y_l)$ and $w_k(Y_l)$ are symmetric with respect to permutations of components of the vector $Y_l = (y_1,\dots,y_l)$, owing to random enumeration of served packets in the system.

We end this section by introducing the necessary notation. Let us denote by P_{loss} the stationary loss probability, i.e., the probability that the incoming packet is lost. Besides, let $\rho = a/(n\mu)$ be the traffic load of the system. Lastly, by $F_*^{(k)}(\cdot)$ we denote the k-fold Stieltjes convolution of any distribution function $F(\cdot)$ of a non-negative random variable with itself, i.e.,

$$\begin{aligned} F_*^{(0)}(y) &\equiv 1,\\ F_*^{(k)}(y) &= \int_0^y F_*^{(k-1)}(y-x)\,\mathrm{d}F(x), \end{aligned} \tag{9}$$

where $k = 1, 2, \dots$.

3. Equations for the classical system without packet dropping

Let us take into consideration the "classical" $M/G/n/m$-type queueing system without packet dropping. Having in mind the notation introduced in the previous section and allowing for the aforementioned symmetry, we can write the following system of differential equations for the unknown functions $\widehat{w}_k(Y_l)$, where $k = 1,\dots,m+n$ and $l = \min(k,n)$:

$$0 = -a\widehat{p}_0 + \frac{\partial \widehat{w}_1(y)}{\partial y}\Big|_{y=0}, \tag{10}$$

$$\begin{aligned} &-\frac{\partial \widehat{w}_1(y)}{\partial y} + \frac{\partial \widehat{w}_1(y)}{\partial y}\Big|_{y=0}\\ &= a\widehat{p}_0 B(y) - a\widehat{w}_1(y) + 2\frac{\partial \widehat{w}_2(y,u)}{\partial u}\Big|_{u=0}, \end{aligned} \tag{11}$$

$$\begin{aligned} &-\sum_{i=1}^{k}\left[\frac{\partial \widehat{w}_k(Y_k)}{\partial y_i} - \frac{\partial \widehat{w}_k(Y_k)}{\partial y_i}\Big|_{y_i=0}\right]\\ &= \frac{a}{k}\sum_{i=1}^{k} \widehat{w}_{k-1}(Y_k^i)B(y_i)\\ &\quad - a\widehat{w}_k(Y_k) + (k+1)\frac{\partial \widehat{w}_{k+1}(Y_k,u)}{\partial u}\Big|_{u=0}, \end{aligned} \tag{12}$$

where $k = 2,\dots,n-1$, and, moreover,

$$\begin{aligned} &-\sum_{i=1}^{n}\left[\frac{\partial \widehat{w}_n(Y_n)}{\partial y_i} - \frac{\partial \widehat{w}_n(Y_n)}{\partial y_i}\Big|_{y_i=0}\right]\\ &= \frac{a}{n}\sum_{i=1}^{n} \widehat{w}_{n-1}(Y_n^i)B(y_i)\\ &\quad - a\widehat{w}_n(Y_n) + n\frac{\partial \widehat{w}_{n+1}(Y_{n-1},u)}{\partial u}\Big|_{u=0} B(y_n), \end{aligned} \tag{13}$$

$$-\sum_{i=1}^{n}\left[\frac{\partial\widehat{w}_k(Y_n)}{\partial y_i}-\frac{\partial\widehat{w}_k(Y_n)}{\partial y_i}\Big|_{y_i=0}\right]$$
$$=a\widehat{w}_{k-1}(Y_n)-a\widehat{w}_k(Y_n)$$
$$+n\frac{\partial\widehat{w}_{k+1}(Y_{n-1},u)}{\partial u}\Big|_{u=0}B(y_n), \quad (14)$$

where $k=n+1,\dots,n+m-1$, and

$$-\sum_{i=1}^{n}\left[\frac{\partial\widehat{w}_{n+m}(Y_n)}{\partial y_i}-\frac{\partial\widehat{w}_{n+m}(Y_n)}{\partial y_i}\Big|_{y_i=0}\right]$$
$$=a\widehat{w}_{n+m-1}(Y_n). \quad (15)$$

For the system (10)–(15) the following boundary conditions hold true:

$$a\widehat{w}_k(Y_k)=(k+1)\frac{\partial\widehat{w}_{k+1}(Y_k,u)}{\partial u}\Big|_{u=0}, \quad (16)$$

where $k=1,\dots,n-1$, and

$$a\widehat{w}_k(Y_n)=n\frac{\partial\widehat{w}_{k+1}(Y_{n-1},u)}{\partial u}\Big|_{u=0}B(y_n), \quad (17)$$

where $k=n,\dots,n+m-1$.

4. Queue-size distribution in the original system with packet dropping

Let us now take into consideration the original $M/G/n/(m,V)$-type queueing system with n independent servers, m places for waiting in the queue and the total volume of packets in the system bounded by V. We implement here the AQM algorithm of packet enqueueing defined as follows. Let $r(\cdot)$ be a right-hand continuous and nonincreasing function defined on the interval $[0,V]$, having the properties $r(0)\le 1$ and $r(V)\ge 0$. Assume that the arriving packet is characterized by a volume x, while the total volume of the system at the pre-arrival epoch equals y.

Later on, we shall analyze the following two types of packet dropping system $M/G/n/(m,V)$ “behavior”: (i) the incoming packet is “qualified” for service with the probability $r(x+y)$ and deleted with the probability $1-r(x+y)$; (ii) the incoming packet is “qualified” for service with the probability $r(y)$ and deleted with the probability $1-r(y)$. Moreover, each packet is lost if $x+y>V$, or if the number of packets present in the system at the pre-arrival epoch equals $m+n$.

If the packet arriving at time t is dropped, in our notation we have $\eta(t)=\eta(t^-)$ and $\sigma(t)=\sigma(t^-)$. In the case of the acceptance of the arriving packet, we have $\eta(t)=\eta(t^-)+1$ and $\sigma(t)=\sigma(t^-)+x$.

For the steady-state behavior of the first-type $M/G/n/(m,V)$ system, we obtain the following Kolmogorov-type equations:

$$0=-ap_0\int_0^V r(v)\,\mathrm{d}L(v)+\frac{\partial w_1(y)}{\partial y}\Big|_{y=0}, \quad (18)$$

$$-\frac{\partial w_1(y)}{\partial y}+\frac{\partial w_1(y)}{\partial y}\Big|_{y=0}$$
$$=ap_0B(y)\int_0^V r(v)\,\mathrm{d}L(v)$$
$$-a\int_0^V g_1(x,y)\int_0^{V-x} r(x+v)\,\mathrm{d}L(v)\,\mathrm{d}x$$
$$+2\frac{\partial w_2(y,u)}{\partial u}\Big|_{u=0}, \quad (19)$$

$$-\sum_{i=1}^{k}\left[\frac{\partial w_k(Y_k)}{\partial y_i}-\frac{\partial w_k(Y_k)}{\partial y_i}\Big|_{y_i=0}\right]$$
$$=\frac{a}{k}\sum_{i=1}^{k}B(y_i)$$
$$\times\int_0^V g_{k-1}(x,Y_k^i)\int_0^{V-x} r(x+v)\,\mathrm{d}L(v)\,\mathrm{d}x$$
$$-a\int_0^V g_k(x,Y_k)\int_0^{V-x} r(x+v)\,\mathrm{d}L(v)\,\mathrm{d}x$$
$$+(k+1)\frac{\partial w_{k+1}(Y_k,u)}{\partial u}\Big|_{u=0},$$
$$k=2,\dots,n-1, \quad (20)$$

$$-\sum_{i=1}^{n}\left[\frac{\partial w_n(Y_n)}{\partial y_i}-\frac{\partial w_n(Y_n)}{\partial y_i}\Big|_{y_i=0}\right]$$
$$=\frac{a}{n}\sum_{i=1}^{n}B(y_i)$$
$$\times\int_0^V g_{n-1}(x,Y_n^i)\int_0^{V-x} r(x+v)\,\mathrm{d}L(v)\,\mathrm{d}x$$
$$-a\int_0^V g_n(x,Y_n)\int_0^{V-x} r(x+v)\,\mathrm{d}L(v)\,\mathrm{d}x$$
$$+n\frac{\partial w_{n+1}(Y_{n-1},u)}{\partial u}\Big|_{u=0}B(y_n), \quad (21)$$

$$-\sum_{i=1}^{n}\left[\frac{\partial w_k(Y_n)}{\partial y_i}-\frac{\partial w_k(Y_n)}{\partial y_i}\Big|_{y_i=0}\right]$$
$$=a\int_0^V g_{k-1}(x,Y_n)\int_0^{V-x} r(x+v)\,\mathrm{d}L(v)\,\mathrm{d}x$$
$$-a\int_0^V g_k(x,Y_n)\int_0^{V-x} r(x+v)\,\mathrm{d}L(v)\,\mathrm{d}x$$
$$+n\frac{\partial w_{k+1}(Y_{n-1},u)}{\partial u}\Big|_{u=0}B(y_n),$$
$$k=n+1,\dots,n+m-1, \quad (22)$$

$$-\sum_{i=1}^{n}\left[\frac{\partial w_{n+m}(Y_n)}{\partial y_i}-\frac{\partial w_{n+m}(Y_n)}{\partial y_i}\Big|_{y_i=0}\right]$$
$$=a\int_0^V g_{n+m-1}(x,Y_n)\int_0^{V-x} r(x+v)\,\mathrm{d}L(v)\,\mathrm{d}x. \quad (23)$$

The boundary conditions are the following:

$$a\int_0^V g_k(x,Y_k)\int_0^{V-x} r(x+v)\,\mathrm{d}L(v)\,\mathrm{d}x = (k+1)\frac{\partial w_{k+1}(Y_k,u)}{\partial u}\Big|_{u=0}, \tag{24}$$

where $k=1,\dots,n-1$, and

$$a\int_0^V g_k(x,Y_n)\int_0^{V-x} r(x+v)\,\mathrm{d}L(v)\,\mathrm{d}x = n\frac{\partial w_{k+1}(Y_{n-1},u)}{\partial u}\Big|_{u=0}B(y_n) \tag{25}$$

for $k=n,\dots,n+m-1$.

Let

$$R(z)=\int_0^z r(V-z+v)\,\mathrm{d}L(v). \tag{26}$$

The function $R(z)$ has the following probability sense: it is the probability that an arriving packet is qualified for service if the free space in the buffer equals z at the pre-arrival epoch. With this notation, the final formulas for the queue-size distribution will be written directly using the convolution of the function $R(\cdot)$ (see also (47)). Inserting (26) into the system (18)–(25), we obtain

$$0=-ap_0R(V)+\frac{\partial w_1(y)}{\partial y}\Big|_{y=0}, \tag{27}$$

$$\begin{aligned}-\frac{\partial w_1(y)}{\partial y}+\frac{\partial w_1(y)}{\partial y}\Big|_{y=0} &= ap_0B(y)R(V)-a\int_0^V g_1(x,y)R(V-x)\,\mathrm{d}x\\ &\quad+2\frac{\partial w_2(y,u)}{\partial u}\Big|_{u=0},\end{aligned} \tag{28}$$

$$\begin{aligned}-\sum_{i=1}^{k}\left[\frac{\partial w_k(Y_k)}{\partial y_i}-\frac{\partial w_k(Y_k)}{\partial y_i}\Big|_{y_i=0}\right] &= \frac{a}{k}\sum_{i=1}^{k}B(y_i)\int_0^V g_{k-1}(x,Y_k^i)R(V-x)\,\mathrm{d}x\\ &\quad-a\int_0^V g_k(x,Y_k)R(V-x)\,\mathrm{d}x\\ &\quad+(k+1)\frac{\partial w_{k+1}(Y_k,u)}{\partial u}\Big|_{u=0},\\ &\qquad k=2,\dots,n-1,\end{aligned} \tag{29}$$

$$\begin{aligned}-\sum_{i=1}^{n}\left[\frac{\partial w_n(Y_n)}{\partial y_i}-\frac{\partial w_n(Y_n)}{\partial y_i}\Big|_{y_i=0}\right] &= \frac{a}{n}\sum_{i=1}^{n}B(y_i)\int_0^V g_{n-1}(x,Y_n^i)R(V-x)\,\mathrm{d}x\\ &\quad-a\int_0^V g_n(x,Y_n)R(V-x)\,\mathrm{d}x\\ &\quad+n\frac{\partial w_{n+1}(Y_{n-1},u)}{\partial u}\Big|_{u=0}B(y_n),\end{aligned} \tag{30}$$

$$\begin{aligned}-\sum_{i=1}^{n}\left[\frac{\partial w_k(Y_n)}{\partial y_i}-\frac{\partial w_k(Y_n)}{\partial y_i}\Big|_{y_i=0}\right] &= a\int_0^V g_{k-1}(x,Y_n)R(V-x)\,\mathrm{d}x\\ &\quad-a\int_0^V g_k(x,Y_n)R(V-x)\,\mathrm{d}x\\ &\quad+n\frac{\partial w_{k+1}(Y_{n-1},u)}{\partial u}\Big|_{u=0}B(y_n),\\ &\qquad k=n+1,\dots,n+m-1,\end{aligned} \tag{31}$$

$$-\sum_{i=1}^{n}\left[\frac{\partial w_{n+m}(Y_n)}{\partial y_i}-\frac{\partial w_{n+m}(Y_n)}{\partial y_i}\Big|_{y_i=0}\right] = a\int_0^V g_{n+m-1}(x,Y_n)R(V-x)\,\mathrm{d}x. \tag{32}$$

The boundary conditions (24)–(25) can be rewritten as follows:

$$a\int_0^V g_k(x,Y_k)R(V-x)\,\mathrm{d}x = (k+1)\frac{\partial w_{k+1}(Y_k,u)}{\partial u}\Big|_{u=0}, \tag{33}$$

where $k=1,\ \dots,n-1$, and

$$a\int_0^V g_k(x,Y_n)R(V-x)\,\mathrm{d}x = n\frac{\partial w_{k+1}(Y_{n-1},u)}{\partial u}\Big|_{u=0}B(y_n) \tag{34}$$

for $k=n,\dots,n+m-1$.

From the systems (10)–(17) and (27)–(34), the following main theorem follows.

Theorem 1. *Let the probability* $\widehat{p}_0$ *and the functions*

$$\widehat{w}_k(Y_l),\quad k=1,\dots,n+m,\quad l=\min(k,n),$$

satisfy the system of equations (10)–(17) and the normalization condition $\widehat{p}_0+\sum_{k=1}^{n+m}\widehat{w}_k(\infty_l)=1$, $l=\min(k,n)$, *for the classical system* $M/G/n/m$, *and* C *be an arbitrary real constant. Then the number* $p_0=C\widehat{p}_0$ *and the functions* $g_k(x,Y_l)$, *such that*

$$g_k(x,Y_l)\,\mathrm{d}x=C\widehat{w}_k(Y_l)\,\mathrm{d}R_*^{(k)}(x), \tag{35}$$

satisfy the system of equations (27)–(34) for the system with packet dropping $M/G/n/(m,V)$.

Proof. From (35) it follows that

$$\begin{aligned} w_k(Y_l) &= g_k(V, Y_l) \\ &= \int_0^V g_k(x, Y_l)\,\mathrm{d}x \\ &= C\widehat{w}_k(Y_l)R_*^{(k)}(V). \end{aligned} \tag{36}$$

Now, we can easily prove that, if we substitute $p_0 = C\widehat{p}_0$ and the functions $g_k(x, Y_l)$, $w_k(Y_l)$ from the relations (35) and (36) into the system of equations (27)–(34), we obtain the equations (10)–(17) for the number $\widehat{p}_0$ and functions $\widehat{w}_k(Y_l)$. ■

From Theorem 1 and the normalization condition $\sum_{k=0}^{m+n} p_k = 1$, the following corollary follows.

Corollary 1. *The stationary queue-size distribution p_k in the $M/G/n/(m, V)$ queueing system with packet dropping can be expressed as*

$$p_k = C\widehat{p}_k R_*^{(k)}(V), \quad k = 0, 1, \ldots, m+n, \tag{37}$$

where

$$C = \left[\sum_{k=0}^{n+m} \widehat{p}_k R_*^{(k)}(V)\right]^{-1},$$

$\widehat{p}_k$, $k = 0, 1, \ldots, m+n$, are stationary probabilities in the classical $M/G/n/m$ system, and $R_*^{(k)}(\cdot)$ is the k-fold Stieltjes convolution of the function $R(\cdot)$ with itself defined in (9).

Corollary 2. *The loss probability $P_{\rm loss}$ for the system $M/G/n/(m, V)$ can be computed as*

$$\begin{aligned} P_{\rm loss} = 1 - \frac{1}{a}\Bigg[&\sum_{k=1}^{n} i\frac{\partial w_k(\infty_{k-1}, u)}{\partial u}\Big|_{u=0} \\ &+ n\sum_{k=n+1}^{n+m} \frac{\partial w_k(\infty_{n-1}, u)}{\partial u}\Big|_{u=0}\Bigg], \end{aligned} \tag{38}$$

where a is the arrival intensity and functions $w_k(Y_l)$ can be obtained from the relation (36).

Proof. The formula (38) is a consequence of the stability condition

$$\begin{aligned} a(1 - P_{\rm loss}) = &\sum_{i=1}^{n} i\frac{\partial w_i(\infty_{i-1}, u)}{\partial u}\Big|_{u=0} \\ &+ n\sum_{i=n+1}^{n+m} \frac{\partial w_i(\infty_{n-1}, u)}{\partial u}\Big|_{u=0}. \end{aligned}$$

■

Note that in our investigation it was assumed that $m < \infty$. It is clear that, in the same way, we can analyze $M/G/n/(\infty, V)$-type queueing systems, but only under the condition that $\rho < 1$, because the essential point of our approach is the existence of a stationary queue-size distribution for the classical $M/G/n$-type system.

For the second type of system behavior, we have to replace the system (18)–(25) by the following equations:

$$0 = -ap_0 r(0)L(V) + \frac{\partial w_1(y)}{\partial y}\Big|_{y=0}, \tag{39}$$

$$\begin{aligned} &-\frac{\partial w_1(y)}{\partial y} + \frac{\partial w_1(y)}{\partial y}\Big|_{y=0} \\ &\quad= ap_0 r(0)L(V) \\ &\qquad - a\int_0^V g_1(x, y)r(x)L(V-x)\,\mathrm{d}x \\ &\qquad + 2\frac{\partial w_2(y, u)}{\partial u}\Big|_{u=0}, \end{aligned} \tag{40}$$

$$\begin{aligned} &-\sum_{i=1}^{k}\left[\frac{\partial w_k(Y_k)}{\partial y_i} - \frac{\partial w_k(Y_k)}{\partial y_i}\Big|_{y_i=0}\right] \\ &\quad= \frac{a}{k}\sum_{i=1}^{k} B(y_i)g_{k-1}(x, Y_k^i)r(x)L(V-x)\,\mathrm{d}x \\ &\qquad - a\int_0^V g_k(x, Y_k)r(x)L(V-x)\,\mathrm{d}x \\ &\qquad + (k+1)\frac{\partial w_{k+1}(Y_k, u)}{\partial u}\Big|_{u=0}, \\ &\qquad\qquad k = 2, \ldots, n-1, \end{aligned} \tag{41}$$

$$\begin{aligned} &-\sum_{i=1}^{n}\left[\frac{\partial w_n(Y_n)}{\partial y_i} - \frac{\partial w_n(Y_n)}{\partial y_i}\Big|_{y_i=0}\right] \\ &\quad= \frac{a}{n}\sum_{i=1}^{n} B(y_i)\int_0^V g_{n-1}(x, Y_n^i)r(x)L(V-x)\,\mathrm{d}x \\ &\qquad - a\int_0^V g_n(x, Y_n)r(x)L(V-x)\,\mathrm{d}x \\ &\qquad + n\frac{\partial w_{n+1}(Y_{n-1}, u)}{\partial u}\Big|_{u=0} B(y_n), \end{aligned} \tag{42}$$

$$\begin{aligned} &-\sum_{i=1}^{n}\left[\frac{\partial w_k(Y_n)}{\partial y_i} - \frac{\partial w_k(Y_n)}{\partial y_i}\Big|_{y_i=0}\right] \\ &\quad= a\int_0^V g_{k-1}(x, Y_n)r(x)L(V-x)\,\mathrm{d}x \\ &\qquad - a\int_0^V g_k(x, Y_n)r(x)L(V-x)\,\mathrm{d}x \\ &\qquad + n\frac{\partial w_{k+1}(Y_{n-1}, u)}{\partial u}\Big|_{u=0} B(y_n), \\ &\qquad\qquad k = n+1, \ldots, n+m-1, \end{aligned} \tag{43}$$

$$-\sum_{i=1}^{n}\left[\frac{\partial w_{n+m}(Y_n)}{\partial y_i}-\frac{\partial w_{n+m}(Y_n)}{\partial y_i}\Big|_{y_i=0}\right.$$
$$=a\int_0^V g_{n+m-1}(x,Y_n)r(x)L(V-x)\,\mathrm{d}x, \tag{44}$$

$$a\int_0^V g_k(x,Y_k)r(x)L(V-x)\,\mathrm{d}x = (k+1)\frac{\partial w_{k+1}(Y_k,u)}{\partial u}\Big|_{u=0}, \quad k=1,\dots,n-1, \tag{45}$$

$$a\int_0^V g_k(x,Y_n)r(x)L(V-x)\,\mathrm{d}x = n\frac{\partial w_{k+1}(Y_{n-1},u)}{\partial u}\Big|_{u=0}B(y_n), \quad k=n,\dots,n+m-1. \tag{46}$$

Setting

$$R(z)=r(V-z)L(z), \tag{47}$$

we can rewrite Eqns. (39)–(46) in the form of (27)–(34). Therefore, in this case, we obtain the solution in the form of (36), (37), where $R(z)$ is determined by the relation (47).

It is clear that classical AQM is a special case of the second type of system "behavior". Indeed, let the numbers $r(i)$, $i=0,\dots,m+n-1$, be accepting probabilities for an incoming packet, on condition that there are i packets in the system just before the arriving epoch. In this case, we have $L(x)=0$ for $x\le 1$, and $L(x)=1$ for $x>1$, $V=n+m<\infty$. Then Eqns. (39)–(46) take the following form:

$$0=-ap_0r(0)+\frac{\partial w_1(y)}{\partial y}\Big|_{y=0}, \tag{48}$$

$$-\frac{\partial w_1(y)}{\partial y}+\frac{\partial w_1(y)}{\partial y}\Big|_{y=0} = ap_0r(0)B(y) - aw_1(y)r(1)+2\frac{\partial w_2(y,u)}{\partial u}\Big|_{u=0}, \tag{49}$$

$$-\sum_{i=1}^{k}\left[\frac{\partial w_k(Y_k)}{\partial y_i}-\frac{\partial w_k(Y_k)}{\partial y_i}\Big|_{y_i=0}\right] = \frac{a}{k}\sum_{i=1}^{k}w_{k-1}(Y_k^i)r(k-1)B(y_i) - aw_k(Y_k)r(k)+(k+1)\frac{\partial w_{k+1}(Y_k,u)}{\partial u}\Big|_{u=0}, \quad k=2,\dots,n-1, \tag{50}$$

$$-\sum_{i=1}^{n}\left[\frac{\partial w_n(Y_n)}{\partial y_i}-\frac{\partial w_n(Y_n)}{\partial y_i}\Big|_{y_i=0}\right] = \frac{a}{n}\sum_{i=1}^{n}w_{n-1}(Y_n^i)r(n-1)B(y_i) - aw_n(Y_n)r(n) + n\frac{\partial w_{n+1}(Y_{n-1},u)}{\partial u}\Big|_{u=0}B(y_n), \tag{51}$$

$$-\sum_{i=1}^{n}\left[\frac{\partial w_k(Y_n)}{\partial y_i}-\frac{\partial w_k(Y_n)}{\partial y_i}\Big|_{y_i=0}\right] = aw_{k-1}(Y_n)r(k-1)-aw_kr(k)(Y_n) + n\frac{\partial w_{k+1}(Y_{n-1},u)}{\partial u}\Big|_{u=0}B(y_n), \quad k=n+1,\dots,n+m-1, \tag{52}$$

$$-\sum_{i=1}^{n}\left[\frac{\partial w_{n+m}(Y_n)}{\partial y_i}-\frac{\partial w_{n+m}(Y_n)}{\partial y_i}\Big|_{y_i=0}\right] = aw_{n+m-1}(Y_n)r(n+m-1), \tag{53}$$

$$aw_k(Y_k)r(k)=(k+1)\frac{\partial w_{k+1}(Y_k,u)}{\partial u}\Big|_{u=0}, \quad k=1,\dots,n-1, \tag{54}$$

$$aw_k(Y_n)r(k)=n\frac{\partial w_{k+1}(Y_{n-1},u)}{\partial u}\Big|_{u=0}B(y_n), \quad k=n,\dots,n+m-1. \tag{55}$$

It can be easily proved by direct substitution that the solution of the system (48)–(55) has the form

$$p_0=C\widehat{p}_0, \quad p_k=C\widehat{p}_k\prod_{i=0}^{k-1}(1-d_i), \tag{56}$$

where $k=1,\dots,n+m$, and $\widehat{p}_k$, $k=0,\dots,n+m$, are the stationary probabilities in the classical $M/G/n/m$ system,

$$C=\left[\sum_{k=0}^{n+m}\widehat{p}_k\prod_{i=0}^{k-1}(1-d_i)\right]^{-1}, \tag{57}$$

and $d_i=1-r(i)$, $i=0,1,\dots,m+n-1$, denotes the "typical" dropping function (Kempa, 2011).

5. Numerical results

In this section we present some numerical results illustrating theoretical formulae for stationary queue-size distributions, for various scenarios determined by queueing models and accepting functions, and under different values of the traffic load ρ. Below we consider, successively, the classical model with a finite waiting room, single arrivals and a "typical" dropping function, next we derive results for the single-server $M/E_2/1/m$ system and the $M/E_2/1/(\infty, V)$-type model with the exponential distribution of packet volumes. Finally, we investigate $M/D/2/(\infty, V)$ under two different traffic loads and means of exponentially distributed packet volumes.

5.1. $M/D/2/8$-type queue with single arrivals and "typical" dropping. Investigate, firstly, the impact of the accepting function on the stationary queue-size distribution in the classical $M/G/n/m$-type queueing model with a finite waiting room and single arrivals. Choose two different types of the accepting function, a linear and a quadratic one, respectively, defined as follows:

$$r_1(i) = 1 - \frac{i}{m+n}, \qquad r_2(i) = 1 - \Big(\frac{i}{m+n}\Big)^2, \tag{58}$$

where $i = 0, 1, \ldots, m+n$. Of course, now $r_k(i) = 1 - d_i^{(k)}$, $k = 1, 2$, where $d_i^{(k)}$ stands for the appropriate "typical" dropping function representing the probability of deleting the incoming packet if the number of packets present in the system at the pre-arrival epoch equals i (Kempa, 2011).

Let us examine numerically the $M/D/2/8$-type model in which packets arrive according to a Poisson process with intensity a and are being processed with constant service time t_0. We start the procedure with finding the steady-state probabilities $\widehat{p}_k$, $k = 0, \ldots, 10$, for the system without AQM-type dropping. For the $M/D/2$-type system with an infinite waiting room and without packet dropping, stationary probabilities $\widetilde{p}_k$ can be derived by using the following algorithm (Bocharov *et al.*, 2004):

$$\widetilde{p}_0 = -\frac{2z_1(1-\rho)}{1-z_1}, \qquad \widetilde{p}_1 = 2(1-\rho-\widetilde{p}_0), \tag{59}$$

where z_1 is a negative solution of the equation

$$z^2 e^{2\rho(1-z)} - 1 = 0 \;\Leftrightarrow\; \ln z^2 + 2\rho(1-z) = 0. \tag{60}$$

Probabilities $\widetilde{p}_k$ for $k \geq 2$ can be obtained using the recursive formula

$$\widetilde{p}_{2+i} = \frac{1}{\beta_0}\Big[\widetilde{p}_i - \beta_i(\widetilde{p}_0 + \widetilde{p}_1) - \sum_{k=1}^{i} \beta_k \widetilde{p}_{2+i-k}\Big], \tag{61}$$

where $i = 0, 1, \ldots,$

$$\beta_j = \frac{(2\rho)^j}{j!} e^{-2\rho}, \quad j = 0, 1, \ldots, \tag{62}$$

and $\rho = at_0/2 < 1$. Substituting $\widetilde{p}_k$ computed for the first $k = 0, \ldots, 10$ into (56)–(57), we generate $\widehat{p}_k$, $k = 0, \ldots, 10$, for the finite-queue $M/D/2/8$ model without AQM, taking $r(i) \equiv 1$. Next, substituting $\widehat{p}_k$ again into (56)–(57) for proper $r(\cdot)$, we find probability distributions for the case of linear and quadratic dropping separately.

In Table 1 we present stationary probabilities for the "pure" (without dropping) $M/D/2/8$-type system and for the case of linear ($r_1(\cdot)$) and quadratic ($r_2(\cdot)$) accepting functions, for $\rho = 0.75$. The case of $\rho = 0.99$ (critical loading) is presented in Table 2. Results from Tables 1–2 are visualized in Figs. 1 and 2. As can be noted, using the accepting function significantly reduces probabilities of high lengths of the queue of packets. The reduction is more visible for linear-type dropping.

Table 1. Queue-size distributions in the $M/D/2/8$ system for $\rho = 0.75$ (no dropping, linear dropping, quadratic dropping).

k	none	linear ($r_1(\cdot)$)	quadratic ($r_2(\cdot)$)
0	0.133176	0.173320	0.146221
1	0.236276	0.307498	0.259420
2	0.227400	0.266352	0.247178
3	0.163638	0.153335	0.170756
4	0.102221	0.0670491	0.0970668
5	0.0602224	0.0237008	0.0480362
6	0.0348450	6.85670×10^{-3}	0.0208455
7	0.0200941	1.58162×10^{-3}	7.69344×10^{-3}
8	0.0115877	2.73624×10^{-4}	2.26266×10^{-3}
9	6.68378×10^{-3}	3.15651×10^{-5}	4.69836×10^{-4}
10	3.85543×10^{-3}	1.82079×10^{-6}	5.14934×10^{-5}

Table 2. Queue-size distributions in the $M/D/2/8$ system for $\rho = 0.99$ (no dropping, linear dropping, quadratic dropping).

k	none	linear ($r_1(\cdot)$)	quadratic ($r_2(\cdot)$)
0	0.0251723	0.0701711	0.0443032
1	0.0643240	0.179312	0.113210
2	0.0928201	0.232874	0.161729
3	0.104896	0.210536	0.175459
4	0.107202	0.150615	0.163178
5	0.105988	0.0893459	0.135518
6	0.103980	0.0438266	0.0997128
7	0.101906	0.0171809	0.0625430
8	0.0998765	5.05159×10^{-3}	0.0312615
9	0.0978905	9.90237×10^{-4}	0.0110305
10	0.0959457	9.70564×10^{-5}	2.05415×10^{-3}

5.2. Single-server $M/E_2/1/9$-type queueing model. Consider now the effect of using the accepting function in the classical single-server $M/E_2/1/9$-type model, in which packets occur according to the Poisson process with intensity a and are being served in time having a 2-Erlang

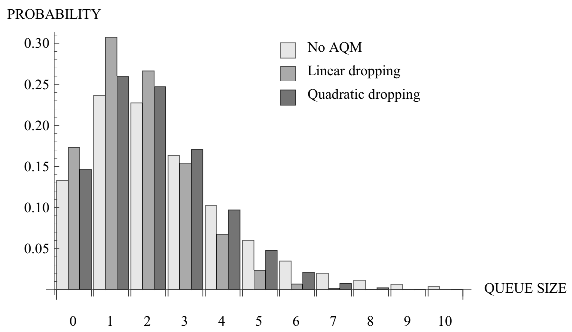

Fig. 1. Queue-size distributions in the $M/D/2/8$ system for $\rho = 0.75$ (no dropping, linear dropping, quadratic dropping).

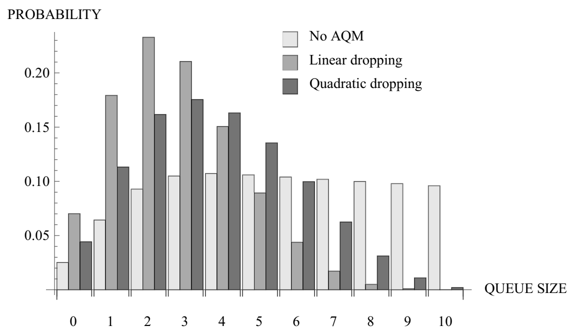

Fig. 2. Queue-size distributions in the $M/D/2/8$ system for $\rho = 0.99$ (no dropping, linear dropping, quadratic dropping).

distribution with parameter μ. We can derive steady-state probabilities $\widetilde{p}_n$, $n = 0, 1, \ldots,$ for the corresponding $M/E_2/1$ system with an infinite waiting room and without AQM-type dropping using the following method. Under the stability condition $\rho = 2a/\mu < 1$, it can be shown (Adan and Resing, 2002) that the equation

$$(a + \mu)x^2 = a + \mu x^3 \tag{63}$$

has exactly two different roots x_i, $i = 1, 2$, such that $|x_i| < 1$ (obviously, the third root $x_3 = 1$). If $\widetilde{q}_n$ denotes the stationary number of exponential phases of the service "present" in the system, then we have (Adan and Resing, 2002)

$$\widetilde{q}_n = c_1 x_1^n + c_2 x_2^n, \quad n = 0, 1, \ldots, \tag{64}$$

where

$$c_1 = \frac{1-\rho}{1 - x_2/x_1}, \quad c_2 = \frac{1-\rho}{1 - x_1/x_2}. \tag{65}$$

Finally, we easily find the steady-state queue-size distribution as

$$\widetilde{p}_0 = \widetilde{q}_0, \quad \widetilde{p}_n = \widetilde{q}_{2n} + \widetilde{q}_{2n-1}, \quad i = 1, 2, \ldots. \tag{66}$$

Next, using the approach described in the previous subsection (for the case of the $M/D/2/8$ queue), we obtain stationary probabilities for the "pure" $M/E_2/1/9$ model without dropping and for the case of linear and quadratic accepting functions defined in (59), taking the same levels of the traffic load, namely, 0.75 and 0.99. The results are given in Tables 3–4 and also presented in Figs. 3–4, and the interpretation is similar to that in the previous subsection.

Table 3. Queue-size distributions in the single-server $M/E_2/1/9$ system for $\rho = 0.75$ (no dropping, linear dropping, quadratic dropping).

k	none	linear ($r_1(\cdot)$)	quadratic ($r_2(\cdot)$)
0	0.254533	0.326808	0.283243
1	0.226694	0.291063	0.252263
2	0.166105	0.191944	0.182992
3	0.116059	0.107290	0.122743
4	0.0800062	0.0517729	0.0769992
5	0.0549348	0.0213293	0.0444109
6	0.0376754	7.31406×10^{-3}	0.0228434
7	0.0258295	2.00574×10^{-3}	0.0100230
8	0.0177063	4.12485×10^{-4}	3.50413×10^{-3}
9	0.0121374	5.65504×10^{-5}	8.64730×10^{-4}
10	8.31990×10^{-3}	3.87641×10^{-6}	1.12623×10^{-4}

Table 4. Queue-size distributions in the single-server $M/E_2/1/9$ system for $\rho = 0.99$ (no dropping, linear dropping, quadratic dropping).

k	none	linear ($r_1(\cdot)$)	quadratic ($r_2(\cdot)$)
0	0.0751184	0.178256	0.123162
1	0.0927732	0.220151	0.152109
2	0.0961712	0.205393	0.156103
3	0.0960421	0.164094	0.149658
4	0.0950501	0.113679	0.134782
5	0.0938565	0.0673510	0.111795
6	0.0926255	0.0332338	0.0827467
7	0.0913976	0.0131173	0.0522558
8	0.0901827	3.88289×10^{-3}	0.0262962
9	0.0889833	7.66248×10^{-4}	9.34073×10^{-3}
10	0.0877995	7.56055×10^{-5}	1.75113×10^{-3}

5.3. Single-server $M/E_2/1/(\infty, 10)$-type model.

Investigate now the reduction of the queue in the $M/E_2/1/(\infty, V)$ model with an infinite waiting room and the total capacity bounded by $V = 10$, under the assumption that volumes of the incoming packets are exponentially distributed, i.e., we have

$$L(x) = 1 - e^{-\alpha x}, \quad x \geq 0. \tag{67}$$

Below we present through numerical examples different scenarios of the operation of such a system. We investigate separately two types of the theoretical model

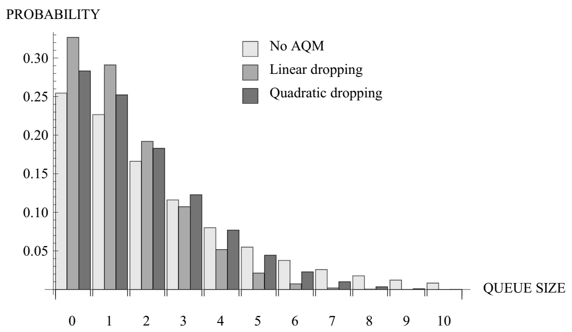

Fig. 3. Queue-size distributions in the $M/E_2/1/9$ system for $\rho = 0.75$ (no dropping, linear dropping, quadratic dropping).

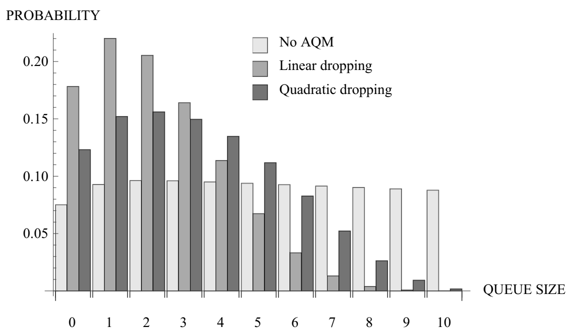

Fig. 4. Queue-size distributions in the $M/E_2/1/9$ system for $\rho = 0.99$ (no dropping, linear dropping, quadratic dropping).

introduced in Section 4, with functions $R(\cdot)$ defined in (26) (Type I) and (47) (Type II), respectively, and compare the results with these obtained for the system without packet dropping. In computations we use the algorithm for finding the stationary queue-size in the "pure" $M/E_2/1$ model described in (63)–(66) and then the formula (37) from Corollary 1. We analyze separately cases of $\rho = 0.75$ and $\rho = 0.99$, and also consider two different values of the parameter α, namely, 1 and 2 (i.e., two different means of packet volumes, 1 and 0.5). As an accepting function, we introduce the linear one defined as

$$r(x) = \frac{V - x}{V}, \quad x \in [0, V]. \tag{68}$$

The results of simulations are shown in Tables 5–8 and also presented in Fig. 5–8 (for $k = 0, \dots, 10$). As can be noted, the reduction of the queue size is more visible in the case of the II type system, in which the function $R(\cdot)$ is defined in (47).

5.4. Two-server $M/D/2/(\infty, V)$-type model. Let us consider now the $M/D/2/(\infty, V)$-type model with deterministic processing times and investigate the behavior of the system under the same parameters as previously: $V = 10$, exponential service times (with means 1 and 0.5), two levels of the traffic load $\rho = 0.75$ and $\rho = 0.99$, and linear accepting function $r(\cdot)$ defined in (68). The appropriate results for the "pure" system and for I and II type AQM dropping are given in Tables 9 and 10 and visualized in Figs. 9–12 (for $k = 0, \dots, 10$). Steady-state probabilities for the "pure" $M/D/2$-type model with an infinite waiting room are derived based on the algorithm given in (59)–(62). Next, the appropriate

Table 5. Queue-size distributions in $M/E_2/1/(\infty, V)$ for exponential packet volumes with mean 1 and $\rho = 0.75$ (no AQM, I type AQM, II type AQM).

k	no AQM	I type AQM	II type AQM
0	0.250000	0.435020	0.438790
1	0.222656	0.348698	0.390779
2	0.163147	0.187363	0.137435
3	0.113992	0.0264491	0.0285465
4	0.0785813	2.32380×10^{-3}	4.00924×10^{-3}
5	0.0539564	1.39224×10^{-4}	4.06995×10^{-4}
6	0.0370044	6.03553×10^{-6}	3.11450×10^{-5}
7	0.0253695	1.97587×10^{-7}	1.85290×10^{-6}
8	0.0173909	5.04706×10^{-9}	8.78290×10^{-8}
9	0.0119212	1.03235×10^{-10}	3.38519×10^{-9}
10	8.17173×10^{-3}	1.72707×10^{-12}	1.07938×10^{-10}

Table 6. Queue-size distributions in $M/E_2/1/(\infty, V)$ for exponential packet volumes with mean 1 and $\rho = 0.99$ (no AQM, I type AQM, II type AQM).

k	no AQM	I type AQM	II type AQM
0	0.100000	0.317415	0.325239
1	0.0123502	0.352816	0.401660
2	0.0128026	0.268202	0.199850
3	0.0127854	0.0541142	0.0593311
4	0.0126534	6.82564×10^{-3}	0.0119629
5	0.0124945	5.88094×10^{-4}	1.74643×10^{-3}
6	0.0123306	3.66863×10^{-5}	1.92311×10^{-4}
7	0.0121671	1.72860×10^{-6}	1.64670×10^{-5}
8	0.0120054	6.35552×10^{-8}	1.12352×10^{-6}
9	0.0118457	1.87123×10^{-9}	6.23320×10^{-8}
10	0.0116882	4.50610×10^{-11}	2.86083×10^{-9}

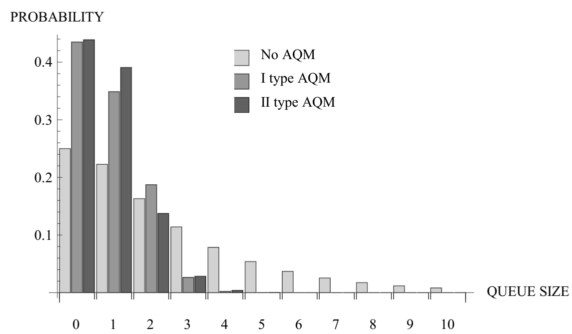

Fig. 5. Queue-size distributions in $M/E_2/1/(\infty, V)$ for exponential packet volumes with mean 1 and $\rho = 0.75$ (no AQM, I type AQM, II type AQM).

Table 7. Queue-size distributions in $M/E_2/1/(\infty, V)$ for exponential packet volumes with mean 0.5 and $\rho = 0.75$ (no AQM, I type AQM, II type AQM).

k	no AQM	I type AQM	II type AQM
0	0.250000	0.402116	0.434776
1	0.222656	0.340227	0.387222
2	0.163147	0.213869	0.140446
3	0.113992	0.0386566	0.0317024
4	0.0785813	4.68979×10^{-3}	5.14244×10^{-3}
5	0.0539564	4.12635×10^{-4}	6.41602×10^{-4}
6	0.0370044	2.76104×10^{-5}	6.40216×10^{-5}
7	0.0253695	1.45222×10^{-6}	5.24157×10^{-6}
8	0.0173909	6.15608×10^{-8}	3.58652×10^{-7}
9	0.0119212	2.14547×10^{-9}	2.08032×10^{-8}
10	8.17173×10^{-3}	6.24877×10^{-11}	1.03474×10^{-9}

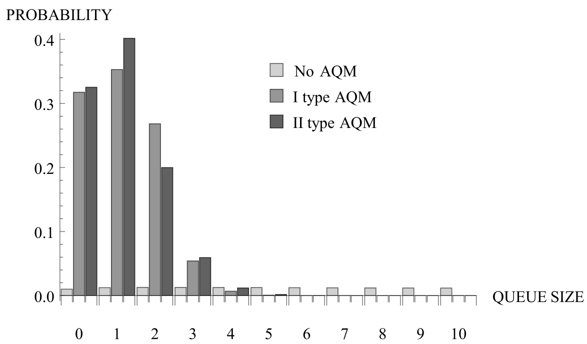

Fig. 6. Queue-size distributions in $M/E_2/1/(\infty, V)$ for exponential packet volumes with mean 1 and $\rho = 0.99$ (no AQM, I type AQM, II type AQM).

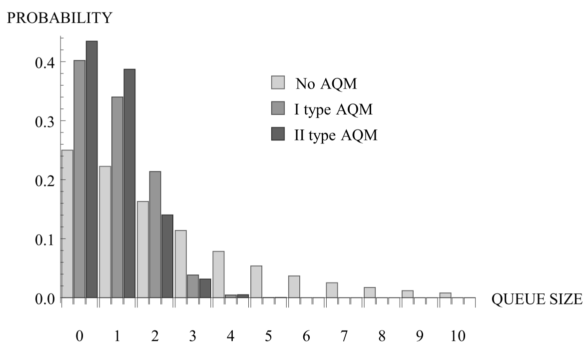

Fig. 7. Queue-size distributions in $M/E_2/1/(\infty, V)$ for exponential packet volumes with mean 0.5 and $\rho = 0.75$ (no AQM, I type AQM, II type AQM).

results for the system with packet dropping are found by applying (37).

As one can observe, the reduction in the queue size is similar in the cases of systems of the I and II type. Note that in the case of exponentially distributed packet volumes the procedure of AQM-type packet dropping reduces the queue of packets emphatically: the probability that the queue size will exceed four packets is negligible.

Table 8. Queue-size distributions in $M/E_2/1/(\infty, V)$ for exponential packet volumes with mean 0.5 and $\rho = 0.99$ (no AQM, I type AQM, II type AQM).

k	no AQM	I type AQM	II type AQM
0	0.100000	0.282505	0.319411
1	0.0123502	0.331456	0.394481
2	0.0128026	0.294770	0.202420
3	0.0127854	0.0761520	0.0653070
4	0.0126534	0.0132634	0.0152083
5	0.0124945	1.67825×10^{-3}	2.72876×10^{-3}
6	0.0123306	1.61591×10^{-4}	3.91816×10^{-4}
7	0.0121671	1.22328×10^{-5}	4.61704×10^{5}
8	0.0120054	7.46405×10^{-7}	4.54729×10^{-6}
9	0.0118457	3.74438×10^{-8}	3.79661×10^{-7}
10	0.0116882	1.56979×10^{-9}	2.71823×10^{-8}

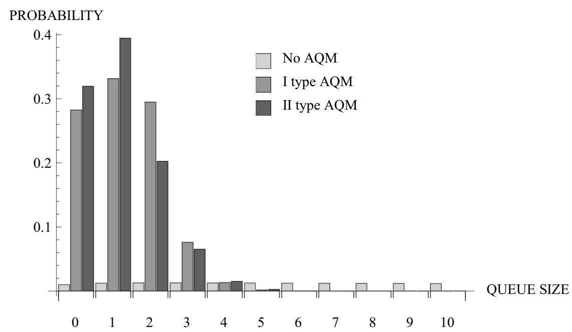

Fig. 8. Queue-size distributions in $M/E_2/1/(\infty, V)$ for exponential packet volumes with mean 0.5 and $\rho = 0.99$ (no AQM, I type AQM, II type AQM).

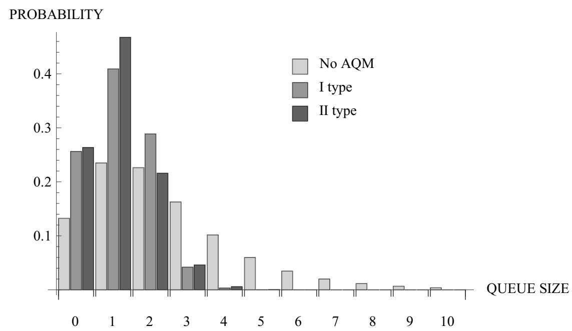

Fig. 9. Queue-size distributions for $\rho = 0.75$ (exponential packet volumes with mean 1, linear accepting function).

6. Conclusion

In the article, a queueing model of the $M/G/n$-type, $n \geq 1$, with a bounded total volume was investigated. It was assumed that the volumes of the arriving packets were generally distributed random variables and the AQM-type mechanism was implemented to control the actual buffer level. Namely, each of the arriving packets was dropped with a probability depending on its volume and the occupied volume of the system at the pre-arrival moment. Compact-form representations

Table 9. Queue-size distributions for $\rho = 0.75$ and $\rho = 0.99$ (exponential packet volumes with mean 1, linear accepting function).

k	p_k for $\rho = 0.75$		p_k for $\rho = 0.99$	
	I type	II type	I type	II type
0	0.256326	0.263621	0.157032	0.164917
1	0.409291	0.467686	0.361146	0.421403
2	0.288864	0.216047	0.382158	0.291869
3	0.0419972	0.0462171	0.0872551	0.0980540
4	0.00334361	0.00588193	0.0113652	0.0204161
5	0.000171880	0.000512320	0.000980444	0.00298422
6	6.29×10^{-6}	0.0000330760	0.0000608005	0.000326672
7	1.73×10^{-7}	1.66×10^{-6}	2.85×10^{-6}	0.0000277822
8	3.72×10^{-9}	6.60×10^{-8}	1.04×10^{-7}	1.88×10^{-6}
9	6.40×10^{-11}	2.14×10^{-9}	3.04×10^{-9}	1.04×10^{-7}
10	9.01×10^{-13}	5.74×10^{-11}	7.27×10^{-11}	4.73×10^{-9}

Table 10. Queue-size distributions for $\rho = 0.75$ and $\rho = 0.99$ (exponential packet volumes with mean 0.5, linear accepting function).

k	p_k for $\rho = 0.75$		p_k for $\rho = 0.99$	
	I type	II type	I type	II type
0	0.228995	0.259866	0.133485	0.160239
1	0.385962	0.461046	0.324046	0.409466
2	0.318676	0.219644	0.401152	0.292477
3	0.0593233	0.0510626	0.117275	0.106782
4	0.00652174	0.00750565	0.0210928	0.0256787
5	0.000492347	0.000803488	0.00267226	0.00461317
6	0.0000277939	0.0000676413	0.000255781	0.000658481
7	1.23×10^{-6}	4.66×10^{-6}	0.0000192318	0.0000770670
8	4.39×10^{-8}	2.68×10^{-7}	1.17×10^{-6}	7.54×10^{-6}
9	1.29×10^{-9}	1.31×10^{-8}	5.81×10^{-8}	6.25×10^{-7}
10	3.15×10^{-11}	5.48×10^{-10}	2.42×10^{-9}	4.45×10^{-8}

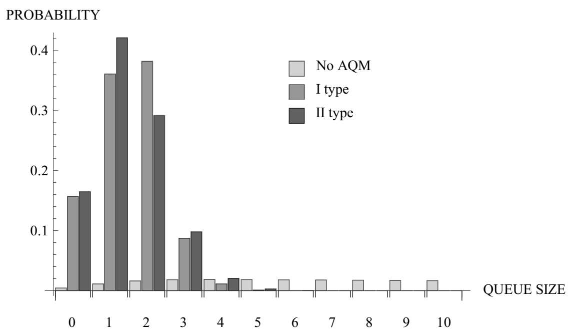

Fig. 10. Queue-size distributions for $\rho = 0.99$ (exponential packet volumes with mean 1, linear accepting function).

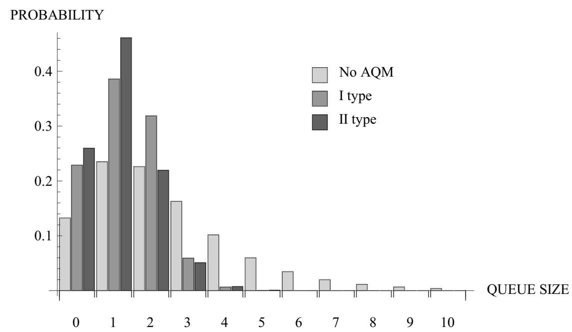

Fig. 11. Queue-size distributions for $\rho = 0.75$ (exponential packet volumes with mean 0.5, linear accepting function).

for the steady-state queue-size distribution and the loss probability were found. Numerical examples were given as well. In the future, it is planned to consider a model in which the processing time depends on the size of the arriving packet; however, as it seems, the solution may require a significant extension of the theoretical tools or the development of a new methodology. Moreover, it would be interesting to analyze the accepting function in the context of a discrete-time model.

Acknowledgment

The authors wish to thank the referees for their valuable and helpful comments which allowed improving the paper and avoiding mistakes.

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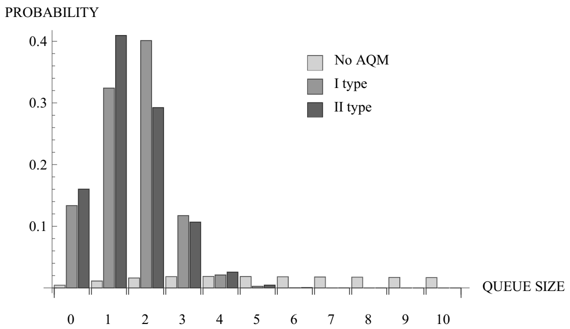

Fig. 12. Queue-size distributions for $\rho = 0.99$ (exponential packet volumes with mean 0.5, linear accepting function).

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Received: 10 December 2015
Revised: 19 April 2016
Accepted: 6 June 2016