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CONDITIONS AND A COMPUTATION METHOD OF THE CONSTRAINED REGULATION PROBLEM FOR A CLASS OF FRACTIONAL–ORDER NONLINEAR CONTINUOUS–TIME SYSTEMS

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The constrained regulation problem (CRP) for fractional-order nonlinear continuous-time systems is investigated. New existence conditions of a linear feedback control law for a class of fractional-order nonlinear continuous-time systems under constraints are proposed. A computation method for solving the CRP for fractional-order nonlinear systems is also presented. Using the comparison principle and positively invariant set theory, conditions guaranteeing positive invariance of a polyhedron for fractional-order nonlinear systems are established. A linear feedback controller model and the corresponding algorithm of the CRP for fractional nonlinear systems are also proposed by using the obtained conditions. The presented model of the CRP is formulated as a linear programming problem, which can be easily implemented from a computational point of view. Numerical examples illustrate the proposed method.

Keywords: fractional-order nonlinear systems, constrained regulation, positively invariant set, linear programming.

1. Introduction

Fractional order calculus and fractional differential equations appear in many fields, including science and engineering (Ma *et al.*, 2019; Karthikeyan *et al.*, 2017; Yepez-Martinez and Gomez-Aguilar, 2018). They have been more and more often applied to real-world engineering problems such as mechanical systems, signal processing and systems identification, control and robotics and so on; for more applications, refer to the works of Zhao *et al.* (2017) or Dastjerdi *et al.* (2019) and the references therein.

The problem of stability analysis and control synthesis of fractional-order systems is important in theory and applications. Many stability conditions have been proposed for fractional-order linear systems (Kaczorek, 2018; Song and Zhen, 2013; Jiao *et al.*, 2013; Chen *et al.*, 2015; Ammour *et al.*, 2015), fractional-order nonlinear systems (Li *et al.*, 2010; Fernandez-Anaya *et al.*, 2016; Kaczorek, 2019), fractional-order neural networks (Zhang *et al.*, 2017; 2015b), fractional-order switched linear systems (Balochian, 2015; Hao and Jiang, 2016), fractional-order singular systems (Yin *et al.*, 2015; Liu *et al.*, 2016), positive fractional-order systems (Kaczorek, 2010; Shen and Lam, 2016; Yang and Jia, 2019; Si and Yang, 2021) and fractional-order chaotic complex networks systems (Zhang *et al.*, 2016).

On the other hand, the control input, the state and/or the output variables have to be bounded in practice either for safety reasons or because of physical limitations. The most realistic representation of these limitations is by considering hard constraints on these variables. New stability conditions and methods of controller synthesis to ensure the stability performance of the resulting system under constraints, which is called the constraint regulation

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problem (CRP for brevity), are interesting topics in fractional-order systems theory and applications.

In recent years, more attention has been paid to the design of state feedback controllers for fractional order nonlinear systems (Lenka and Banerjee, 2016; Zhang *et al.*, 2015a; Wang *et al.*, 2016). Common methods to derive the stability conditions for nonlinear systems are mainly based on the Laplace transform (Lenka and Banerjee, 2016), Mittag-Leffler functions (Zhang *et al.*, 2015a) and the Bellman–Gronwall inequality (Wang *et al.*, 2016). In those methods, linear matrix inequalities (LMIs) play an important role. In the works of Wang *et al.* (2014), Martinezfuentes and Martinezguerra (2018) or Li *et al.* (2019), stability and stabilization of fractional order nonlinear systems are studied via the comparison principle. Benzaouia *et al.* (2014) propose a linear programming method to determine stability and to find a linear feedback controller under state and control constraints. It is an interesting and appealing method from the computational point of view in that linear programming can be performed readily using any mathematical software.

Based on the above observations and background, research on the CRP for fractional order nonlinear continuous systems is not only necessary, but also more challenging than that on integer order systems. Motivated by methods of Wang *et al.* (2014), Martinezfuentes and Martinezguerra (2018), Li *et al.* (2019), Benzaouia *et al.* (2014) or Yang and Hu (2020), the CRP for a class of fractional order nonlinear continuous-time systems is studied in this paper. Our main contribution consists in establishing new conditions on the positive invariance of a polyhedron for fractional order nonlinear systems considered through the comparison principle and positively invariant set theory; a model and the corresponding algorithm of a linear feedback controller for the CRP of fractional order nonlinear systems are also proposed. Numerical examples show that our method is effective.

The paper is organized as follows. Section 2 gives some preliminaries and the problem formulation. Positively invariant sets and existence conditions of a controller for the fractional order nonlinear systems considered is investigated in Section 3; we also propose a new method to find the comparison system for the fractional order system. By virtue of the comparison system, the CRP for the fractional nonlinear systems considered is formulated as a linear programming one in Section 4, which can be easily implemented because of its intrinsic quality. Section 5 presents two numerical examples to illustrate the method proposed in this paper. Section 6 contains conclusions and future research topics.

Notation. $\mathbb{R}^n$ is the n dimensional space of real vectors, $\mathbb{R}^{n\times n}$ is the space of $n\times n$ matrices with real entries. For $\rho \in \mathbb{R}^n$, $\rho > 0$ (resp. $\rho \geq 0$) means all components of ρ are positive (nonnegative). M_n is the set of Metzler matrices with their off-diagonal elements nonnegative; $\operatorname{rank}(A)$ denotes the rank of the matrix A.

2. Preliminaries and problem formulation

In this section, some definitions and lemmas are presented for fractional-order nonlinear dynamical systems.

2.1. Class of fractional-order nonlinear continuous-time systems. Consider the following class of fractional-order nonlinear continuous-time systems:

$$\begin{cases} {}_0D_t^\alpha x(t) = Ax(t) + Bu(t) + f(x(t)), & t > 0, \\ x(t) = x_0, \quad -\infty < t \leq 0, \end{cases} \tag{1}$$

where $0 < \alpha < 1$, $x(0) = x_0$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ is the pseudo state and control input, respectively, $f(x(t)) \in \mathbb{R}^n$ is a continuous function such that $f(0) = 0$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. Throughout the paper, we assume that the system (1) satisfies the conditions of solution existence and has a unique solution with the given initial condition $x(0)$.

Definition 1. The expression ${}_0D_t^\alpha x(t)$ represents the Riemann–Liouville fractional-order derivative of $x(t)$ defined by

$${}_0D_t^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^t \frac{x(\tau)}{(t-\tau)^\alpha}\,\mathrm{d}\tau\Big), \tag{2}$$

where $0 < \alpha < 1$ is the order of the fractional derivative, while the Gamma function is defined by

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,\mathrm{d}t, \quad Re(z) \in \mathbb{R}_+.$$

Definition 2. A nonempty set $P \subseteq \mathbb{R}^n$ is a *positive invariant set* of the system (1) if and only if $x_0 \in P$ implies $x(t; 0, x_0) \in P, t > 0$, where $x(t; 0, x_0)$ is the trajectory of the system with the initial conditions $(0, x_0)$.

Definition 3. (*Athanasopoulos* et al.*, 2010*) A function $h(y), h : \mathbb{R}^m \to \mathbb{R}^m$ is said to be *monotone nondecreasing* if all of its components are nondecreasing with respect to all the components of the vector y.

2.2. Constrained regulation problem. In practice, the control input, the state and/or the output variables have to be bounded by considering hard constraints on these variables. For the fractional-order nonlinear system (1), suppose that the initial state x_0 belongs to the polyhedral set defined by

$$P(Q, \rho) = \{x_0 \in \mathbb{R}^n : Qx_0 \leq \rho\}, \tag{3}$$

where $Q \in \mathbb{R}^{q\times n}$ with $\mathrm{rank}(Q) = n$, $q \geq n$, and $\rho > 0$. The state constraints $x(t)$ are constituted by the polyhedral set

$$P(M,c) = \{x(t) \in \mathbb{R}^n : Mx(t) \leq c\}, \tag{4}$$

where $M \in \mathbb{R}^{s\times n}$ with $\mathrm{rank}(M) = n$, $s \geq n$, and $c > 0$.

Suppose that the control input $u(t)$ is subject to linear constraints of the form

$$-\underline{\omega} \leq u(t) \leq \overline{\omega}, \tag{5}$$

where $\underline{\omega}$ and $\overline{\omega}$ are real vectors with positive components. If there exists a state feedback control law K and an input $u(t) = Kx(t)$ for the system (1), let $\overline{A} = A + BK$; then the system (1) becomes

$$\begin{cases} {}_0D_t^{\alpha}x(t) = \overline{A}x(t) + f(x(t)), & t > 0, \\ x(t) = x_0, \quad -\infty < t \leq 0, \end{cases} \tag{6}$$

where $0 < \alpha < 1$, $x(0) = x_0$.

Remark 1. Information of $x(t)$ at $t = 0$ is not sufficient to predict the future behavior of the system. The description of (1) and (6) is thus not strictly a state-space description and is termed a pseudo state-space (Sabatier *et al.*, 2013). Under the condition of $x(t) = \varphi(t)$ for $t \in (-\infty, 0]$, the solution of the system (6) is given by

$$\begin{aligned} x(t) = &-\int_0^t e_{\alpha}^{\overline{A}(t-\tau)}\psi(x,\alpha,-\infty,0,\tau)\,\mathrm{d}\tau \\ &+ \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\overline{A}(t-\tau)^{\alpha})f(x(\tau))\,\mathrm{d}\tau. \end{aligned}$$

It was demonstrated by Sabatier *et al.* (2013) with a counterexample (Sabatier *et al.*, 2010) that Caputo's definition does not permit one to obtain the pseudo trajectories of the exact system if an initial condition such as $x(t_0) = x_0, t_0 > 0$ is taken into account in (1) and (6). We study the system (6) with a constant history, that is, $x(t) = x_0$ for $t \in (-\infty, 0]$, for which the initialization function is given by (Lim and Ahn, 2013)

$$\psi(x,\alpha,-\infty,0,\tau) = -\frac{x_0 t^{-\alpha}}{\Gamma(1-\alpha)}.$$

Thus, the solution of (6) is given by

$$\begin{aligned} x(t) = &E_{\alpha}(\overline{A}t^{\alpha})x_0 \\ &+ \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\overline{A}(t-\tau)^{\alpha})f(x(\tau))\,\mathrm{d}\tau. \end{aligned}$$

The constrained regulation problem of fractional-order nonlinear continuous-time system (1) consists in determining a state feedback control law K and an input $u(t) = Kx(t)$ such that, for all the initial states x_0 satisfying the inequality (3), the system (6) is asymptotically stable, while the corresponding trajectory $x(t; 0, x_0)$ of (6) does not violate the state constraint (4) as well as the control input constraint (5) for any $t \geq 0$.

3. Positively invariant sets and the existence condition for a controller

In this section, conditions guaranteeing the positive invariance of a polyhedron for fractional-order nonlinear systems and the existence conditions for a constrained controller are investigated by means of the comparison principle and positively invariant set theory.

3.1. Positively invariant sets of a fractional-order nonlinear system. Consider a continuous vector-valued function $v(x) : \mathbb{R}^n \to \mathbb{R}^q$ and a monotone nondecreasing function $h(y) : \mathbb{R}^q \to \mathbb{R}^q$ such that

$${}_0D_t^{\alpha}v(x(t)) = v(\overline{A}x(t) + f(x(t))) \leq h(v(x(t))). \tag{7}$$

Then the system

$${}_0D_t^{\alpha}y(t) = h(y(t)) \tag{8}$$

is called the comparison system of (6), where

$$\begin{aligned} h_i(y(t)) &\geq h_i(v(t)) \\ &= \max_{v(x(t))\leq y(t)}\{v_i(\overline{A}x(t) + f(x(t)))\}, \\ &\qquad i = 1,2,\dots,q. \end{aligned}$$

Definition 4. A vector-valued function $v(x) : \mathbb{R}^n \to R^s$ is said to be of class $\mathscr{B}$ if, for any $y \geq 0$, the set $\{x \in \mathbb{R}^n : v(x) \leq y\}$ is closed. We assume that $v(x)$ in this paper is of class $\mathscr{B}$, which guarantees the existence of $h(y)$ in this paper.

Lemma 1. (Yan *et al.*, 2010; Lenka, 2018) *Let* ${}_0D_t^{\alpha}y(t) \geq {}_0D_t^{\alpha}v(x(t))$ *and* $y(0) = v(x(0))$*, where* $0 < \alpha < 1$*. Then* $y(t) \geq v(x(t))$*.*

Remark 2. From the proof of the fractional comparison principle (Yan *et al.*, 2010), we find that Lemma 1 also holds when $y(0) \geq v(x(0))$. This conclusion can also be obtained with the proof of the fractional comparison principle (Lenka, 2018). By applying

$${}_0D_t^{\alpha}x(t) = {}_0D_t^{\alpha}x(t) + \frac{x_0 t^{-\alpha}}{\Gamma(1-\alpha)},$$

Lemma 1 can be stated as follows: If (8) is a comparison system of (6), then $v(x(0)) \leq y(0)$ implies $v(x(t)) \leq y(t)$ for all $t \geq 0$.

Theorem 1. *The polyhedral set*

$$P(v,\rho) = \{x(t) \in \mathbb{R}^n : v(x(t)) \leq \rho\},$$

with $v(x(t)) : \mathbb{R}^n \to \mathbb{R}^q, \rho \in \mathbb{R}^q$ *and* $\rho > 0$*, is a positively invariant set for the fractional-order nonlinear system (6) if and only if there exists a comparison system*

$${}_0D_t^{\alpha}y(t) = h(y(t)) \tag{9}$$

such that

$$h(\rho) \le -\varepsilon\rho, \quad \varepsilon \in [0,1], \tag{10}$$

where

$$\overline{A}x(t) = y(t),$$

$$h_i(y(t)) \ge h_i(v(t)) = \max_{v(x(t))\le y(t)} \{v_i(\overline{A}x(t) + f(x(t)))\}, \quad i = 1,2,\dots,q.$$

Proof. For the set $P(v,\rho) = \{x(t) \in \mathbb{R}^n : v(x(t)) \le \rho\}$, we have that, for the trajectory of $v(x(t))$,

$$v_i(\overline{A}x(t) + f(x(t))) \le y_i(t) \le \rho_i, \quad i = 1,2,\dots,q,$$

for all $x(t) \in P(v, y(t))$ and $y(t) \le \rho$.

Let $\overline{A}x(t) = y(t)$. Consider functions $h(y)$ such that

$$h_i(y(t)) \ge h_i(v(t)) = \max_{v(x(t))\le y(t)} \{v_i(\overline{A}x(t) + f(x(t)))\}, \quad i = 1,2,\dots,q.$$

There must exist $h(y)$ for all $y(t) \le \rho$ from the assumption of Definition 4. Since $h(y)$ is a monotone nondecreasing function and $v(\overline{A}x(t) + f(x(t))) \le h(v(x(t)))$, there exists a comparison system ${}_0D_t^\alpha y(t) = h(y(t))$.

Moreover, if $h(\rho) > 0$, then, for the initial state $x(0)$, which $v(x(0)) \le \rho$, we have the trajectory $v(x(t))$ of the following comparison system:

$$\begin{aligned} &{}_0D_t^\alpha v(x(t)) \\ &= \max_{v(x(t))\le y(t)} \{v(\overline{A}x(t) + f(x(t)))\} \\ &= h(\rho) > 0, \end{aligned}$$

while it is in contradiction with the fact that the polyhedral set $P(v,\rho)$ is a positively invariant set. Hence, the polyhedral set $P(v,\rho)$ is a positively invariant set of the fractional-order nonlinear system (6) if and only if $h(\rho) \le -\varepsilon\rho$, $\varepsilon \in [0,1]$. ■

Remark 3. Suppose $v(x(t))$ is a Lyapunov function such that $v(x(t)) \le \rho$ if ${}_0D_t^\alpha v(x(t)) \le h(v(x(t)))$ and $h(\rho) < 0$; then the system (8) is asymptotically stable.

Let $v(x(t)) = Qx(t)$. We obtain a condition guaranteeing the positive invariance of a polyhedron for the fractional-order nonlinear system (6).

Theorem 2. *The polyhedral set*

$$P(Q,\rho) = \{x(t) \in \mathbb{R}^n : Qx(t) \le \rho\},$$

with $Q \in \mathbb{R}^{q\times n}$, $\rho \in \mathbb{R}^q$ and $\rho > 0$, is a positively invariant set of the system (6) if and only if there exists a matrix $H \in M_m$, such that

$$Q\overline{A} = HQ \tag{11}$$

and

$$h(\rho) \le -\varepsilon\rho, \quad \varepsilon \in [0,1], \tag{12}$$

where

$$h_i(y(t)) \ge h_i(v(t)) = \max_{Qx(t)\le y(t)} \{(HQx(t) + Qf(x(t)))_i\},$$

$i = 1,2,\dots,q$.

Proof.
(Sufficiency) Let $v(x(t)) = Qx(t)$. From (11), we have

$$\begin{aligned} {}_0D_t^\alpha v_i(x(t)) &= \{Q({}_0D_t^\alpha x(t)\}_i \\ &= \{Q\overline{A}x(t) + Qf(x(t))\}_i \\ &= \{HQx(t) + Qf(x(t))\}_i. \end{aligned} \tag{13}$$

Then ${}_0D_t^\alpha y(t) = h(y(t))$ is the comparison system of (6), where

$$h_i(y(t)) \ge h_i(v(t)) = \max_{Qx(t)\le y(t)} \{(HQx(t) + Qf(x(t)))_i\},$$

$i = 1,2,\dots,q$, such that

$$h(\rho) \le -\varepsilon\rho, \quad \varepsilon \in [0,1].$$

Hence, from Theorem 1, the polyhedral set $P(Q,\rho)$ is a positively invariant set of the system (6).

(Necessity) From (8), we have

$${}_0D_t^\alpha v(x(t)) = Q\overline{A}x(t) + Qf(x(t)) \le h(v(x(t))).$$

Let

$$h_i(y(t)) \ge h_i(v(t)) = \max_{Qx(t)\le y(t)} \{(Q\overline{A}x(t) + Qf(x(t)))_i\}, \tag{14}$$

$i = 1,2,\dots,q$, and the comparative system of (6) is obtained.

$P(Q,\rho)$ is a positively invariant set of the system (6). Then

$$Qx(0) = 0 \Rightarrow Q\overline{A}x(t) + Qf(x(t)) = 0. \tag{15}$$

Since

$$\lim_{\|x(t)\|\to 0} \frac{\|f(x(t))\|}{\|x(t)\|} = 0,$$

from (15) we have

$$Qx(0) = 0 \Rightarrow Q\overline{A}x(t) = 0. \tag{16}$$

Then there exists a matrix H such that $Q\overline{A} = HQ$. Thus,

$$\begin{aligned} h_i(y(t)) &\geq h_i(v(t)) \\ &= \max_{Qx(t)\leq y(t)} \{(HQx(t) + Qf(x(t)))_i\}, \end{aligned}$$

$i = 1, 2, \ldots, q$. Moreover, by Theorem 1, $h(\rho) \leq -\varepsilon\rho,\ \ \varepsilon \in [0, 1]$. ■

Let

$$\begin{aligned} H_{ij}^{+} &= \begin{cases} h_{ij} & \text{for } i = j, \\ \max(H_{ij}, 0) & \text{for } i \neq j, \end{cases} \\ H_{ij}^{-} &= \begin{cases} 0 & \text{for } i = j, \\ \max(-H_{ij}, 0) & \text{for } i \neq j. \end{cases} \end{aligned} \tag{17}$$

Then, a direct consequence of Theorem 2 is the following result.

Theorem 3. *The polyhedral set*

$$U(K, -\underline{w}, \overline{w}) = \{x(t) \in \mathbb{R}^n : -\underline{w} \leq Kx \leq \overline{w}\},$$

with $K \in \mathbb{R}^{m\times n}, \underline{w} \in \mathbb{R}^m$, $\overline{w} \in \mathbb{R}^m$ and $\underline{w} > 0, \overline{w} > 0$, is a positively invariant set of the system (6) if and only if there exists a matrix $H \in M_m$, such that

$$K\overline{A} - HK = 0 \tag{18}$$

and

$$h(\widehat{w}) \leq -\varepsilon\widehat{w}, \quad \varepsilon \in [0, 1], \tag{19}$$

where

$$\begin{aligned} h_i(y) &\geq h_i(v(t)) \\ &= \max_{Kx(t)\leq y} \left\{ \begin{array}{c} (HKx(t) + Kf(x(t)))_i \\ (-HKx(t) - Kf(x(t)))_i \end{array} \right\}, \\ &\qquad i = 1, 2, \ldots, m \end{aligned}$$

and

$$\widehat{w} = \begin{bmatrix} \overline{w} \\ \underline{w} \end{bmatrix}.$$

Proof. From (17) we get

$$H = H^{+} - H^{-}. \tag{20}$$

Replace H in (18) with H in (20), so that

$$\begin{aligned} K\overline{A} &= HK = (H^{+} - H^{-})K, \\ -K\overline{A} &= H(-K) = (H^{+} - H^{-})(-K) \\ &= (H^{-} - H^{+})K. \end{aligned}$$

Hence

$$\begin{bmatrix} K \\ -K \end{bmatrix} \overline{A} = \begin{bmatrix} H^{+} & H^{-} \\ H^{-} & H^{+} \end{bmatrix} \begin{bmatrix} K \\ -K \end{bmatrix}. \tag{21}$$

Let

$$Q = \begin{bmatrix} K \\ -K \end{bmatrix}, \quad \rho = \widehat{w} = \begin{bmatrix} \overline{w} \\ \underline{w} \end{bmatrix}.$$

Then

$$\begin{aligned} h_i(y) &\geq h_i(v(t)) \\ &= \max_{Qx(t)\leq y} \{(HQx(t) + Qf(x))_i\}, \\ &\qquad i = 1, 2, \ldots, m, \end{aligned}$$

from (19) and (21), and there exists a matrix

$$\widehat{H} = \begin{bmatrix} H^{+} & H^{-} \\ H^{-} & H^{+} \end{bmatrix} \in M_{2m},$$

such that

$$Q\overline{A} = \widehat{H}Q, \quad h(\rho) \leq -\varepsilon\rho, \quad \varepsilon \in [0, 1].$$

By means of Theorem 2, the polyhedral set $P(Q, \rho) = \{x(t) \in \mathbb{R}^n : Qx(t) \leq \rho\}$ is a positively invariant set of system (6). Furthermore, the set $U(K, -\underline{w}, \overline{w})$ can be reformulated in the form of $P(Q, \rho)$. We have the conclusion that $U(K, -\underline{w}, \overline{w})$ is a positively invariant set of the system (6). ■

By virtue of the Farkas lemma, we obtain the condition of $P(Q, \rho) \subseteq U(K, -\underline{w}, \overline{w})$.

Theorem 4. *The polyhedral set $P(Q, \rho)$ and $U(K, -\underline{w}, \overline{w})$ satisfy*

$$P(Q, \rho) \subseteq U(K, -\underline{w}, \overline{w})$$

if and only if there exists a nonnegative matrix $L \in \mathbb{R}^{2m\times 2q}$ such that

$$\begin{aligned} L \begin{pmatrix} Q \\ Q \end{pmatrix} &= \begin{pmatrix} K \\ -K \end{pmatrix}, \\ L \begin{pmatrix} \rho \\ \rho \end{pmatrix} &\leq \begin{pmatrix} \overline{w} \\ \underline{w} \end{pmatrix}. \end{aligned} \tag{22}$$

Proof. This theorem has a different form in the work of Athanasopoulos *et al*. (2010), and the proof in that work is based on the Farkas lemma. We give a simple proof here since the set has a different form $P(Q, \rho) \subseteq U(K, \overline{w})$ is equivalent to the existence of a nonnegative matrix $L_1 \in \mathbb{R}^{m\times q}$ such that $L_1 Q = K$ and $L_1\rho = \overline{w}$. On the other hand, $P(Q, \rho) \subseteq U(-K, \underline{w})$ is equivalent to the existence of a nonnegative matrix $L_2 \in \mathbb{R}^{m\times q}$ such that $L_2 Q = -K$ and $L_2\rho = \underline{w}$.

Hence, $P(Q, \rho) \subseteq U(K, -\underline{w}, \overline{w})$ if and only if there exists a nonnegative matrix $L \in \mathbb{R}^{2m\times 2q}$ such that (22) holds. ■

3.2. Existence conditions for a constrained controller.

For a state feedback control $u = Kx(t)$, by means of the inequality (5), we have

$$U(K, -\underline{w}, \overline{w}) = \{x \in \mathbb{R}^n : -\underline{w} \le Kx(t) \le \overline{w}\}.$$

It is obvious that a control law $u = Kx(t)$ is a solution to the CRP for the fractional-order nonlinear continuous-time system (1) if and only if the closed-loop system (6) is asymptotically stable at the origin for all initial states x_0 satisfying (3), while the corresponding trajectory $x(t; t_0, x_0)$ does not violate the state constraint (4) as well as the control input constraint (5) for any $t \ge t_0$.

We formulate this conclusion as the following result.

Theorem 5. *The control law $u = Kx(t)$ with $K \in \mathbb{R}^{m\times n}$ is a solution to the CRP for the system (1) if and only if*

(i) $u = Kx(t)$ stabilizes the system (1);

(ii) there exists a positively invariant set $\Omega \in \mathbb{R}^n$ of the closed-loop system (6) such that $P(Q, \rho) \subseteq \Omega \subseteq P(M, c)$ and $\Omega \subseteq U(K, -\underline{w}, \overline{w})$.

Proof. Determine a state feedback control law K and input $u(t) = Kx(t)$ such that the system (6) is asymptotically stable at the origin for all initial states x_0 satisfying the inequality (3) if and only if $u = Kx(t)$ can stabilize the system (1).

The corresponding trajectory $x(t; 0, x_0)$ violates neither the state constraint (4) nor the control input constraint (5) for any $t \ge 0$ if and only if there exists a positively invariant set $\Omega \in \mathbb{R}^n$ for the closed-loop system (6) such that $P(Q, \rho) \subseteq \Omega \subseteq P(M, c)$ and $\Omega \subseteq U(K, -\underline{w}, \overline{w})$.

Thus, if the conditions (i) and (ii) are satisfied, the control law $u(t) = Kx(t)$ is a solution to the CRP for system (1). ■

If $P(Q, \rho)$ is a positively invariant set of the system (6), from Definition 2, it must satisfy $P(Q, \rho) \subseteq P(M, c)$, if $P(Q, \rho)$ is equal to Ω. Then we have the following result:

Corollary 1. *The control law $u = Kx(t)$ with $K \in \mathbb{R}^{m\times n}$ is a solution to the CRP if*

(i) $u = Kx(t)$ stabilizes the system (1);

(ii) $P(Q, \rho)$ is a positively invariant set of the closed-loop system (6) and $P(Q, \rho) \subseteq U(F, -\underline{w}, \overline{w})$.

This condition can also be in the form of the following theorem.

Theorem 6. *The control law $u(t) = Kx(t)$ is a solution to the CRP for the system (1) if and only if there exist matrices $K \in \mathbb{R}^{m\times n}$, a nonnegative matrix $L \in \mathbb{R}^{2m\times 2q}$ and $H \in M_m$ such that*

$$\begin{aligned} Q\overline{A} &= HQ, \\ H\rho + \eta(\rho) &< -\varepsilon\rho, \\ L\begin{pmatrix} Q \\ Q \end{pmatrix} &= \begin{pmatrix} K \\ -K \end{pmatrix}, \\ L\begin{pmatrix} \rho \\ \rho \end{pmatrix} &\le \begin{pmatrix} \overline{w} \\ \underline{w} \end{pmatrix}. \end{aligned} \tag{23}$$

Proof. If there is a state feedback control law K such that $\overline{A} = A + BK$, then the system (1) becomes (6). If

$$\eta_i(y(t)) \ge \max_{Qx(t)\le y(t)} \{Qf(x(t))\}_i,$$

by virtue of (13), we have

$$\begin{aligned} {}_0D_t^\alpha v(x(t)) &= \{HQx(t) + Qf(x(t))\} \\ &\le \max_{Qx(t)\le y(t)} \{HQx(t) + Qf(x(t))\} \\ &= h(v(x(t))) \\ &\le \max_{Qx(t)\le y(t)} \{Hy(t) + \eta(y(t))\} \\ &= h(y(t)) = {}_0D_t^\alpha y(t) \end{aligned} \tag{24}$$

and $h(\rho) = H\rho + \eta(\rho) < -\varepsilon\rho$. By virtue of Remark 3, system (6) is asymptotically stable, which means that $u = Kx(t)$ stabilizes system (1).

From Theorems 2 and 4, (23) and (24), $P(Q, \rho)$ is a positively invariant set of the closed-loop system (6) such that $P(Q, \rho) \subseteq U(F, -\underline{w}, \overline{w})$. By virtue of Corollary 1, Theorem 6 is verified. ■

If the initial state $x_0 \in U(K, -\underline{w}, \overline{w})$, and $U(K, -\underline{w}, \overline{w})$ is the positively invariant set of system (6), by Definition 2, we have $U(K, -\underline{w}, \overline{w}) \subseteq P(M, c)$. When $U(K, -\underline{w}, \overline{w})$ is equal to Ω, we have the following sufficient condition.

Corollary 2. *The control law $u = Kx(t)$ with $K \in \mathbb{R}^{m\times n}$ is a solution to the CRP for the system (1) if*

(i) $u = Kx(t)$ stabilizes the system (1);

(ii) $U(K, -\underline{w}, \overline{w})$ is a positively invariant set of the closed-loop the system (6).

Theorem 7. *The control law $u(t) = Kx(t)$ is a solution to the CRP for the system (1) if and only if there exist matrices $K \in \mathbb{R}^{m\times n}$ and $H \in M_m$ such that*

$$\begin{aligned} K\overline{A} &= HK, \\ \widehat{H}\widehat{w} + \eta(\widehat{w}) &< -\varepsilon\widehat{w}, \\ -\underline{w} \le Kx(t) &\le \overline{w}, \end{aligned} \tag{25}$$

where

$$\eta_i(y(t)) \geq \max_{Kx(t)\leq y(t)} \{Kf(x(t))\}_i$$

and

$$\widehat{H} = \begin{bmatrix} H^+ & H^- \\ H^- & H^+ \end{bmatrix},$$

$$\widehat{w} = \begin{bmatrix} \overline{w} \\ \underline{w} \end{bmatrix}.$$

Proof. If there is a state feedback control law K, let $\overline{A} = A + BK$. Then the system (1) becomes (6). If

$$\eta_i(y(t)) \geq \max_{Kx(t)\leq y(t)} \{Kf(x(t))\}_i$$

and the initial state $x_0 \in U(K, -\underline{w}, \overline{w})$,

$$v(x(t)) : \begin{cases} Kx(t) \leq \overline{w}, \\ -Kx(t) \leq \underline{w}, \end{cases}$$

we have

$$\begin{aligned} &{}_0D_t^{\alpha} v(x(t)) \\ &= \begin{Bmatrix} HKx(t) + Kf(x(t)) \\ -HKx(t) - Kf(x(t)) \end{Bmatrix} \\ &\leq \max_{Kx(t)\leq y(t)} \left\{ \widehat{H} \begin{pmatrix} Kx(t) \\ -Kx(t) \end{pmatrix} + \begin{pmatrix} Kf(x(t)) \\ -Kf(x(t)) \end{pmatrix} \right\} \\ &= h(v(x(t))) \\ &\leq \max_{Kx(t)\leq y(t)} \left\{ \widehat{H} \begin{pmatrix} y(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} \eta(y(t)) \\ \eta(y(t)) \end{pmatrix} \right\} \\ &= h(y(t)) = {}_0D_t^{\alpha} y(t), \end{aligned} \tag{26}$$

and $h(\widehat{w}) = \widehat{H}\widehat{w} + \eta(\widehat{w}) < -\varepsilon\widehat{w}$. From Remark 3, the system (6) is asymptotically stable and $u = Kx(t)$ stabilizes the system (1).

From Theorem 2 and (25), we have that $U(K, -\underline{w}, \overline{w})$ is a positively invariant set of the closed-loop system (6). From Corollary 2, the proof of Theorem 7 is completed. ■

4. Controller design method

We discuss the controller design method in two cases.

Case 1. If the initial state x_0 satisfies (3) and the state satisfies the state constraint (4) as well as the control constraint (5), by Theorem 6, we obtain a sufficient and necessary condition for the solution to CRP for the system (1). However, for the system (1), it is important not only to guarantee stability, but also to increase the rate of convergence to the equilibrium. This can be done by defining a linear programming problem with the objective function

$$S(K, H, \varepsilon) = \varepsilon \tag{27}$$

and the constraints

$$\begin{cases} Q\overline{A} = HQ, \\ H\rho + \eta(\rho) < -\varepsilon\rho, \\ L\begin{pmatrix} Q \\ Q \end{pmatrix} = \begin{pmatrix} K \\ -K \end{pmatrix}, \\ L\begin{pmatrix} \rho \\ \rho \end{pmatrix} \leq \begin{pmatrix} \overline{w} \\ \underline{w} \end{pmatrix}. \end{cases} \tag{28}$$

Case 2. If the initial state x_0 satisfies the constraint (3) and the control input satisfies the constraint (5), by Theorem 7, a solution can be obtained by defining a linear programming problem with the objective function

$$S(K, H, \varepsilon) = \varepsilon \tag{29}$$

and the constraints

$$\begin{cases} K\overline{A} = HK, \\ \widehat{H}\widehat{w} + \eta(\widehat{w}) < -\varepsilon\widehat{w}, \\ -\underline{w} \leq Kx(t) \leq \overline{w}. \end{cases} \tag{30}$$

Hence, maximizing ε increases the rate of convergence to the origin.

5. Numerical examples

Example 1. Consider the continuous-times fractional order nonlinear systems

$$\begin{cases} {}_0D_t^{\alpha} x(t) = Ax(t) + Bu(t) + f(x(t)), \quad t > 0, \\ x(0) = x_0, \quad -\infty < t \leq 0, \\ \alpha = 0.7 \end{cases} \tag{31}$$

with

$$A = \begin{bmatrix} -2.5 & 0.8 \\ -0.85 & 1.4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix},$$

$$f(x(t)) = 0.02 \begin{pmatrix} x_1^2 + x_1x_2 \\ x_2^2 + x_1x_2 \end{pmatrix},$$

and a set of initial states $P(Q, \rho)$ defined by

$$\begin{aligned} 2x_2 &\leq 0.8, \\ x_1 + 2x_2 &\leq 2.4, \\ -0.5x_1 + 4x_2 &\geq -1.8. \end{aligned} \tag{32}$$

The state vector $P(M, c)$ satisfies the constraints

$$\begin{aligned} -5 \leq x_1 \leq 7, \\ -1 \leq x_2 \leq 1, \end{aligned} \tag{33}$$

and the control vector $u(t) \in U(K, -\underline{w}, \overline{w})$ satisfies the constraint

$$-6 \leq u(t) \leq 9. \tag{34}$$

The system (31) is unstable since the eigenvalues of A are $\lambda_1 = -2.3171$ and $\lambda_2 = 1.2171$. The CRP for the fractional-order nonlinear continuous-time system (31)–(34) is to determine a state feedback control law K and an input $u(t) = Kx(t)$ such that for all initial states x_0 satisfiying inequality (32) the closed-loop system is asymptotically stable, the corresponding trajectory $x(t;0,x_0)$ does not violate state constraint (33), and the control input does not violate constraint (34) for any $t \geq 0$.

Let $Q = \begin{bmatrix} q_{i1} & q_{i2} \end{bmatrix}$, $i = 1,2,3$. From (27) and (28), we obtain

$$\begin{aligned} &\max_{Qx(t)\leq y(t)} \{Qf(x(t))\} \\ &= 0.02 \max_{Qx(t)\leq y(t)} \Big(q_{i1}(x_1^2 + x_1x_2) \\ &\quad + q_{i2}(x_2^2 + x_1x_2)\Big) \\ &\leq 0.02 y_i \max\left\{ \frac{y_1}{q_{12}}, \frac{y_2}{\min(q_{21},q_{22})}, \frac{y_3}{\min(q_{31},q_{32})} \right\}. \end{aligned} \tag{35}$$

Then

$$\begin{aligned} \eta(y(t)) = 0.02\,(y_i) \max\Big\{ &\frac{y_1}{q_{12}}, \frac{y_2}{\min(q_{21},q_{22})}, \\ &\frac{y_3}{\min(q_{31},q_{32})} \Big\}. \end{aligned}$$

Thus,

$$\begin{aligned} \eta(\rho) &= 0.02 \begin{pmatrix} 0.8 \\ 2.4 \\ 1.8 \end{pmatrix} \max\left\{ \frac{0.8}{2}, \frac{2.4}{1}, \frac{1.8}{0.5} \right\} \\ &= \begin{pmatrix} 0.0576 & 0.1728 & 0.1296 \end{pmatrix}^T, \end{aligned}$$

and the proposed method (27), (28) is formulated as a linear programming problem with the objective function

$$S(K,H,\varepsilon) = \varepsilon$$

under the constraints

$$\begin{cases} \begin{bmatrix} 0 & 2 \\ -1 & 2 \\ 0.5 & -4 \end{bmatrix} \left(\begin{bmatrix} -2.5 & 0.8 \\ -0.85 & 1.4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} K \right) \\ \qquad = H \begin{bmatrix} 0 & 2 \\ -1 & 2 \\ 0.5 & -4 \end{bmatrix}, \\ H \begin{bmatrix} 0.8 \\ 2.4 \\ 1.8 \end{bmatrix} + \begin{pmatrix} 0.0576 \\ 0.1728 \\ 0.1296 \end{pmatrix} \leq -\varepsilon \begin{bmatrix} 0.8 \\ 2.4 \\ 1.8 \end{bmatrix}, \\ L \begin{pmatrix} 0 & 2 \\ -1 & 2 \\ 0.5 & -4 \\ 0 & 2 \\ -1 & 2 \\ 0.5 & -4 \end{pmatrix} = \begin{pmatrix} K \\ -K \end{pmatrix}, \; L \begin{pmatrix} 0.8 \\ 2.4 \\ 1.8 \\ 0.8 \\ 2.4 \\ 1.8 \end{pmatrix} \leq \begin{pmatrix} 9 \\ 6 \end{pmatrix}, \\ \qquad h_{ij} \geq 0, \quad \varepsilon > 0. \end{cases}$$

The solution of this linear programming problem is

$$\varepsilon = 0.128,$$

$$K = \begin{bmatrix} 1.5, -4 \end{bmatrix},$$

$$L = \begin{bmatrix} 0 & 0 & 0 & 4 & 0 & 3 \\ 0 & 0 & 0 & 0.5 & 1.5 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} -0.8 & 0.2 & 0 \\ 1.8 & -0.8 & 0 \\ 0 & 0 & -0.2 \end{bmatrix}.$$

The resulting system (6) for (31) is stable because the eigenvalues of $\overline{A}$ are $\lambda_1 = -1.4$ and $\lambda_2 = -0.2$.

The trajectory $x(t)$ of (6) for the system (31) under the initial state $x(0) = [-1.6 \quad 0.4]^T$ satisfies the state constraint inequalities (33) and the control constraint inequalities (34), which is shown in Fig.1.

In Fig. 2, the trajectory emanating from the initial state $x(0) = [-1.6 \;\; 0.4]^T$ is shown; it is in the constraint region. The resulting system is asymptotically stable to the origin with the time increasing just as well.

The result in Fig. 3 shows that the feedback control law emanating from the initial state is also in the domain of the constraint region (34).

These results show that there exists a linear state feedback control $u(t) = Kx(t)$, which satisfies the inequalities (34) and yields an asymptotically stable system during the interval $t \in [0, 120]$, and the state constraint (33) as well as the control constraint inequalities (34) are satisfied. Thus, $u(t) = 1.5x_1 - 4x_2$ is a solution to the CRP of the system (1). The behavior of the resulting system is asymptotically stable to the origin with the time increasing. Example 1 also shows that our controller design method is effective and can find the solution of CRP for the system (31). ♦

Example 2. Consider the continuous-time fractional order linear system

$$\begin{cases} {}_0D_t^\alpha x(t) = Ax(t) + Bu(t) + f(x(t)), \quad t > 0, \\ x(0) = x_0, \quad -\infty < t \leq 0, \\ \alpha = 0.5 \end{cases} \tag{36}$$

with the

$$A = \begin{bmatrix} 1 & -3 \\ -1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -1 & 0.5 \end{bmatrix},$$

$$f(x(t)) = 0.05 \begin{pmatrix} x_1x_2 \\ x_2^2 \end{pmatrix}.$$

The control vector $u(t) \in U(K, -\underline{w}, \overline{w})$ satisfies the constraint

$$\begin{bmatrix} -1 \\ -2 \end{bmatrix} \leq Kx(t) \leq \begin{bmatrix} 2 \\ 4 \end{bmatrix}. \tag{37}$$

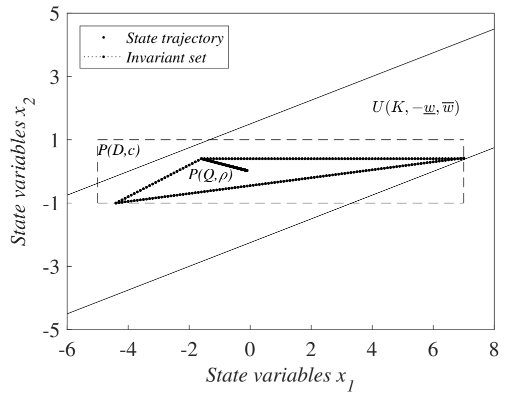

Fig. 1. State trajectory of the system.

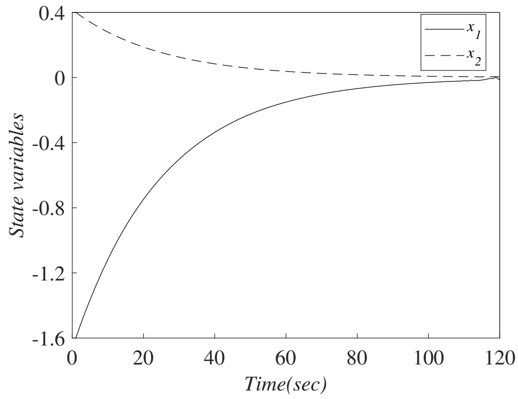

Fig. 2. Trajectory emanating from the initial state $x(0) = [-1.6 \;\; 0.4]^T$.

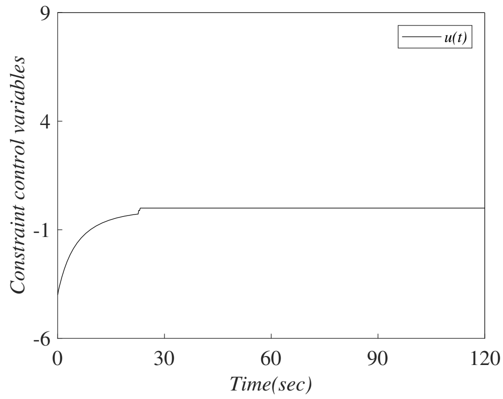

Fig. 3. Constrained control for the trajectory emanating from the initial state $x(0) = [0.654 \;\; 0.449]^T$.

The system (36) is unstable because the eigenvalues of A are $\lambda_1 = 0.3542$ and $\lambda_2 = 5.6458$. The CRP of the fractional-order nonlinear continuous-time system (36) is to determine a state feedback control law K and input $u(t) = Kx(t)$ such that for all initial states x_0 satisfying the inequality (37) the closed-loop system is asymptotically stable, and the corresponding trajectory $x(t; 0, x_0)$ does not violate (37) for any $t \geq 0$.

Let

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}.$$

From (29) and (30), we obtain

$$\begin{aligned}
&\max_{Kx(t)\leq y(t)} \{Kf(x(t))\} \\
&= 0.05 \max_{Kx(t)\leq y(t)} \begin{pmatrix} k_{11}x_1x_2 + k_{12}x_2^2 \\ k_{21}x_1x_2 + k_{22}x_2^2 \end{pmatrix} \\
&\leq 0.05 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \max\left\{\frac{y_1}{k_{12}}, \frac{y_2}{k_{22}}\right\}.
\end{aligned} \tag{38}$$

Then

$$\eta(y(t)) = 0.05 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \max\left\{\frac{y_1}{k_{12}}, \frac{y_2}{k_{22}}\right\}$$

and

$$\eta(\widehat{w}) = 0.05 \left\{ \begin{matrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \max\left\{\frac{1}{k_{12}}, \frac{2}{k_{22}}\right\} \\ \begin{pmatrix} 2 \\ 4 \end{pmatrix} \max\left\{\frac{2}{k_{12}}, \frac{4}{k_{22}}\right\} \end{matrix} \right\}.$$

The CRP for (36) is formulated as the linear programming problem with the objective function

$$S(K, H, \varepsilon) = \varepsilon$$

under the constraints

$$\begin{cases}
K\left(\begin{bmatrix} 1 & -3 \\ -1 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & 0.5 \end{bmatrix} K\right) = HK, \\
\begin{bmatrix} H^+ & H^- \\ H^- & H^+ \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix} + \\
0.05 \left\{ \begin{matrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \max\left\{\frac{1}{k_{12}}, \frac{2}{k_{22}}\right\}, \\ \begin{pmatrix} 2 \\ 4 \end{pmatrix} \max\left\{\frac{2}{k_{12}}, \frac{4}{k_{22}}\right\} \end{matrix} \right\} \leq -\varepsilon \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}, \\
\begin{bmatrix} -1 \\ -2 \end{bmatrix} \leq u(t) \leq \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \\
\varepsilon > 0.
\end{cases}$$

We have the solution

$$\varepsilon = 0.9112,$$

$$K = \begin{bmatrix} -2.051 & 7.444 \\ -1.255 & 2.629 \end{bmatrix},$$

$$H = \begin{bmatrix} -4.11 & 0.43 \\ 0.33 & -1.96 \end{bmatrix}.$$

The resulting system (36) is stable because the eigenvalues of $\overline{A}$ are $\lambda_1 = -1.8938$ and $\lambda_2 = -4.1707$.

The trajectory $x(t)$ of the system (6) under the initial state $x(0) = [0.654 \quad 0.449]^T$ as well as the set $U(K, -\underline{w}, \overline{w})$ are shown in Fig. 4. They all satisfy the constraints and in the domain of the constraint. The behavior of the resulting system is asymptotically stable to the origin with the time increase.

In Fig. 5, the control law evolution emanating from the initial state is shown; it is in the domain of the constraint (37). With the initial state $x(0) = [0.654 \quad 0.449]^T$, in Fig. 6, the constrained control for the trajectory emanating from the initial state also remains in the domain of constraint (37). The behavior of the resulting system is asymptotically stable to the origin with the time increasing.

From Example 2, we have that there exists a linear state feedback control $u(t) = Kx(t)$ for the system (36), which makes the trajectory under the initial state satisfying the control constraint (37) asymptotically stable during the interval $t \in [0, 60]$. Therefore,

$$u(t) = \begin{bmatrix} -2.051 & 7.444 \\ -1.255 & 2.629 \end{bmatrix} x(t)$$

is a solution to the CRP of the system (1). The behavior of the resulting system is asymptotically stable to the origin with the time increasing just as well. Example 2 also shows that our method is effective and can find a solution of the CRP for the system (36). ♦

Remark 4. Designing control systems that maintain stability and performance with state and/or control constraints is a topic of continuous interest. the proposed methods are based on factors such as the l_1 norm, predictive control, polynomial approaches and positive invariance. Most of these methods are discussed only for linear systems. For nonlinear systems, especially for fractional order nonlinear systems, analysis and synthesis approaches of control systems subjected to constraints are proposed through the Lyapunov method, or by using the saturation control method. Both depend on linear matrix inequalities (LMIs) as a tool, and are more complex than the method proposed in this paper especially when the dimension of the state variables is large. But linear programming can deal with problems of high dimensional variables, and can be operated by most mathematical packages. In this aspect, we claim that the method proposed in this paper is effective and more implementable than other available methods from the computational point of view.

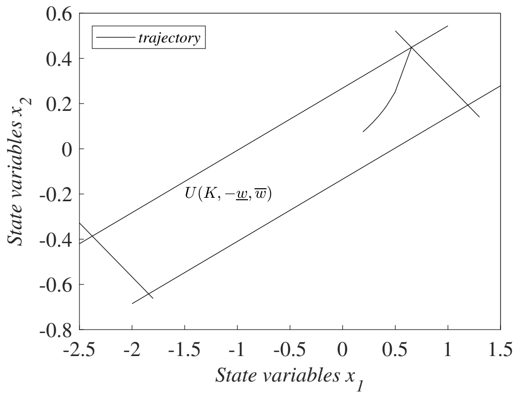

Fig. 4. Trajectory and the positively invariant set of the system.

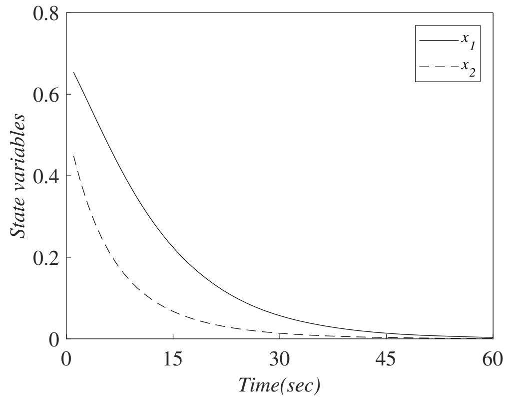

Fig. 5. Trajectory emanating from the initial state.

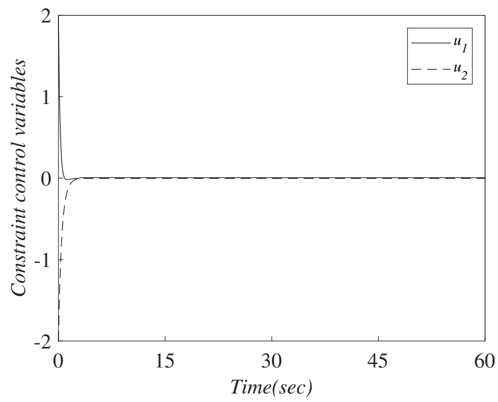

Fig. 6. Constrained control for the trajectory emanating from the initial state $x(0) = [0.654 \;\; 0.449]^T$.

6. Conclusion

In this paper, the constrained regulation problem for a class of fractional-order nonlinear continuous-time systems was studied. By virtue of the comparison principle and positively invariant set theory, conditions guaranteeing positive invariance of a polyhedron for a class of fractional-order nonlinear system were established. We also provided an LP model and the corresponding algorithm for determining a linear state feedback controller for the CRP. Compared with other available methods, our algorithm is readily implemented from a computational point of view since LPs can be solved by any off-the-shelf mathematical software. Numerical examples show that the proposed method is effective.

We conjecture that our approach will still work for higher-order (fractional order) fractional systems since it depends on positively invariant sets and a Metzler matrix and is independent of the system order. We will investigate the validity of the proposed approach for higher-order fractional systems and the effects of external disturbances on the method in the future. Regarding higher-dimensional systems, we think that this is exactly the advantage of our method. The proposed method can deal with higher dimensional systems readily because of the intrinsic quality of the LP.

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