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QUALITATIVE ANALYSIS OF SYMMETRIC FUZZY STOCHASTIC DIFFERENTIAL EQUATIONS

MAREK T. MALINOWSKI [a]

[a] Department of Applied Mathematics
Tadeusz Kościuszko Cracow University of Technology
ul. Warszawska 24, 31-155 Kraków, Poland
e-mail: marek.malinowski@pk.edu.pl

In this paper, a special approach to stochastic differential equations is explored. Specifically, the values of the mappings involved are fuzzy sets, rather than the usual single values on the real line. Additionally, the equations under consideration are symmetric, meaning that the terms of drift and diffusion appear on both sides of the equation, which is crucial for the properties of the solutions. The primary goal of this paper is to establish certain qualitative results, such as the existence of a unique solution and stability of the solution. These results are obtained under the assumption that the coefficients of the equation satisfy a condition that is weaker than the standard Lipschitz condition. It is also noted that the results obtained can be applied to symmetric fuzzy random integral equations and deterministic symmetric fuzzy integral equations.

Keywords: fuzzy mathematics, fuzzy stochastic differential equations, fuzzy stochastic process, existence of solution, uniqueness of solution.

1. Introduction

Stochastic differential equations have become a well-explored area of research, as evidenced by numerous probabilistic studies (e.g., Arnold, 1974; Gihman and Skorohod, 1972; Kloeden and Platen, 1992; Øksendal, 2003; Jackowska-Zduniak, 2022) as well as analytic approaches (e.g., Lee *et al.*, 2022). This interest stems from their extensive applications in modeling the dynamics of phenomena influenced by stochastic disturbances. Typically, the states of such phenomena are represented by single values, often as real numbers. At any given moment, the states are unpredictable due to inherent randomness. Thus, stochastic equations can be described as equations that embody uncertainty, specifically of a stochastic nature. However, it is possible that not all parameters of the phenomenon in question, such as the initial value, are precisely defined by a single real number. Instead, we might know that the initial value lies within the interval $[90, 110]$, or we might have information in the form of a linguistic expression, such as ”approximately 100”. This also introduces uncertainty regarding the initial value, but this uncertainty is not stochastic in nature. The shift from dealing with numbers to dealing with words is facilitated by fuzzy sets (e.g., Zadeh, 1965; 2002; Xia *et al.*, 2022). By considering two types of uncertainty, namely stochasticity and fuzziness, fuzzy stochastic differential equations can be formulated. The author has conducted extensive research in this new field (see Malinowski, 2013; 2014; 2016b; 2016a; 2020) and established a framework for studying such equations. Fuzziness has been incorporated into stochastic differential equations in such a way that the values of the stochastic processes that solve these equations are fuzzy sets, meaning the solution is essentially a fuzzy stochastic process. Although the concept is straightforward, the mathematical foundation is quite complex. Within the framework established by Malinowski (2012b; 2012a; 2013; 2014; 2016b; 2016a; 2020), other researchers are also expanding this theory and exploring its potential applications, as seen in e.g., the works of Bandyopadhyay and Kar (2019), Priyadharsini and Balasubramaniam (2020), Arhrrabi *et al.* (2021), Arshad and Shafqat (2022), Jafari and Malinowski (2023), Jafari and Farahani (2023), Luo *et al.* (2023), Sarhan and Ismail (2023), or Wen *et al.* (2024).

In the papers by Malinowski (2012b; 2012a; 2013; 2014), the structure of the fuzzy stochastic differential equations considered naturally mirrors and extends the form known from classical theory (e.g., Arnold, 1974; Gihman and Skorohod, 1972; Kloeden and Platen, 1992; Øksendal, 2003) to the case of values in fuzzy sets. Without delving into specifics, the integral form is given by

$$x(t) = x_0 \oplus \int_0^t a(s, x(s))\, \mathrm{d}s \oplus \Big\langle \int_0^t b(s, x(s))\, \mathrm{d}B(s) \Big\rangle,$$

where x_0 is a fuzzy random variable, a is a fuzzy set-valued drift coefficient that is random, b is a single-valued random diffusion coefficient, and B is the real-valued Brownian motion. However, as demonstrated by Malinowski (2013), if such equations have solutions, each solution retains the fuzziness in its values over time (the fuzziness is non-decreasing). This characteristic could be quite restrictive, necessitating the construction of equations whose solutions would overcome this limitation. Consequently, Malinowski (2016b) proposed a slightly modified form of the equation, which is as follows:

$$x(t) \oplus (-1) \odot \int_0^t a(s, x(s))\, \mathrm{d}s \\ \oplus \Big\langle (-1) \int_0^t b(s, x(s))\, \mathrm{d}B(s) \Big\rangle = x_0.$$

This seemingly minor modification results in solutions whose fuzziness does not increase over time. This indicates that dealing with fuzzy stochastic differential equations reveals interesting nuances, highlighting the greater subtlety of such equations. To embrace both mentioned cases of equations without having to consider each separately, the author introduced symmetric equations (Malinowski, 2016a; 2020), where fuzzy stochastic integrals appear on both sides of the equation, i.e.,

$$x(t) \oplus \int_0^t a_1(s, x(s))\, \mathrm{d}s \oplus \Big\langle \int_0^t b_1(s, x(s))\, \mathrm{d}B_1(s) \Big\rangle \\ = x_0 \oplus \int_0^t a_2(s, x(s))\, \mathrm{d}s \oplus \Big\langle \int_0^t b_2(s, x(s))\, \mathrm{d}B_2(s) \Big\rangle.$$

This paper will study such equations. Our main goal will be to provide the qualitative properties of symmetric fuzzy stochastic differential equations such as the existence of a unique solution and the stability of solution understood as continuous dependence of the solution with respect to the coefficients of the equation. Like in classical single-valued analysis, the assertion of the existence of a unique solution is a crucial issue because finding an explicit solution to the stochastic differential equation is typically not possible. Consequently, this paper does not focus on solving specific examples or deriving explicit solutions. Instead, it lays the groundwork for future efforts to find approximate solutions to these equations, ensuring the existence and uniqueness of the solutions discussed here. We will involve the weakest conditions used so far for the fuzzy stochastic differential equations, which will guarantee such results. The vast majority of research on stochastic differential equations is conducted with the global Lipschitz condition imposed on the coefficients of the equation (see, e.g., Arnold, 1974; Gihman and Skorohod, 1972; Kloeden and Platen, 1992; Øksendal, 2003). In this paper, we go much further than this standard Lipschitz condition and use a much weaker condition with a function entangled in some integral inequality. Some ideas of this type have been proposed for the classical equations of Yamada (1981) or Taniguchi (1992). We not only achieve the existence of a solution, but we show that it is not very sensitive to small changes in the data in the equation.

Finally, we also note that the analysis performed can be applied effectively to both random (see Malinowski 2009; Vu, 2017, Long, 2018; Srivastava *et al.*, 2022; Atyia *et al.*, 2023) and deterministic fuzzy differential equations (see Kaleva, 1987; Berger and Schwarz, 1995; Pedro *et al.*, 2023; Gomes *et al.*, 2015; Mazandarani and Xiu, 2021). We note this fact because the latter equations can be the subject of completely separate studies from fuzzy stochastic differential equations.

2. Preliminaries

This section of the paper will compile the mathematical frameworks necessary for constructing fuzzy stochastic differential equations. Although these details are available in the papers by Malinowski (2012b; 2012a; 2013), we present them here for the reader's convenience.

By $\mathcal{P}(\mathbb{R}^d)$, we refer to the collection of all non-empty, convex, and compact subsets of $\mathbb{R}^d$. When $d = 1$, the collection $\mathcal{P}(\mathbb{R})$ comprises intervals. The distance between elements in $\mathcal{P}(\mathbb{R}^d)$ is determined using the Hausdorff metric H, which is defined as

$$H(A, B) := \max\Big\{ \sup_{x\in A} \inf_{y\in B} \|x-y\|, \sup_{y\in B} \inf_{x\in A} \|x-y\| \Big\},$$

where $\|\cdot\|$ denotes a norm in $\mathbb{R}^d$. A fuzzy set u in $\mathbb{R}^d$ (see Zadeh, 1965; 2002) can be represented by its membership function (also denoted by u), which maps $\mathbb{R}^d$ to the interval $[0, 1]$. The symbol $\mathcal{F}(\mathbb{R}^d)$ refers to a collection of fuzzy sets $u\colon \mathbb{R}^d \to [0,1]$ such that $[u]^\alpha \in \mathcal{P}(\mathbb{R}^d)$ for every $\alpha \in [0, 1]$, where

$$[u]^\alpha := \{\, x \in \mathbb{R}^d : u(x) \geq \alpha \,\}$$

for $\alpha \in (0, 1]$ and

$$[u]^0 := \mathrm{cl}\{\, x \in \mathbb{R}^d : u(x) > 0 \,\}.$$

The set $[u]^0$ is called the support of the fuzzy set u. A straightforward measure of the fuzziness of u is

$\mathrm{Fuzz}(u) := \mathrm{diam}([u]^0) = \sup\{\|x-y\| : x, y \in [u]^0\}$. If u is a crisp (single-valued) set, then $\mathrm{Fuzz}(u) = 0$. Besides fuzzy sets from the family $\mathcal{F}(\mathbb{R}^d)$, we will also exploit fuzzy sets from its subfamily

$$\mathcal{F}_c(\mathbb{R}^d) = \{u \in \mathcal{F}(\mathbb{R}^d) : \alpha \mapsto [u]^\alpha \text{ is an } H\text{-continuous map}\}.$$

The notation $\langle r \rangle$ represents the characteristic function of the singleton $\{r\}$, where $r \in \mathbb{R}^d$. Clearly, $\langle r \rangle$ belongs to $\mathcal{F}(\mathbb{R}^d)$. The operations of addition $u \oplus v$ and scalar multiplication $r \odot u$ for $u, v \in \mathcal{F}(\mathbb{R}^d)$ and $r \in \mathbb{R}$ are defined levelwise, meaning

$$[u \oplus v]^\alpha = [u]^\alpha + [v]^\alpha$$

(where the right side denotes the Minkowski sum of sets), $[r \odot u]^\alpha = r \cdot [u]^\alpha$, where $\alpha \in [0,1]$. The fuzzy set w is referred to as the Hukuhara difference of the fuzzy sets u and v if $u = v \oplus w$, and w is then denoted by $u \ominus v$. It is important to note that the Hukuhara difference may not always exist, and $u \ominus v \neq u \oplus (-1) \odot v$.

The most frequently used metric in $\mathcal{F}(\mathbb{R}^d)$ is

$$d_\infty(u, v) := \sup_{\alpha \in [0,1]} H([u]^\alpha, [v]^\alpha).$$

For random variables taking values in $\mathcal{F}(\mathbb{R}^d)$, we also need to establish some settings. Let $(\Omega, \mathcal{A}, P)$ be a complete probability space. The mapping $F\colon \Omega \to \mathcal{P}(\mathbb{R}^d)$ is called an $\mathcal{A}$-measurable set-valued random variable (or random set, for short) if

$$\{\omega \in \Omega : F(\omega) \cap O \neq \emptyset\} \in \mathcal{A}$$

for every open set $O \in \mathbb{R}^d$. The set $\mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{P}(\mathbb{R}^d))$ denotes the collection of random sets F that are L^p-integrally bounded, meaning that $\omega \mapsto H(F(\omega), \{0\})$ belongs to $L^p(\Omega, \mathcal{A}, P; \mathbb{R})$.

A mapping $x\colon \Omega \to \mathcal{F}(\mathbb{R}^d)$ is considered a fuzzy random variable if $[x]^\alpha\colon \Omega \to \mathcal{P}(\mathbb{R}^d)$ is a random set for every $\alpha \in [0,1]$. It has been demonstrated that $x\colon \Omega \to \mathcal{F}_c(\mathbb{R}^d)$ qualifies as a fuzzy random variable if and only if

$$x\colon (\Omega, \mathcal{A}) \to (\mathcal{F}_c(\mathbb{R}^d), \mathcal{B}_{d_\infty}) \text{ is } \mathcal{A}|\mathcal{B}_{d_\infty}\text{-measurable},$$

where $\mathcal{B}_{d_\infty}$ represents the σ-algebra generated by the topology induced by the metric d_∞. A fuzzy random variable $x\colon \Omega \to \mathcal{F}(\mathbb{R}^d)$ is said to be L^p-integrally bounded, for $p \geq 1$, if $[x]^0$ is an L^p-integrally bounded random set. The set $\mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{F}(\mathbb{R}^d))$ denotes all L^p-integrally bounded fuzzy random variables.

Let $I := [0,T]$, where T is a fixed positive number. Consider the probability space with a filtration $\{\mathcal{A}_t\}_{t \in I}$ that satisfies the usual hypotheses. The mapping $x\colon I \times \Omega \to \mathcal{F}(\mathbb{R}^d)$ is referred to as a fuzzy stochastic process if, for every $t \in I$, the mapping $x(t,\cdot)\colon \Omega \to \mathcal{F}(\mathbb{R}^d)$ is a fuzzy random variable. This process is said to be d_∞-continuous if almost all of its trajectories (with respect to the probability measure P), i.e., the mappings $x(\cdot,\omega)\colon I \to \mathcal{F}(\mathbb{R}^d)$, are d_∞-continuous functions. A fuzzy stochastic process x is considered nonanticipating if, for every $\alpha \in [0,1]$, the mapping $[x(\cdot,\cdot)]^\alpha$ is measurable with respect to the σ-algebra $\mathcal{N} := \{A \in \mathcal{B}(I) \otimes \mathcal{A} : A^t \in \mathcal{A}_t \text{ for every } t \in I\}$, where $A^t = \{\omega : (t,\omega) \in A\}$. A fuzzy stochastic process x is termed L^p-integrally bounded ($p \geq 1$) if $\mathbb{E}\int_I H^p([x(s,\omega)]^0, \{0\})\,\mathrm{d}s < \infty$. The set $\mathcal{L}^p(I \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}^d))$ denotes the collection of nonanticipating and L^p-integrally bounded fuzzy stochastic processes.

For $t \in I$ and $a \in \mathcal{L}^1(I \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}^d))$ we can define (see Malinowski, 2012b; 2012a) the fuzzy stochastic Lebesgue–Aumann integral $\Omega \ni \omega \mapsto \int_0^t a(s,\omega)\,\mathrm{d}s \in \mathcal{F}(\mathbb{R}^d)$, which is a fuzzy random variable. This is done in such a way that

$$\left[\int_0^t a(s,\omega)\,\mathrm{d}s\right]^\alpha = \int_0^t [a(s,\omega)]^\alpha\,\mathrm{d}s$$

for $\alpha \in [0,1]$ and $\omega \in \Omega$. The integral on the right is the well-known Aumann integral for set-valued functions. The primary characteristics of the fuzzy stochastic Lebesgue–Aumann integral are outlined in the following statement.

Lemma 1. (See Malinowski, 2012b; 2012a) *Let $p \geq 1$. If $a, b \in \mathcal{L}^p(I \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}^d))$ then*

(i) *$I \times \Omega \ni (t,\omega) \mapsto \int_0^t a(s,\omega)\,\mathrm{d}s \in \mathcal{F}(\mathbb{R}^d)$ belongs to $\mathcal{L}^p(I \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}^d))$,*

(ii) *the fuzzy stochastic process $(t,\omega) \mapsto \int_0^t a(s,\omega)\,\mathrm{d}s$ is d_∞-continuous,*

(iii) *P-a.e. for every $t \in I$*

$$\sup_{u \in [0,t]} d_\infty^p\Big(\int_0^u a(s,\omega)\,\mathrm{d}s, \int_0^u b(s,\omega)\,\mathrm{d}s\Big) \leq t^{p-1} \int_0^t d_\infty^p\big(a(s,\omega), b(s,\omega)\big)\,\mathrm{d}s,$$

(iv) *for every $t \in I$*

$$\mathbb{E} \sup_{u \in [0,t]} d_\infty^p\Big(\int_0^u a(s)\,\mathrm{d}s, \int_0^u b(s)\,\mathrm{d}s\Big) \leq t^{p-1} \mathbb{E} \int_0^t d_\infty^p\big(a(s), b(s)\big)\,\mathrm{d}s.$$

Unfortunately, the method used to define the fuzzy stochastic Lebesgue integral cannot be replicated for the construction of the Itô integral for fuzzy stochastic processes. As demonstrated by Ogura (2008) and Zhang (2008), when integrating ordinary set-valued mappings, the set-valued Itô integral cannot be defined as a bounded set, rendering it entirely impractical. Consequently, to address fuzzy stochastic differential equations of the Itô type, the author suggested retaining the stochastic Itô integral in its classical, single-valued form.

Having laid out the mathematical foundation, we can now proceed to the main section of the paper.

3. Existence of a unique solution

Consider $0 < T < \infty$, $I = [0, T]$, and let $(\Omega, \mathcal{A}, P)$ be a complete probability space with a filtration $\{\mathcal{A}_t\}_{t\in I}$ that satisfies the usual conditions. Let $B_1 = \{B_1(t)\}_{t\in I}$ and $B_2 = \{B_2(t)\}_{t\in I}$ be two one-dimensional $\{\mathcal{A}_t\}$-Brownian motions (not necessarily independent) defined on this space.

We start this section by clearly presenting the form of the equation under study. We will examine symmetric fuzzy stochastic differential equations, which can be expressed in their symbolic differential form as follows:

$$\begin{aligned} &\mathrm{d}x(t) \oplus a_1(t, x(t))\,\mathrm{d}t \oplus \langle b_1(t, x(t))\mathrm{d}B_1(t)\rangle \\ &\quad \overset{I\ P.1}{=} a_2(t, x(t))\,\mathrm{d}t \oplus \langle b_2(t, x(t))\mathrm{d}B_2(t)\rangle \qquad (1) \\ &\quad \text{with initial condition } x(0) \overset{P.1}{=} x_0, \end{aligned}$$

and where $a_1, a_2 \colon I \times \Omega \times \mathcal{F}(\mathbb{R}^d) \to \mathcal{F}(\mathbb{R}^d)$, $b_1, b_2 \colon I \times \Omega \times \mathcal{F}(\mathbb{R}^d) \to \mathbb{R}^d$ and $x_0 \colon \Omega \to \mathcal{F}(\mathbb{R}^d)$ is a fuzzy random variable. The symbol "$I\ P.1$" above the equal sign indicates that this equality is valid for every $t \in I$ and occurs with probability P equal to 1. Likewise, the symbol "$P.1$" above the equal sign signifies that this equality holds with probability P equal to 1.

The symbolic representation (1) of this differential equation necessitates a precise definition of what constitutes a solution to such an equation.

Definition 1. A solution on the interval I to equation (1) is defined as a fuzzy stochastic process $x \colon I \times \Omega \to \mathcal{F}(\mathbb{R}^d)$ that satisfies the following conditions:

(i) $x \in \mathcal{L}^2(I \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}^d))$,

(ii) x is d_∞-continuous,

(iii) it holds

$$\begin{aligned} &x(t) \oplus \int_0^t a_1(s, x(s))\,\mathrm{d}s \oplus \left\langle \int_0^t b_1(s, x(s))\,\mathrm{d}B_1(s) \right\rangle \qquad (2) \\ &\overset{I\ P.1}{=} x_0 \oplus \int_0^t a_2(s, x(s))\,\mathrm{d}s \oplus \left\langle \int_0^t b_2(s, x(s))\,\mathrm{d}B_2(s) \right\rangle. \end{aligned}$$

The terms $\int a_1\,\mathrm{d}s$ and $\int a_2\,\mathrm{d}s$ in (2) represent the fuzzy stochastic Lebesgue–Aumann integrals discussed in the previous section, while $\int b_1 \mathrm{d}B_1(s)$ and $\int b_2 \mathrm{d}B_2(s)$ denote the classical $\mathbb{R}^d$-valued stochastic Itô integrals (see, e.g., Arnold, 1974; Gihman and Skorohod, 1972). Ogura (2008) demonstrates that defining the fuzzy stochastic Itô integral in the same manner as the fuzzy stochastic Lebesgue–Aumann integral is not feasible. Therefore, in our equations, the diffusion parts are considered as single-valued Itô stochastic integrals. Additionally, from the representation (2) for the solution x, we obtain the form

$$\begin{aligned} x(t) &\overset{I\ P.1}{=} \left[x_0 \oplus \int_0^t a_2(s, x(s))\,\mathrm{d}s \right] \\ &\ominus \int_0^t a_1(s, x(s))\,\mathrm{d}s \oplus \left\langle \int_0^t b_2(s, x(s))\mathrm{d}B_2(s) \right. \qquad (3) \\ &\left. - \int_0^t b_1(s, x(s))\mathrm{d}B_1(s) \right\rangle, \end{aligned}$$

where the appearance of Hukuhara differences is inevitable. This necessitates certain inherent assumptions to obtain the result regarding the existence of a solution.

We have already defined the solution of (1), but it is also important to clarify what it means for the solution to be unique.

Definition 2. A solution $x \colon I \times \Omega \to \mathcal{F}(\mathbb{R}^d)$ of equation (1) is considered unique if $d_\infty\big(x(t), y(t)\big) \overset{I\ P.1}{=} 0$, where $y \colon I \times \Omega \to \mathcal{F}(\mathbb{R}^d)$ is any other solution of (1).

One of the primary objectives of this section is to establish the existence and uniqueness of solutions to (1) under conditions that are less stringent than the global Lipschitz condition. The global Lipschitz condition was employed, for example, by Malinowski (2012b; 2012a; 2016a). We now consider the following conditions:

(a0) $x_0 \in \mathcal{L}^2(\Omega, \mathcal{A}_0, P; \mathcal{F}_c(\mathbb{R}^d))$,

(a1) the mappings $a_1, a_2 \colon (I \times \Omega) \times \mathcal{F}_c(\mathbb{R}^d) \to \mathcal{F}_c(\mathbb{R}^d)$ are $\mathcal{N} \otimes \mathcal{B}_{d_\infty} | \mathcal{B}_{d_\infty}$-measurable and $b_1, b_2 \colon (I \times \Omega) \times \mathcal{F}_c(\mathbb{R}^d) \to \mathbb{R}^d$ are $\mathcal{N} \otimes \mathcal{B}_{d_\infty} | \mathcal{B}(\mathbb{R}^d)$-measurable,

(a2) there is $\eta \colon I \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

(i) $\eta(\cdot, x)$ is integrable for every $x \in \mathbb{R}_+$,

(ii) $\eta(t, \cdot)$ is continuous, nondecreasing and concave for every $t \in I$,

(iii) $\eta(t, 0) = 0$ for every $t \in I$,

(iv) if for $\zeta \colon I \to \mathbb{R}_+$ it holds $\zeta(0) = 0$ and

$$\zeta(t) \le M \int_0^t \eta(s, \zeta(s))\,\mathrm{d}s, \quad t \in I,$$

where M is a positive constant, then $\zeta(t) = 0$ for $t \in I$,

(v) with $P.1$ for every $t \in I$ and for every $u, v \in \mathcal{F}_c(\mathbb{R}^d)$

$$\begin{aligned}&\max\{d_\infty^2\big(a_1(t,\omega,u), a_1(t,\omega,v)\big),\\&\quad d_\infty^2\big(a_2(t,\omega,u), a_2(t,\omega,v)\big),\\&\quad \|b_1(t,\omega,u) - b_1(t,\omega,v)\|^2,\\&\quad \|b_2(t,\omega,u) - b_2(t,\omega,v)\|^2\}\\&\leq \eta(t, d_\infty^2(u,v)),\end{aligned}$$

(a3) there exist integrable functions $\gamma, \delta\colon I \to \mathbb{R}_+$ such that with $P.1$ for every $t \in I$ and for every $u \in \mathcal{F}_c(\mathbb{R}^d)$

$$\begin{aligned}&\max\{d_\infty^2(a_1(t,\omega,u), \langle 0\rangle),\\&\quad d_\infty^2(a_1(t,\omega,u), \langle 0\rangle),\\&\quad \|b_1(t,\omega,u)\|^2, \|b_2(t,\omega,u)\|^2\}\\&\quad\leq \gamma(t) + \delta(t) d_\infty^2(u, \langle 0\rangle),\end{aligned}$$

(a4) there exists $\tilde{T} \in (0, T]$ such that for every $n = 0, 1, 2, \dots$ the mappings $x_n\colon \tilde{I} \times \Omega \to \mathcal{F}_c(\mathbb{R}^d)$, where $\tilde{I} = [0, \tilde{T}]$, described as

$$x_0(t) \overset{\tilde{I}\ P.1}{=} x_0$$

and

$$\begin{aligned}x_n(t) &\overset{\tilde{I}\ P.1}{=} \Big[x_0 \oplus \int_0^t a_2(s, x_{n-1}(s))\,\mathrm{d}s\Big]\\&\ominus \int_0^t a_1(s, x_{n-1}(s))\,\mathrm{d}s\\&\oplus \Big\langle \int_0^t b_2(s, x_{n-1}(s))\,\mathrm{d}B_2(s)\\&- \int_0^t b_1(s, x_{n-1}(s))\,\mathrm{d}B_1(s)\Big\rangle\end{aligned}$$

are properly defined (specifically, the Hukuhara differences exist).

Condition (a4) arises from the structure of the symmetric equation (1) we are examining and is essential due to the form (3) of the solution. When this condition is met, the x_n's are well-defined d_∞-continuous fuzzy stochastic processes from $\mathcal{L}^2(\tilde{I} \times \Omega, \mathcal{N}; \mathcal{F}_c(\mathbb{R}^d))$. The processes x_n will be utilized to obtain the solution of (1) using the Picard method. Since x_n, $n = 0, 1, 2, \dots$ are defined in the interval $\tilde{I}$, it is expected that the solution will also be defined in this interval. Therefore, it will not be a global solution but rather a local one. For symmetric fuzzy stochastic differential equations, in general, solutions defined on a half-line cannot be considered unless a very restrictive assumption (a4) about the existence of Hukuhara differences for any $t \in [0, \infty)$ is made. We aim to avoid such restrictive conditions. Therefore, in assumption (a4), we posit the existence of a number $\tilde{T}$ so that the Hukuhara differences exist for $t \in [0, \tilde{T}]$.

The generalized Lipschitz condition in (a2)(v) is expressed using the η function, which appears in a specific integral inequality in (a2)(iv). In Taniguchi (1992), a result is presented that confirms the validity of this integral inequality, and we refer to it here.

Remark 1. (*Taniguchi, 1992*) Consider the function $\eta\colon I \times \mathbb{R}_+ \to \mathbb{R}_+$. Suppose that

- for every $x \in \mathbb{R}_+$ the function $\eta(\cdot, x)\colon I \to \mathbb{R}_+$ is integrable,
- for every $t \in I$ the function $\eta(t, \cdot)\colon \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and continuous,
- for every $t \in I$ it holds $\eta(t, 0) = 0$,
- the differential equation $\mathrm{d}x(t) = \eta(t, x(t))\,\mathrm{d}t$ has the solution $x(\cdot)$ with the following property:
- if there exists $t^* \in [0, T)$ such that $x(t^*) = 0$, then $x(t) = 0$ for every $t \in [t^*, T]$.

Then, if a continuous function $\zeta\colon I \to \mathbb{R}_+$ satisfies $\zeta(0) = 0$ and

$$\zeta(t) \leq \int_0^t \eta(s, \zeta(s))\,\mathrm{d}s \quad \text{for every} \quad t \in I'$$

then $\zeta(t) = 0$ for every $t \in I$.

Remark 2. (*Taniguchi, 1992*) Let $\eta\colon I \times \mathbb{R}_+ \to \mathbb{R}_+$ be a function defined as $\eta(t, x) = f(t)g(x)$, where

- $f\colon I \to \mathbb{R}_+$ is an integrable function,
- $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and nondecreasing function such that $g(0) = 0$ and $\int_{0+} \frac{\mathrm{d}x}{g(x)} = \infty$.

Under these assumptions, η satisfies condition (a2)(iv).

Furthermore, if $f \equiv L$ (with $L > 0$ as a constant) and g satisfies the properties mentioned in Remark 2, including being concave, then a condition described by Malinowski (2020) emerges. Specifically, for the case $\eta(t, x) = Lx$, the global Lipschitz condition (a2)(v) aligns with that introduced by Malinowski (2016a).

There are various intriguing examples of functions g that are fitted to Remark 2, such as those found in the work of Yamada (1981), i.e.,

$$g_1(x) = \begin{cases} x \ln \frac{1}{x}, & 0 \leq x \leq \varepsilon,\\ \varepsilon \ln \frac{1}{\varepsilon} + g_1'(\varepsilon-)(x - \varepsilon), & x > \varepsilon,\end{cases}$$

$$g_2(x) = \begin{cases} x \ln \frac{1}{x} \ln\ln \frac{1}{x}, & 0 \le x \le \varepsilon, \\ \varepsilon \ln \frac{1}{\varepsilon} \ln\ln \frac{1}{\varepsilon} + g_2'(\varepsilon-)(x-\varepsilon), & x > \varepsilon, \end{cases}$$

where $\varepsilon \in (0,1)$ is a sufficiently small positive number, and the expressions $g_1'(\varepsilon-)$ and $g_2'(\varepsilon-)$ represent the left-hand derivatives of the functions g_1 and g_2 (respectively) at the point ε.

Following an initial discussion of assumptions (a0)–(a4) concerning the properties of the coefficients in Eqn. (1), we will demonstrate that the processes x_n are uniformly bounded together. This property will be utilized to prove the main theorem of this section.

Lemma 2. *Assume that conditions (a0)–(a4) are satisfied. Then for every $n \in \mathbb{N}$ we have*

$$\mathbb{E} \sup_{t\in\tilde{I}} d_\infty^2(x_n(t), \langle 0\rangle) \le Q, \quad \textit{where } Q > 0.$$

Proof. Let us choose $n \in \mathbb{N}$ and $t \in \tilde{I}$. Then

$$\begin{aligned}
&\mathbb{E} \sup_{u\in[0,t]} d_\infty^2(x_n(u), \langle 0\rangle) \\
&\quad = \mathbb{E} \sup_{u\in[0,t]} d_\infty^2\Bigg(\Big[x_0 \oplus \int_0^t a_2(s, x_{n-1}(s))\,\mathrm{d}s\Big] \\
&\quad \ominus \int_0^t a_1(s, x_{n-1}(s))\,\mathrm{d}s \\
&\quad \oplus \Big\langle \int_0^t b_2(s, x_{n-1}(s))\mathrm{d}B_2(s) \\
&\quad - \int_0^t b_1(s, x_{n-1}(s))\mathrm{d}B_1(s)\Big\rangle, \langle 0\rangle\Bigg) \\
&\le \quad 5\mathbb{E} d_\infty^2(x_0, \langle 0\rangle) \\
&\quad + 5\mathbb{E} \sup_{u\in[0,t]} d_\infty^2\Big(\int_0^u a_1(s, x_{n-1}(s))\,\mathrm{d}s, \langle 0\rangle\Big) \\
&\quad + 5\mathbb{E} \sup_{u\in[0,t]} d_\infty^2\Big(\int_0^u a_2(s, x_{n-1}(s))\,\mathrm{d}s, \langle 0\rangle\Big) \\
&\quad + 5\mathbb{E} \sup_{u\in[0,t]} \Big\|\int_0^u b_1(s, x_{n-1}(s))\mathrm{d}B_1(s)\Big\|^2 \\
&\quad + 5\mathbb{E} \sup_{u\in[0,t]} \Big\|\int_0^u b_2(s, x_{n-1}(s))\mathrm{d}B_2(s)\Big\|^2.
\end{aligned}$$

Applying Lemma 1 along with the Doob inequality, we obtain

$$\begin{aligned}
&\mathbb{E} \sup_{u\in[0,t]} d_\infty^2(x_n(u), \langle 0\rangle) \\
&\quad \le 5\mathbb{E} d_\infty^2(x_0, \langle 0\rangle) \\
&\quad + 5t\mathbb{E} \int_0^t d_\infty^2\big(a_1(s, x_{n-1}(s)), \langle 0\rangle\big)\,\mathrm{d}s \\
&\quad + 5t\mathbb{E} \int_0^t d_\infty^2\big(a_2(s, x_{n-1}(s)), \langle 0\rangle\big)\,\mathrm{d}s \\
&\quad + 20\mathbb{E} \int_0^t \big\|b_1(s, X_{n-1}(s))\big\|^2\,\mathrm{d}s \\
&\quad + 20\mathbb{E} \int_0^t \big\|b_2(s, X_{n-1}(s))\big\|^2\,\mathrm{d}s.
\end{aligned}$$

Based on assumption (a3) and applying the Fubini theorem, we can express

$$\begin{aligned}
&\mathbb{E} \sup_{u\in[0,t]} d_\infty^2(x_n(u), \langle 0\rangle) \\
&\le 5\mathbb{E} d_\infty^2(x_0, \langle 0\rangle) + (10t + 40)\int_0^t \gamma(s)\,\mathrm{d}s \\
&\quad + (10t+40)\mathbb{E}\int_0^t \delta(s) d_\infty^2(x_{n-1}(s), \langle 0\rangle)\,\mathrm{d}s \\
&\le 5\mathbb{E} d_\infty^2(x_0, \langle 0\rangle) + (10\tilde{T} + 40)\int_0^{\tilde{T}} \gamma(s)\,\mathrm{d}s \\
&\quad + (10\tilde{T}+40)\int_0^t \delta(s) \\
&\quad \times \mathbb{E} \sup_{u\in[0,s]} d_\infty^2(x_{n-1}(u), \langle 0\rangle)\,\mathrm{d}s.
\end{aligned}$$

Consequently, we can deduce that for $k \in \mathbb{N}$

$$\begin{aligned}
&\max_{1\le n\le k} \mathbb{E} \sup_{u\in[0,t]} d_\infty^2(x_n(u), \langle 0\rangle) \\
&\le 5\mathbb{E} d_\infty^2(x_0, \langle 0\rangle) + (10\tilde{T} + 40)\int_0^{\tilde{T}} \gamma(s)\,\mathrm{d}s \\
&\quad + (10\tilde{T}+40)\int_0^t \delta(s) \\
&\quad \times \max_{1\le n\le k} \mathbb{E} \sup_{u\in[0,s]} d_\infty^2(x_{n-1}(u), \langle 0\rangle)\,\mathrm{d}s.
\end{aligned}$$

Noting that

$$\begin{aligned}
&\max_{1\le n\le k} \mathbb{E} \sup_{u\in[0,t]} d_\infty^2(x_{n-1}(u), \langle 0\rangle) \\
&\quad \le \mathbb{E} d_\infty^2(x_0, \langle 0\rangle) + \max_{1\le n\le k} \mathbb{E} \sup_{u\in[0,t]} d_\infty^2(x_n(u), \langle 0\rangle),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\max_{1\le n\le k} \mathbb{E} \sup_{u\in[0,t]} d_\infty^2(x_n(u), \langle 0\rangle) \\
&\le 5\mathbb{E} d_\infty^2(x_0, \langle 0\rangle) + (10\tilde{T} + 40)\int_0^{\tilde{T}} \gamma(s)\,\mathrm{d}s \\
&+ (10\tilde{T} + 40)\mathbb{E} d_\infty^2(x_0, \langle 0\rangle)\int_0^{\tilde{T}} \delta(s)\,\mathrm{d}s \\
&+ (10\tilde{T}+40)\int_0^t \delta(s) \max_{1\le n\le k} \mathbb{E} \sup_{u\in[0,s]} d_\infty^2(x_n(u), \langle 0\rangle)\,\mathrm{d}s.
\end{aligned}$$

Now, by applying the Gronwall inequality, we deduce that

$$\max_{1\le n\le k} \mathbb{E} \sup_{u\in[0,t]} d_\infty^2(x_n(u),\langle 0\rangle) \le K \exp\{(10\tilde{T}+40)\int_0^t \delta(s)\,\mathrm{d}s\}, \quad t\in\tilde{I},$$

where

$$K = 5\mathbb{E}d_\infty^2(x_0,\langle 0\rangle) + (10\tilde{T}+40)\int_0^{\tilde{T}} \gamma(s)\,\mathrm{d}s + (10\tilde{T}+40)\mathbb{E}d_\infty^2(x_0,\langle 0\rangle)\int_0^{\tilde{T}} \delta(s)\,\mathrm{d}s.$$

Hence, we can deduce that for all $n\in\mathbb{N}$

$$\mathbb{E} \sup_{u\in[0,\tilde{T}]} d_\infty^2(x_n(u),\langle 0\rangle) \le K\exp\{(10\tilde{T}+40)\int_0^{\tilde{T}} \delta(s)\,\mathrm{d}s\},$$

which concludes the derivation. ∎

At this point, we present the main theorem concerning the existence of a unique strong solution.

Theorem 1. *Assume that conditions (a0)–(a4) are satisfied. Then the symmetric fuzzy stochastic differential equation* (1) *has a unique strong solution x in the interval $\tilde{I}=[0,\tilde{T}]$.*

Proof. To establish the existence of a solution to (1), we will utilize the sequence of fuzzy stochastic processes $\{x_n\}_{n=0}^\infty$ outlined in (a4).

Let us choose $t\in\tilde{I}$. Observe that for all $n,\ell\in\mathbb{N}\cup\{0\}$, it holds that

$$\begin{aligned}
&\mathbb{E} \sup_{u\in[0,t]} d_\infty^2(x_{n+1}(u),x_{\ell+1}(u))\\
&= \mathbb{E} \sup_{u\in[0,t]} d_\infty^2\Big(\big[\int_0^u a_1(s,x_n(s))\,\mathrm{d}s \ominus \int_0^u a_2(s,x_n(s))\,\mathrm{d}s\big] \oplus\Big\langle \int_0^u b_1(s,x_n(s))\mathrm{d}B_1(s) - \int_0^u b_2(s,x_n(s))\mathrm{d}B_2(s)\Big\rangle,\\
&\qquad \big[\int_0^u a_1(s,x_\ell(s))\,\mathrm{d}s \ominus \int_0^u a_2(s,x_\ell(s))\,\mathrm{d}s\big] \oplus\Big\langle \int_0^u b_1(s,x_\ell(s))\mathrm{d}B_1(s) - \int_0^u b_2(s,x_\ell(s))\mathrm{d}B_2(s)\Big\rangle\Big)\\
&\le 4\mathbb{E} \sup_{u\in[0,t]} d_\infty^2\Big(\int_0^u a_1(s,x_n(s))\,\mathrm{d}s, \int_0^u a_1(s,x_\ell(s))\,\mathrm{d}s\Big)\\
&\quad + 4\mathbb{E} \sup_{u\in[0,t]} d_\infty^2\Big(\int_0^u a_2(s,x_n(s))\,\mathrm{d}s, \int_0^u a_2(s,x_\ell(s))\,\mathrm{d}s\Big)\\
&\quad + 4\mathbb{E} \sup_{u\in[0,t]} \Big\|\int_0^u b_1(s,x_n(s))\mathrm{d}B_1(s) - \int_0^u b_1(s,x_\ell(s))\mathrm{d}B_1(s)\Big\|^2\\
&\quad + 4\mathbb{E} \sup_{u\in[0,t]} \Big\|\int_0^u b_2(s,x_n(s))\mathrm{d}B_2(s) - \int_0^u b_2(s,x_\ell(s))\mathrm{d}B_2(s)\Big\|^2\\
&\le 4t\mathbb{E}\int_0^t d_\infty^2\big(a_1(s,x_n(s)),a_1(s,x_\ell(s))\big)\,\mathrm{d}s\\
&\quad + 4t\mathbb{E}\int_0^t d_\infty^2\big(a_2(s,x_n(s)),a_2(s,x_\ell(s))\big)\,\mathrm{d}s\\
&\quad + 16\mathbb{E}\int_0^t \big\|b_1(s,x_n(s)) - b_1(s,x_\ell(s))\big\|^2\,\mathrm{d}s\\
&\quad + 16\mathbb{E}\int_0^t \big\|b_2(s,x_n(s)) - b_2(s,x_\ell(s))\big\|^2\,\mathrm{d}s.
\end{aligned}$$

Based on (a2), the Fubini theorem, and the Jensen inequality, we obtain

$$\begin{aligned}
&\mathbb{E} \sup_{u\in[0,t]} d_\infty^2(x_{n+1}(u),x_{\ell+1}(u))\\
&\le (8t+32)\mathbb{E}\int_0^t \eta\big(s,d_\infty^2(x_n(s),x_\ell(s))\big)\,\mathrm{d}s\\
&\le (8t+32)\int_0^t \mathbb{E}\eta\big(s,\sup_{u\in[0,s]} d_\infty^2(x_n(u),x_\ell(u))\big)\,\mathrm{d}s\\
&\le (8T+32)\int_0^t \eta\big(s,\mathbb{E}\sup_{u\in[0,s]} d_\infty^2(x_n(u),x_\ell(u))\big)\mathrm{d}s.
\end{aligned}$$

Using Lemma 2, we can define a function $\zeta\colon \tilde{I}\to\mathbb{R}_+$ as follows:

$$\zeta(t) = \limsup_{n,\ell\to\infty}\Big(\mathbb{E}\sup_{u\in[0,t]} d_\infty^2(x_n(u),x_\ell(u))\Big), \quad t\in\tilde{I}.$$

Applying the Fatou lemma and invoking the continuity of $\eta(s,\cdot)$, we obtain

$$\zeta(t) \le (8T+32)\int_0^t \eta(s,\zeta(s))\,\mathrm{d}s \quad \text{for } t\in\tilde{I},$$

which indicates (based on assumption (a2)(iv)) that $\zeta(t)=0$ for all $t\in\tilde{I}$. Therefore, for every $t\in\tilde{I}$

$$\lim_{n,\ell\to\infty} \mathbb{E}d_\infty^2(x_n(t),x_\ell(t)) = 0.$$

Taking into account the metric

$$\rho(x,y) = \big(\mathbb{E}d_\infty^2(x,y)\big)^{1/2}$$

in $\mathcal{L}^2(\Omega,\mathcal{A}_t,P;\mathcal{F}_c(\mathbb{R}^d))$, we obtain the metric space (cf. Feng, 1999). Therefore, for each $t\in\tilde{I}$, there exists $x_t\in\mathcal{L}^2(\Omega,\mathcal{A}_t,P;\mathcal{F}_c(\mathbb{R}^d))$ such that

$$\mathbb{E}d_\infty^2(x_n(t),x_t) \longrightarrow 0 \quad \text{as} \quad n\to\infty.$$

Thus, by setting $x(t,\omega)=x_t(\omega)$, we can establish a fuzzy stochastic process $x\colon \tilde{I}\times\Omega\to\mathcal{F}_c(\mathbb{R}^d)$ that is $\{\mathcal{A}_t\}$-adapted. In the following, we will demonstrate that the process x meets the criteria outlined in Definition 1. Indeed, given that

$$\lim_{n,\ell\to\infty}\left(\mathbb{E}\sup_{u\in\tilde{I}} d_\infty^2(x_n(u),x_\ell(u))\right)=0,$$

we can apply Chebyshev's inequality to obtain: for any $\varepsilon>0$

$$P\Big(\sup_{u\in\tilde{I}} d_\infty^2\big(x_n(u),x_\ell(u)\big)>\varepsilon\Big)\longrightarrow 0 \quad \text{as} \quad n,\ell\to\infty.$$

Therefore, we conclude that there exists a subsequence $\{x_{n_k}(\cdot,\cdot)\}$ of the sequence $\{x_n(\cdot,\cdot)\}$ such that

$$\sup_{u\in\tilde{I}} d_\infty\big(x_{n_k}(u),x(u)\big)\xrightarrow{P.1}0 \quad \text{as} \quad k\to\infty.$$

Thus, the process x is d_∞-continuous and, as a result, $\mathcal{B}(\tilde{I})\otimes\mathcal{A}$-measurable. Given that x is also $\{\mathcal{A}_t\}$-adapted, we infer that x is $\mathcal{N}$-measurable. Given that $x(t)\in\mathcal{L}^2(\Omega,\mathcal{A},P;\mathcal{F}_c(\mathbb{R}^d))$ for all $t\in\tilde{I}$, we can express

$$\mathbb{E}\int_{\tilde{I}} d_\infty^2(x(t),\langle 0\rangle)\,\mathrm{d}t \le \tilde{T}\sup_{t\in\tilde{I}}\mathbb{E}d_\infty^2(x(t),\langle 0\rangle)<\infty$$

which indicates that $x\in\mathcal{L}^2(\tilde{I}\times\Omega,\mathcal{N};\mathcal{F}_c(\mathbb{R}^d))$.

We will demonstrate that the recently defined fuzzy stochastic process x is the desired solution to (1). To achieve this, let us observe that for each $t\in\tilde{I}$

$$\begin{aligned}&\mathbb{E}d_\infty^2\Bigg(x(t),\big[x_0\oplus\int_0^t a_2(s,x(s))\,\mathrm{d}s\big]\\ &\ominus\int_0^t a_1(s,x(s))\,\mathrm{d}s\\ &\oplus\Big\langle\int_0^t b_2(s,x(s))\mathrm{d}B_2(s)-\int_0^t b_1(s,x(s))\,\mathrm{d}B_1(s)\Big\rangle\Bigg)\\ &\le 3z_n^1(t)+3z_n^2(t)+3z_n^3(t),\end{aligned}$$

where

$$z_n^1(t)=\mathbb{E}d_\infty^2(x_n(t),x(t)),$$

$$\begin{aligned}z_n^2(t) \;=\;& \mathbb{E}d_\infty^2\Big(x_n,\big[x_0+\int_0^t a_2(s,x_{n-1}(s))\,\mathrm{d}s\big]\\ &\ominus\int_0^t a_1(s,x_{n-1}(s))\,\mathrm{d}s\\ &\oplus\Big\langle\int_0^t b_2(s,x_{n-1}(s))\,\mathrm{d}B_2(s)\\ &-\int_0^t b_1(s,x_{n-1}(s))\mathrm{d}B_1(s)\Big\rangle\Big),\end{aligned}$$

$$\begin{aligned}z_n^3(t) \;=\;& \mathbb{E}d_\infty^2\Big(\big[\int_0^t a_2(s,x_{n-1}(s))\,\mathrm{d}s\\ &\ominus\int_0^t a_1(s,x_{n-1}(s))\,\mathrm{d}s\big]\\ &\oplus\Big\langle\int_0^t b_2(s,x_{n-1}(s))\,\mathrm{d}B_2(s)\\ &-\int_0^t b_1(s,x_{n-1}(s))\mathrm{d}B_1(s)\Big\rangle,\\ &\big[\int_0^t a_2(s,x(s))\,\mathrm{d}s\ominus\int_0^t a_1(s,x(s))\,\mathrm{d}s\big]\\ &\oplus\Big\langle\int_0^t b_2(s,x(s))\,\mathrm{d}B_2(s)\\ &-\int_0^t b_1(s,x(s))\mathrm{d}B_1(s)\Big\rangle\Big).\end{aligned}$$

Clearly, $z_n^1(t)\xrightarrow{n\to\infty}0$ and $z_n^2(t)=0$ for all $t\in\tilde{I}$. Observe that

$$\begin{aligned}z_n^3(t) \;\le\;& 4t\mathbb{E}\int_0^t d_\infty^2\big(a_1(s,x_{n-1}(s)),a_1(s,x(s))\big)\,\mathrm{d}s\\ &+4t\mathbb{E}\int_0^t d_\infty^2\big(a_2(s,x_{n-1}(s)),a_2(s,x(s))\big)\,\mathrm{d}s\\ &+4\mathbb{E}\int_0^t \big\|b_1(s,x_{n-1}(s))-b_1(s,X(s))\big\|^2\,\mathrm{d}s\\ &+4\mathbb{E}\int_0^t \big\|b_2(s,x_{n-1}(s))-b_2(s,x(s))\big\|^2\,\mathrm{d}s\\ \le\;& 8(t+1)\mathbb{E}\int_0^t \eta\big(s,d_\infty^2(x_{n-1}(s),x(s))\big)\,\mathrm{d}s\\ \le\;& 8(t+1)\int_0^t \eta\big(s,\mathbb{E}d_\infty^2(x_{n-1}(s),x(s))\big)\,\mathrm{d}s.\end{aligned}$$

By applying the Lebesgue dominated convergence theorem, the continuity of $\eta(s,\cdot)$ and assumption $\eta(s,0)=0$, we obtain that $z_n^3(t)\xrightarrow{n\to\infty}0$ for $t\in\tilde{I}$. Therefore, for each $t\in\tilde{I}$

$$\begin{aligned}&\mathbb{E}d_\infty^2\Big(x(t),\big[x_0\oplus\int_0^t a_2(s,x(s))\,\mathrm{d}s\big]\ominus\int_0^t a_1(s,x(s))\,\mathrm{d}s\\ &\oplus\Big\langle\int_0^t b_2(s,x(s))\mathrm{d}B_2(s)-\int_0^t b_1(s,x(s))\mathrm{d}B_1(s)\Big\rangle\Big)=0.\end{aligned}$$

Thus, for each $t \in \tilde{I}$

$$d_\infty^2\Big(x(t), \big[x_0 \oplus \int_0^t a_2(s,x(s))\,\mathrm{d}s\big] \ominus \int_0^t a_1(s,x(s))\,\mathrm{d}s \oplus \Big\langle \int_0^t b_2(s,x(s))\mathrm{d}B_2(s) - \int_0^t b_1(s,x(s))\mathrm{d}B_1(s)\Big\rangle\Big) \overset{P.1}{=} 0.$$

However, since the process x is d_∞-continuous, it follows that

$$d_\infty^2\Big(x(t), \big[x_0 \oplus \int_0^t a_2(s,x(s))\,\mathrm{d}s\big] \ominus \int_0^t a_1(s,x(s))\,\mathrm{d}s \oplus \Big\langle \int_0^t b_2(s,x(s))\mathrm{d}B_2(s) - \int_0^t b_1(s,x(s))\mathrm{d}B_1(s)\Big\rangle\Big) \overset{\tilde{I}\ P.1}{=} 0.$$

This precisely indicates that x is a strong solution to (1). Finally, we will demonstrate the uniqueness of x. Let y represent another solution to (1). Then, for each $t \in \tilde{I}$, we have

$$\begin{aligned}
&\mathbb{E} \sup_{u\in[0,t]} d_\infty^2(x(u),y(u)) \\
&\quad\le 4t\mathbb{E}\int_0^t d_\infty^2(a_1(s,x(s)),a_1(s,y(s)))\,\mathrm{d}s \\
&\qquad + 4t\mathbb{E}\int_0^t d_\infty^2(a_2(s,x(s)),a_2(s,y(s)))\,\mathrm{d}s \\
&\qquad + 16\mathbb{E}\int_0^t \|b_1(s,x(s)) - b_1(s,y(s))\|^2\,\mathrm{d}s \\
&\qquad + 16\mathbb{E}\int_0^t \|b_2(s,x(s)) - b_2(s,y(s))\|^2\,\mathrm{d}s \\
&\quad\le (8t+32)\int_0^t \mathbb{E}\eta(s, d_\infty^2(x(s),y(s)))\,\mathrm{d}s \\
&\quad\le (8\tilde{T}+32)\int_0^t \eta(s, \mathbb{E}\sup_{u\in[0,s]} d_\infty^2(x(s),y(s)))\,\mathrm{d}s.
\end{aligned}$$

Based on (a2)(iv), we have

$$\mathbb{E}\sup_{u\in\tilde{I}} d_\infty^2(x(u),y(u)) = 0,$$

which directly implies

$$d_\infty(x(u),y(u)) \overset{\tilde{I}\ P.1}{=} 0.$$

Therefore, the solution x is unique. ■

The above statement is extremely important. Usually, it is not possible to find the solution as a concrete process even in the case of single-valued stochastic differential equations, and it is even more difficult in the fuzzy context presented by Malinowski (2013; 2014; 2016b). Therefore, being sure that the equation has a solution at all is a fundamental aspect of analysis. The significance of this result lies also in the fact that, with the knowledge that the equation has a unique solution, future research can be considered focused on finding approximate solutions through numerical methods.

4. Insensitivity of solution

In this section, we explore additional well-posedness properties. These properties pertain to the validation of the solution's stability for symmetric fuzzy stochastic differential equations. Stability, in this context, refers to the solution's minimal sensitivity to variations in the equation parameters, such as changes in the initial value and modifications in the drift and diffusion coefficients.

Consider the equation (1) with the initial value x_0 and solution x, if it exists. Additionally, consider the following equations:

$$\begin{aligned}
&\mathrm{d}x_n(t) \oplus a_1(t,x_n(t))\,\mathrm{d}t \oplus \langle b_1(t,x_n(t))\mathrm{d}B_1(t)\rangle \\
&\overset{I\ P.1}{=} a_2(t,x_n(t))\,\mathrm{d}t \oplus \langle b_2(t,x_n(t))\mathrm{d}B_2(t)\rangle, \qquad (4) \\
&x_n(0) \overset{P.1}{=} x_{0,n},
\end{aligned}$$

for $n \in \mathbb{N}$, with initial values $x_{0,n}$ and solutions x_n, if they exist.

Theorem 2. *Consider fuzzy random variables $x_0, x_{0,n}\colon \Omega \to \mathcal{F}_c(\mathbb{R}^d)$, $n \in \mathbb{N}$, which satisfy (a0). Let $a_1, a_2\colon I \times \Omega \times \mathcal{F}_c(\mathbb{R}^d) \to \mathcal{F}_c(\mathbb{R}^d)$ and $b_1, b_2\colon I \times \Omega \times \mathcal{F}_c(\mathbb{R}^d) \to \mathbb{R}^d$ satisfy (a1)-(a4), particularly achieving (a4) with the same constant $\tilde{T}$ for x_0 and each $x_{0,n}$. Assume that*

$$\lim_{n\to\infty} \mathbb{E}d_\infty^2(x_{0,n},x_0) = 0.$$

Then the solution x to (1) *remains stable with respect to the solutions x_n to* (4)*, meaning that*

$$\lim_{n\to\infty} \mathbb{E}\sup_{t\in\tilde{I}} d_\infty^2(x_n(t),x(t)) = 0.$$

Proof. Fix $t \in \tilde{I}$. We observe that

$$\begin{aligned}
&\mathbb{E}\sup_{u\in[0,t]} d_\infty^2(x_n(u),x(u)) \\
&\le 5\mathbb{E}d_\infty^2(x_{0,n},x_0) \\
&\quad + 5\mathbb{E}\sup_{u\in[0,t]} d_\infty^2\Big(\int_0^t a_1(s,x_n(s))\,\mathrm{d}s, \int_0^t a_1(s,x(s))\,\mathrm{d}s\Big) \\
&\quad + 5\mathbb{E}\sup_{u\in[0,t]} d_\infty^2\Big(\int_0^t a_2(s,x_n(s))\,\mathrm{d}s, \int_0^t a_2(s,x(s))\,\mathrm{d}s\Big) \\
&\quad + 5\mathbb{E}\sup_{u\in[0,t]}\Big\|\int_0^t b_1(s,x_n(s))\mathrm{d}B_1(s) \\
&\quad - \int_0^t b_1(s,x(s))\mathrm{d}B_1(s)\Big\|^2 \\
&\quad + 5\mathbb{E}\sup_{u\in[0,t]}\Big\|\int_0^t b_2(s,x_n(s))\mathrm{d}B_2(s) \\
&\quad - \int_0^t b_2(s,x(s))\mathrm{d}B_2(s)\Big\|^2 \\
&\le 5\mathbb{E}d_\infty^2(x_{0,n},x_0) \\
&\quad + 5t\mathbb{E}\int_0^t d_\infty^2(a_1(s,x_n(s)),a_1(s,x(s)))\,\mathrm{d}s
\end{aligned}$$

$$+ 5t\mathbb{E}\int_0^t d_\infty^2(a_2(s,x_n(s)),a_2(s,x(s)))\,\mathrm{d}s$$
$$+ 20\mathbb{E}\int_0^t \|b_1(s,x_n(s))-b_1(s,x(s))\|^2\,\mathrm{d}s$$
$$+ 20\mathbb{E}\int_0^t \|b_2(s,x_n(s))-b_2(s,x(s))\|^2\,\mathrm{d}s.$$

Based on assumption (a2) and the characteristics of the function η, we obtain

$$\mathbb{E}\sup_{u\in[0,t]} d_\infty^2(x_n(u),x(u)) \le 5\mathbb{E}d_\infty^2(x_{0,n},x_0) + (10\tilde{T}+40)\int_0^t \eta(s,\mathbb{E}\sup_{u\in[0,s]} d_\infty^2(x_n(u),x(u)))\,\mathrm{d}s.$$

Define

$$\zeta_n(t) := \mathbb{E}\sup_{u\in[0,t]} d_\infty^2(x_n(u),x(u))$$

and

$$\zeta(t) := \lim_{n\to\infty} g_n(t)$$

for $t\in\tilde{I}$. Then

$$\zeta_n(t) \le 5\mathbb{E}d_\infty^2(x_{0,n},x_0) + (10\tilde{T}+40)\int_0^t \eta(s,\zeta_n(s))\,\mathrm{d}s$$

and by the properties of η and the assumption on $\mathbb{E}d_\infty^2(x_{0,n},x_0)$, we have

$$\zeta(t) \le (10\tilde{T}+40)\int_0^t \eta(s,\zeta(s))\,\mathrm{d}s.$$

Based on assumption (a2)(iv), we have $\zeta(t) \equiv 0$. Therefore,

$$\lim_{n\to\infty}\mathbb{E}\sup_{u\in[0,t]} d_\infty^2(x_n(t),x(t)) = 0 \quad \text{for every } t\in\tilde{I}.$$

Consequently,

$$\lim_{n\to\infty}\mathbb{E}\sup_{t\in\tilde{I}} d_\infty^2(x_n(t),x(t)) = 0$$

which concludes the proof. ■

To demonstrate stability when the coefficients a_1, a_2, b_1, b_2 undergo slight changes, we consider Eqn. (1) with initial value x_0 and solution x, if it exists, along with the following equations:

$$\begin{aligned} &\mathrm{d}x_n(t) \oplus a_1^n(t,x_n(t))\,\mathrm{d}t \oplus \langle b_1^n(t,x_n(t))\mathrm{d}B_1(t)\rangle \\ &\quad \overset{I\ P.1}{=} a_2^n(t,x_n(t))\,\mathrm{d}t \oplus \langle b_2^n(t,x_n(t))\mathrm{d}B_2(t)\rangle, \\ &\quad x_n(0) \overset{P.1}{=} x_0, \end{aligned} \tag{5}$$

for $n\in\mathbb{N}$ with the same initial value x_0 and solutions x_n, if they exist.

Theorem 3. *Consider a fuzzy random variable $x_0\colon \Omega\to\mathcal{F}_c(\mathbb{R}^d)$ that satisfies (a0). Assume that the conditions (a1)-(a4) are met, with the same constant $\tilde{T}$ and the same function η, for*

$$a_1,a_2\colon I\times\Omega\times\mathcal{F}_c(\mathbb{R}^d)\to\mathcal{F}_c(\mathbb{R}^d),$$

$$b_1,b_2\colon I\times\Omega\times\mathcal{F}_c(\mathbb{R}^d)\to\mathbb{R}^d,$$

$$a_1^n,a_2^n\colon I\times\Omega\times\mathcal{F}_c(\mathbb{R}^d)\to\mathcal{F}_c(\mathbb{R}^d)$$

and

$$b_1^n,b_2^n\colon I\times\Omega\times\mathcal{F}_c(\mathbb{R}^d)\to\mathbb{R}^d \text{ for every } n=1,2,\dots.$$

Suppose that for every $(t,u)\in\tilde{I}\times\mathcal{F}_c(\mathbb{R}^d)$

$$\lim_{n\to\infty}\mathbb{E}\int_0^{\tilde{T}} d_\infty^2(a_k^n(t,u),a_k(t,u))\,\mathrm{d}t = 0$$

and

$$\lim_{n\to\infty}\mathbb{E}\int_0^{\tilde{T}} \|b_k^n(t,u)-b_k^n(t,u))\|^2\,\mathrm{d}t = 0,\ k=1,2.$$

Then the solution x to (1) *remains stable with respect to the solutions x_n to* (5)*, meaning that*

$$\lim_{n\to\infty}\mathbb{E}\sup_{t\in\tilde{I}} d_\infty^2(x_n(t),x(t)) = 0.$$

Proof. Fix the moment $t\in\tilde{I}$. Note that

$$\begin{aligned} &\mathbb{E}\sup_{u\in[0,t]} d_\infty^2(x_n(u),x(u)) \\ &\le 4\mathbb{E}\sup_{u\in[0,t]} d_\infty^2\Big(\int_0^t a_1^n(s,x_n(s))\,\mathrm{d}s, \int_0^t a_1(s,x(s))\,\mathrm{d}s\Big) \\ &+ 4\mathbb{E}\sup_{u\in[0,t]} d_\infty^2\Big(\int_0^t a_2^n(s,x_n(s))\,\mathrm{d}s, \int_0^t a_2(s,x(s))\,\mathrm{d}s\Big) \\ &+ 4\mathbb{E}\sup_{u\in[0,t]}\Big\|\int_0^t b_1^n(s,x_n(s))\,\mathrm{d}B_1(s) \\ &- \int_0^t b_1(s,x(s))\,\mathrm{d}B_1(s)\Big\|^2 \\ &+ 4\mathbb{E}\sup_{u\in[0,t]}\Big\|\int_0^t b_2^n(s,x_n(s))\,\mathrm{d}B_2(s) \\ &- \int_0^t b_2(s,x(s))\mathrm{d}B_2(s)\Big\|^2 \end{aligned}$$

$$\leq 8\mathbb{E}\sup_{u\in[0,t]} d_\infty^2\Big(\int_0^t a_1^n(s,x_n(s))\,\mathrm{d}s, \int_0^t a_1^n(s,x(s))\,\mathrm{d}s\Big)$$
$$+8\mathbb{E}\sup_{u\in[0,t]} d_\infty^2\Big(\int_0^t a_1^n(s,x(s))\,\mathrm{d}s, \int_0^t a_1(s,x(s))\,\mathrm{d}s\Big)$$
$$+ 8\mathbb{E}\sup_{u\in[0,t]} d_\infty^2\Big(\int_0^t a_2^n(s,x_n(s))\,\mathrm{d}s, \int_0^t a_2^n(s,x(s))\,\mathrm{d}s\Big)$$
$$+ 8\mathbb{E}\sup_{u\in[0,t]} d_\infty^2\Big(\int_0^t a_2^n(s,x(s))\,\mathrm{d}s, \int_0^t a_2(s,x(s))\mathrm{d}s\Big)$$
$$+ 8\mathbb{E}\sup_{u\in[0,t]}\Big\|\int_0^t b_1^n(s,x_n(s))\mathrm{d}B_1(s) - \int_0^t b_1^n(s,x(s))\mathrm{d}B_1(s)\Big\|^2$$
$$+ 8\mathbb{E}\sup_{u\in[0,t]}\Big\|\int_0^t b_1^n(s,x(s))\mathrm{d}B_1(s) - \int_0^t b_1(s,x(s))\mathrm{d}B_1(s)\Big\|^2$$
$$+ 8\mathbb{E}\sup_{u\in[0,t]}\Big\|\int_0^t b_2^n(s,x_n(s))\mathrm{d}B_2(s) - \int_0^t b_2^n(s,x(s))\mathrm{d}B_2(s)\Big\|^2$$
$$+ 8\mathbb{E}\sup_{u\in[0,t]}\Big\|\int_0^t b_2^n(s,x(s))\mathrm{d}B_2(s) - \int_0^t b_2(s,x(s))\mathrm{d}B_2(s)\Big\|^2$$
$$\leq K_n + 8t\mathbb{E}\int_0^t d_\infty^2(a_1^n(s,x_n(s)), a_1^n(s,x(s)))\,\mathrm{d}s$$
$$+ 8t\mathbb{E}\int_0^t d_\infty^2(a_2^n(s,x_n(s)), a_2^n(s,x(s)))\,\mathrm{d}s$$
$$+ 32\mathbb{E}\int_0^t \|b_1^n(s,x_n(s)) - b_1^n(s,x(s))\|^2\,\mathrm{d}s$$
$$+ 32\mathbb{E}\int_0^t \|b_2^n(s,x_n(s)) - b_2^n(s,x(s))\|^2\,\mathrm{d}s,$$

where

$$K_n = 8\tilde{T}\mathbb{E}\int_0^{\tilde{T}} d_\infty^2(a_1^n(s,x(s)), a_1(s,x(s)))\,\mathrm{d}s$$
$$+\ 8\tilde{T}\mathbb{E}\int_0^{\tilde{T}} d_\infty^2(a_2^n(s,x(s)), a_2(s,x(s)))\,\mathrm{d}s$$
$$+\ 32\mathbb{E}\int_0^{\tilde{T}} \|b_1^n(s,x(s)) - b_1(s,x(s))\|^2\,\mathrm{d}s$$
$$+\ 32\mathbb{E}\int_0^{t} \|b_2^n(s,x(s)) - b_2(s,x(s))\|^2\,\mathrm{d}s.$$

Based on assumption, we obtain

$$\lim_{n\to\infty} K_n = 0.$$

Given assumption (a2) and the properties of the function η, we can state

$$\mathbb{E}\sup_{u\in[0,t]} d_\infty^2(x_n(u), x(u))$$
$$\leq K_n + (16\tilde{T}+64)\int_0^t \eta(s, \mathbb{E}\sup_{u\in[0,s]} d_\infty^2(x_n(u),x(u)))\,\mathrm{d}s.$$

Let us redefine

$$\zeta_n(t) := \mathbb{E}\sup_{u\in[0,t]} d_\infty^2(x_n(u), x(u))$$

and

$$\zeta(t) := \lim_{n\to\infty}\zeta_n(t)$$

for $t\in\tilde{I}$. Observe that

$$\zeta_n(t) \leq K_n + (16\tilde{T}+64)\int_0^t \eta(s,\zeta_n(s))\,\mathrm{d}s$$

and based on assumptions about η, we get

$$\zeta(t) \leq (16\tilde{T}+64)\int_0^t \eta(s,\zeta(s))\,\mathrm{d}s$$

which implies that $\zeta(t)\equiv 0$. Thus,

$$\lim_{n\to\infty}\mathbb{E}\sup_{t\in\tilde{I}} d_\infty^2(x_n(t),x(t)) = 0$$

which concludes the proof. ■

Therefore, we have demonstrated that the problems described by the symmetric fuzzy stochastic differential equation (1), with coefficients meeting conditions (a0)–(a4), are well-posed. This implies that there is a unique solution, and it remains stable under minor variations in the coefficients.

5. Connections to random and deterministic symmetric fuzzy differential equations

Assuming $b_1 \equiv 0$ and $b_2 \equiv 0$ in (1), we derive a symmetric fuzzy random differential equation:

$$\mathrm{d}x(t,\omega)\oplus a_1(t,\omega,x(t,\omega))\,\mathrm{d}t \overset{I\ P.1}{=} a_2(t,\omega,x(t,\omega))\,\mathrm{d}t,$$
$$\text{with initial condition } x(0,\omega) \overset{P.1}{=} x_0(\omega), \tag{6}$$

where $a_1, a_2\colon I\times\Omega\times\mathcal{F}_c(\mathbb{R}^d)\to\mathcal{F}_c(\mathbb{R}^d)$ and $x_0\colon\Omega\to\mathcal{F}_c(\mathbb{R}^d)$ is a fuzzy random variable.

Since the Itô stochastic integral is absent in this equation, the solution no longer needs to be an $\mathcal{N}$-measurable process. Instead, the product measurability of process x suffices. Specifically, a fuzzy stochastic process $x\colon I\times\Omega\to\mathcal{F}_c(\mathbb{R}^d)$ is a solution to (6) if

$x \in \mathcal{L}^1(I \times \Omega, \mathcal{B}(I) \otimes \mathcal{A}; \mathcal{F}_c(\mathbb{R}^d))$, x is d_∞-continuous, and

$$x(t,\omega) \oplus \int_0^t a_1(s,\omega,x(s,\omega))\,\mathrm{d}s \overset{I\ P.1}{=} x_0(\omega) \oplus \int_0^t a_2(s,\omega,x(s,\omega))\,\mathrm{d}s.$$

The uniqueness of the solution is understood as per Definition 2. The analysis conducted in the previous sections can be applied to derive the equivalents of Theorems 1, 2, and 3, but this time for equation (6). In this context, the conditions (a0) and (a1) can be slightly relaxed for measurability purposes. Therefore, in (a0), it is sufficient that

• (a0r) $x_0 \in \mathcal{L}^2(\Omega, \mathcal{A}, P; \mathcal{F}_c(\mathbb{R}^d))$,

and in (a1) that

• (a1r) a_1 and a_2 are $(\mathcal{B}(I) \otimes \mathcal{A}) \otimes \mathcal{B}_{d_\infty}|\mathcal{B}_{d_\infty}$-measurable. Then we obtain the counterpart of Theorem 1 below.

Corollary 1. *Suppose that x_0, a_1, a_2 meet the conditions (a0r), (a1r), (a2)–(a4). Then the symmetric fuzzy random differential equation* (6) *has a unique solution.*

The analogs of Theorems 2 and 3 for equation (6) follow directly.

Besides the random equation (6) derived from equation (1), we can also formulate a deterministic symmetric fuzzy differential equation:

$$\mathrm{d}x(t) \oplus a_1(t,x(t))\,\mathrm{d}t = a_2(t,x(t))\,\mathrm{d}t, \quad \text{with initial condition } x(0) = x_0, \tag{7}$$

where $a_1, a_2 \colon I \times \mathcal{F}(\mathbb{R}^d) \to \mathcal{F}(\mathbb{R}^d)$ and $x_0 \in \mathcal{F}(\mathbb{R}^d)$.

A solution to (7) is a deterministic fuzzy mapping $x \colon I \to \mathcal{F}(\mathbb{R}^d)$ that satisfies: x is d_∞-continuous and

$$x(t) \oplus \int_0^t a_1(s,x(s))\,\mathrm{d}s = x_0 \oplus \int_0^t a_2(s,x(s))\,\mathrm{d}s$$

for all $t \in I$. Since x_0 is merely a fuzzy set and not a mapping, a condition like (a0) is no longer necessary. The remaining conditions are:

(a1d) the mappings $a_1, a_2 \colon I \times \mathcal{F}(\mathbb{R}^d) \to \mathcal{F}(\mathbb{R}^d)$ are $\beta(I) \otimes \mathcal{B}_{d_\infty}|\mathcal{B}_{d_\infty}$-measurable, where $\beta(I)$ denotes the Borel σ-algebra of subsets of I,

(a2d) there exists a function $\eta \colon I \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\eta(\cdot, r)$ is integrable for every $r \in \mathbb{R}_+$, $\eta(t,\cdot)$ is continuous, nondecreasing and concave for every $t \in I$, $\eta(t,0) = 0$ for every $t \in I$,

– if for $\zeta \colon I \to \mathbb{R}_+$ it holds $\zeta(0) = 0$ and $\zeta(t) \le M \int_0^t \eta(s,\zeta(s))\,\mathrm{d}s$ for $t \in I$, where M is a positive constant, then $\zeta(t) = 0$ for $t \in I$,

– for every $t \in I$ and for any $u, v \in \mathcal{F}(\mathbb{R}^d)$

$$\max\Big\{ d_\infty^2\big(a_1(t,u), a_1(t,v)\big), d_\infty^2\big(a_2(t,u), a_2(t,v)\big) \Big\} \le \eta(t, d_\infty^2(u,v)),$$

(a3d) there exist integrable functions $\gamma, \delta \colon I \to \mathbb{R}_+$ such that for every $t \in I$ and for every $u \in \mathcal{F}(\mathbb{R}^d)$

$$\max\Big\{ d_\infty^2(a_1(t,u), \langle 0 \rangle), d_\infty^2(a_2(t,u), \langle 0 \rangle) \Big\} \le \gamma(t) + \delta(t) d_\infty^2(u, \langle 0 \rangle),$$

(a4d) there exists $\tilde{T} \in (0,T]$ such that for every $n = 0,1,2,\dots$ the mappings $x_n \colon \tilde{I} \to \mathcal{F}(\mathbb{R}^d)$, where $\tilde{I} = [0,\tilde{T}]$, described as

$$x_0(t) \equiv x_0$$

and

$$x_n(t) = \Big[x_0 \oplus \int_0^t a_2(s, x_{n-1}(s))\,\mathrm{d}s\Big] \ominus \int_0^t a_1(s, x_{n-1}(s))\,\mathrm{d}s \quad t \in \tilde{I}$$

are well-defined.

Corollary 2. *Assume that $a_1, a_2 \colon I \times \mathcal{F}(\mathbb{R}^d) \to \mathcal{F}(\mathbb{R}^d)$ satisfy conditions (a1d)–(a4d). Then the deterministic symmetric fuzzy differential equation* (7) *has a unique solution.*

Analogously, the results regarding the low sensitivity of the solution to minor variations in the equation parameters can be derived. This means that the counterparts of Theorems 2 and 3 for the deterministic equation (7) can be established.

6. Concluding remarks

This paper investigates symmetric fuzzy stochastic differential equations. The term “symmetric” refers to the inclusion of drift and diffusion components on both sides of the equation. In the context of fuzzy equations, this symmetry is particularly meaningful and leads to new properties of the solutions, highlighting the importance of studying such equations. However, this paper primarily focuses on establishing the well-posedness of these equations, which means proving the existence of a unique solution and the stability of the solution with respect to small changes in the equation’s parameters, such as the initial value and the drift and diffusion coefficients. These results were obtained under conditions much weaker than the usual global Lipschitz condition with a constant Lipschitz constant. We assumed that the coefficients of the equation satisfy a condition involving a function within an integral inequality, which significantly generalizes the Lipschitz condition with a constant. The theorems we derived enable the practical application of these equations in problems where the dynamics must be described by differential equations that account for uncertainties from both randomness and ambiguity. The conditions on the coefficients of the equation used in this paper can be weakened in future studies. For example, coefficients that are discontinuous or locally unbounded can be considered, and methods leading to L^p conditions with $p > d/2$ can be applied (cf. Lee *et al.*, 2022). In addition, the results provide a foundation for future research on the discovery of approximate solutions, including the use of numerical methods. The numerical perspective can bring about the study of a convergence rate of stochastic approximation schemes, and this appears interesting.

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Marek T. Malinowski is an associate professor at the Cracow University of Technology, specializing in fuzzy mathematics, multivalued analysis, stochastic differential equations, probability, and artificial intelligence. His latest research focuses on developing a framework for fuzzy and multivalued stochastic differential equations, where he has established fundamental mathematical foundations that contribute to the initiation and advancement of these fields. Throughout his academic career, he has co-supervised three PhD students and authored over 50 papers published in international scientific journals. He has served as an editor for special issues of scientific publications and currently serves as an associate editor for multiple journals. He is a member of the Polish Mathematical Society and is deeply committed to teaching and dissemination of mathematical knowledge to students.

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