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NEURAL NETWORK-BASED NARX MODELS IN NON-LINEAR ADAPTIVE CONTROL

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The applicability of approximate NARX models of non-linear dynamic systems is discussed. The models are obtained by a new version of Fourier analysis-based neural network also described in the paper. This constitutes a reformulation of a known method in a recursive manner, i.e. adapted to account for incoming data on-line. The method allows us to obtain an approximate model of the non-linear system. The estimation of the influence of the modelling error on the discrepancy between the model and real system outputs is given. Possible applications of this approach to the design of BIBO stable closed-loop control are proposed.

Keywords: neural networks, adaptive control, nonlinear systems

1. Introduction

The method described is based on the N-D non-uniform sampling approach which led to a Fourier analysis-based feedforward neural network. The main scientific sources of the approach were: the multi-dimensional Fourier analysis, feedforward neural networks and non-uniform N-D sampling theory. The basic idea was originally proposed by Sanner and Slotine, but was later substantially revised and advanced, resulting in a more widely applicable and more sophisticated algorithm.

Let us briefly describe our approach to identification of a deterministic, non-linear, single-input single-output (SISO) system given by the discrete-time, $t \in \mathbb{Z}_+$, input-output NARX (Leontaritis and Billings, 1985; Chen and Billings, 1989) model

$$y(t+1) = f\big(y(t), \dots, y(t-n+1), u(t), \dots, u(t-m+1)\big), \tag{1}$$

with $y \in [a,b] \subset \mathbb{R}$, $u \in [c,d] \subset \mathbb{R}$ and $f\colon D \to [a,b]$ with the domain of definition $D = [a,b]^n \times [c,d]^m$. It is physically natural that output y and input u assume only finite values on a connected set and can attain their bounds (this does not preclude stability and boundedness issues). In (1) we assume that $f \in L^1(D) \cap L^2(D)$ is unknown, but we can measure current and past inputs $u(t), u(t-1), \dots$ and outputs $y(t), y(t-1), \dots$ at any t. Set $N = m+n$. Given the samples

$$\lambda_k = (\lambda_{1,k_1}, \dots, \lambda_{N,k_N}) = (y_t, \dots, y_{t-n+1}, u_t, \dots, u_{t-m+1})$$

and

$$f(\lambda_k), \tag{2}$$

where we write $y_t = y(t)$ etc. for brevity ($t \in \mathbb{Z}_+$) and $k = (k_1, k_2, \dots, k_N)$, the issue is to reconstruct the multivariable function f, which is a problem from multi-dimensional (N-D) Signal Processing. The approach was introduced by Sanner and Slotine (1992), but they assumed that the multi-dimensional samples are uniform, i.e. regularly distributed in the domain D of f. This seems to be a simplification, as the dynamics of (1) manifest themselves through irregular samples (Dzieliński and Żbikowski, 1995). Sanner and Slotine also required f to be analytic.

Let us now present a brief summary of our approach to the modelling of (1) in the context of N-D irregular sampling (Dzieliński and Żbikowski, 1995).

We are going to use the Fourier transform in several variables (Stein and Weiss, 1971) and therefore $f\colon D \to \mathbb{R}$ must be first extended to $\mathbb{R}^N$. We do it by the space-limited extension

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in D, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Thus $\tilde{f}$ is of bounded support and its Fourier transform is

$$\tilde{F}(w) = \int_{\mathbb{R}^N} \tilde{f}(x) \mathrm{e}^{-j\omega\cdot x}\,\mathrm{d}x = \int_D f(x) \mathrm{e}^{-j\omega\cdot x}\,\mathrm{d}x, \tag{4}$$

where $\omega \cdot x = \sum_{j=1}^N \omega_j x_j$, so that $\tilde{F}$ is a Paley-Wiener function of regular growth (Żbikowski and Dzieliński, 1996).

We are given a *finite* number of non-uniformly spread samples $\tilde{f}(\lambda_k)$, where $\lambda_k = (\lambda_{1,k_1}, \ldots, \lambda_{N,k_N})$, of the non-linear function $\tilde{f} = \tilde{f}(x)$, with $x = (x_1, \ldots, x_N) = (y_t, y_{t-1}, \ldots, y_{t-n+1}, u_t, u_{t-1}, \ldots, u_{t-m+1})$, i.e. $N = m + n$. We want to find the function $\tilde{f}$, which in general can only be done approximately due to the finiteness of data.

The main idea of our method is to replace the non-uniform sampling problem in the space domain by a uniform problem in the Fourier transform domain. The aim is to reconstruct $\tilde{F}$ and then, by its inversion, $\tilde{f}$. Therefore, our solution consists of two basic steps. First, we must find an approximation of $\tilde{F}$ on the basis of given non-uniformly sampled values of $\tilde{f}$, i.e. $\tilde{f}(\lambda_k)$. Then, we must find the Fourier inverse of the approximation of $\tilde{F}$ to get an approximation of $\tilde{f}$.

For simplicity and clarity, we show the reasoning for $D = [a, b] \times [c, d]$, i.e. $m = n = 1$, so that $N = 2$. Cases $m \neq n$ and $N > 2$ are straightforward generalisations of the 2-D derivation, but require more elaborate notations.

The Fourier transform $\tilde{F}$ of $\tilde{f}$ can be represented by its Shannon series (Papoulis, 1962) since $\tilde{f}$ is space-limited, see (3). The actual formula can be computed (following the reasoning in (Petersen and Middleton, 1962)) for a rectangular sampling geometry, i.e. by sampling arguments ω_1 and ω_2 of $\tilde{F}$ independently. It follows that the minimal (Shannon) frequencies are

$$\omega_{S_1} = \frac{2\pi}{b-a}, \tag{5}$$

$$\omega_{S_2} = \frac{2\pi}{d-c}, \tag{6}$$

and the corresponding exact representation of $\tilde{F}$ by the Shannon (cardinal) series is

$$\begin{aligned}\tilde{F}(\omega_1, \omega_2) = {} & \mathrm{e}^{-j\frac{b+a}{2}\omega_1}\mathrm{e}^{-j\frac{d+c}{2}\omega_2} \\ & \times \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} \mathrm{e}^{j\frac{b+a}{2}k_1\omega_{S_1}}\mathrm{e}^{j\frac{d+c}{2}k_2\omega_{S_2}} \\ & \times \tilde{F}(k_1\omega_{S_1}, k_2\omega_{S_2})\mathrm{sinc}\Big[\frac{b-a}{2}(\omega_1 - k_1\omega_{S_1})\Big] \\ & \times \mathrm{sinc}\Big[\frac{d-c}{2}(\omega_2 - k_2\omega_{S_2})\Big], \end{aligned} \tag{7}$$

where $\mathrm{sinc}(x) = (\sin x)/x$. Note that if D is centred at the origin, i.e. $a = -b$ and $c = -d$, then (7) becomes the (iterated) standard reconstruction formula:

$$\begin{aligned}\tilde{F}(\omega_1, \omega_2) = {} & \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} \tilde{F}(k_1\omega_{S_1}, k_2\omega_{S_2}) \\ & \times \mathrm{sinc}\,[b(\omega_1 - k_1\omega_{S_1})]\,\mathrm{sinc}\,[d(\omega_2 - k_2\omega_{S_2})], \end{aligned}$$

where $\omega_{S_1} = \pi/b$ and $\omega_{S_2} = \pi/d$. Also, in order for (7) to uniquely represent $\tilde{F}$ for any $(\omega_1, \omega_2) \in \mathbb{R}^2$, the sampling frequencies ω_{S_1} and ω_{S_2} cannot be smaller than (5) and (6), but may be larger (oversampling).

In principle, Sanner and Slotine's approach (Sanner and Slotine, 1992) of replacing (7) with an approximating Gaussian neural network can be used. While this seems to be motivated in their case by an attempt to relax the assumption of the band-boundedness of f, it is not needed here, due to the space-limited extension (3).

An advantage of representing $\tilde{F}$ by (7) is that by taking the inverse transform of (7) we get a representation of $\tilde{f}$ in terms of samples $\tilde{F}(k_1\omega_{S_1}, k_2\omega_{S_2})$:

$$\begin{aligned}\tilde{f}(x_1, x_2) = {} & \frac{1}{(b-a)(d-c)} \\ & \times \begin{cases} \displaystyle\sum_{k_1}\sum_{k_2} \tilde{F}(k_1\omega_{S_1}, k_2\omega_{S_2})\mathrm{e}^{jk_1\omega_{S_1}x_1} \\ \quad \times \mathrm{e}^{jk_2\omega_{S_2}x_2} \ \text{ for } \ (x_1, x_2) \in D, \\ 0 \ \text{ otherwise.} \end{cases} \end{aligned} \tag{8}$$

Since the summations in (8) are symmetric (from $-\infty$ to $+\infty$), the reconstructed $\tilde{f}$ will be real for all $(x_1, x_2) \in \mathbb{R}^2$. If quantities $\tilde{R}$ and $\tilde{I}$ denote respectively the real and imaginary parts of $\tilde{F}$, $\tilde{F}(\omega_1, \omega_2) = \tilde{R}(\omega_1, \omega_2) + j\tilde{I}(\omega_1, \omega_2)$, then $\tilde{R}$ is even, i.e. $\tilde{R}(\omega_1, \omega_2) = \tilde{R}(-\omega_1, -\omega_2)$, and $\tilde{I}$ is odd, i.e. $\tilde{I}(\omega_1, \omega_2) = -\tilde{I}(-\omega_1, -\omega_2)$.

If $\tilde{f}$ is known, then the above reasoning is a tautology and (8) is not needed. However, our problem is that $\tilde{f}$ is unknown, but we have its samples $\tilde{f}(\lambda_{1,k_1}, \lambda_{2,k_2}) = \tilde{f}(y_{t-1}, u_{t-1})$, where $t = 1, 2, \ldots, T$.

Since $\tilde{f}$ is space limited, its Fourier transform $\tilde{F}$ can be, in principle, reconstructed from (7), for which only samples of $\tilde{F}$ are needed. Thus the core issue is how to obtain these from $\tilde{f}(y_{t-1}, u_{t-1})$. Of course, this can be done only approximately, as we have finite data. From the definition of the Fourier transform we approximate the integral (4) by the finite Riemann-like sum

$$\begin{aligned}\tilde{F}_T(\omega_1, \omega_2) &= \sum_{t=1}^{T} \tilde{f}(y_{t-1}, u_{t-1})\mathrm{e}^{-j(\omega_1 y_{t-1} + \omega_2 u_{t-1})}A_t \\ &= \sum_{t=1}^{T} y_t \mathrm{e}^{-j(\omega_1 y_{t-1} + \omega_2 u_{t-1})}A_t, \end{aligned} \tag{9}$$

where T is the horizon of observation and A_t is the area associated with (y_{t-1}, u_{t-1}), with $\sum_{t=1}^{T} A_t = (b-a)(d-c)$. While the summation in (9) is over one index, t, it is an approximation of the double ($N = 2$) integral (4), which manifests itself in (y_{t-1}, u_{t-1}) and A_t.

Note that (9) is a non-standard approximation of (4) since (y_{t-1}, u_{t-1}) are distributed non-uniformly, i.e. they are not nodes of a rectangular grid (an ordinary Riemann sum). Therefore, we have to allocate area A_t to each

(y_{t-1}, u_{t-1}) according to the density of the points in D, which may be viewed as a weighting of (9). One possible approach is to use Voronoi diagrams, for which linear-time computational methods exist (Okabe *et al.*, 1992). Thus A_t would be the area of the Voronoi polygon generated by (y_{t-1}, u_{t-1}). This can be extended to $N > 2$ dimensions, as well (Dwyer, 1991). Another approach, implemented in our software, is to preprocess the samples to make them quasi-equidistributed and to apply the Monte Carlo method, (Stroud, 1971), effectively setting $A_t = (b-a)(c-d)/T$ (here T is the number of samples after preprocessing).

It should be emphasised that if non-uniformity is such that the points are evidently not equidistributed, the approximation of (9) is inaccurate. An example of this situation is when the points cluster in a few regions of D, leaving the rest of the rectangle with only very few points. Such an outlier gives little information about the values of $\tilde{f}$ in its neighbourhood, but the area allocated to it is a relatively large fraction of $(b-a)(d-c)$, thus amplifying the uncertainty. There are two possible remedies in such a situation. One is to shrink D, eliminating the regions with little information, and another is to try to generate the missing data. As the data are obtained from the dynamical system (1), the latter means designing a better identification experiment, while the former suggests that D is a superset of the true domain.

When a reasonable approximation $\tilde{F}_T$ is obtained, it can be substituted to (7) and then to (8). In order to obtain a computationally feasible approximation, the sums in (8) thus modified must be symmetrically truncated to $(2L_1+1)(2L_2+1)$ terms, say. Then the approximate reconstruction formula is 0 for (x_1, x_2) outside D, and for $(x_1, x_2) \in D$ it yields

$$\begin{aligned}\tilde{f}^a(x_1,x_2) = \frac{1}{(b-a)(d-c)}\Bigg\{ & R(0,0) + 2\sum_{k_1=1}^{L_1} R(k_1\omega_{S_1}, 0) \\ & \times \cos(k_1\omega_{S_1}x_1) - I(k_1\omega_{S_1}, 0)\sin(k_1\omega_{S_1}x_1) \\ & + 2\sum_{k_2=1}^{L_2} R(0, k_2\omega_{S_2})\cos(k_2\omega_{S_2}x_2) \\ & - I(0, k_2\omega_{S_2})\sin(k_2\omega_{S_2}x_2) \\ & + 2\sum_{k_1=1}^{L_1}\sum_{\substack{k_2=-L_2 \\ k_2\neq 0}}^{L_2} \Big[R(k_1\omega_{S_1}, k_2\omega_{S_2}) \\ & \times \cos(k_1\omega_{S_1}x_1 + k_2\omega_{S_2}x_2) \\ & - I(k_1\omega_{S_1}, k_2\omega_{S_2})\sin(k_1\omega_{S_1}x_1 \\ & + k_2\omega_{S_2}x_2)\Big]\Bigg\}. \end{aligned} \tag{10}$$

Here R and I are respectively the real and imaginary parts of $\tilde{F}_T$ of (9), i.e. $\tilde{F}_T(\omega_1, \omega_2) = R(\omega_1, \omega_2) + jI(\omega_1, \omega_2)$, and the assumptions that R is even and I is odd are made.

Note that L_1 and L_2 above are independent of T of (9), as they result from a symmetric truncation of (7), while T defines the number of data points available. In principle, increasing L_1 and L_2 should improve the accuracy of the approximation (10), but there are limitations imposed by the accuracy of $\tilde{F}_T$ of (9) and computational resources available.

Formula (10) is in essence a rectangular partial sum of a multiple Fourier series with coefficients expressed by (an approximation of) the Fourier transform of $\tilde{f}$. Hence Gibbs' phenomenon will occur on ∂D; this can be alleviated by artificially enlarging D.

Finally, note that (10) is an interpolation formula valid for all points of D, but obtained from the samples (y_{t-1}, u_{t-1}). The latter define R and I in (10) via (9).

The main features of the algorithm are as follows:

- the use of real-world data from an input-output, discrete-time NARX model;
- relative computational simplicity: only (9) and (10) are needed;
- mild assumptions: m, n in (1) known, $f \in L^1(D) \cap L^2(D)$ and the availability of measurements (2).

2. Adaptive Version of the Identification Method

The presented method seems to be a promising tool in the area of non-linear dynamic systems modelling. The version discussed so far is based on the assumption of the availability of all samples of the function f. This means it is *off-line* in character. This assumption has a direct influence on the definition of Riemann sums in (9). To define the areas A_t we need all the λ_k's. Let us note that the order in which these values appear in the model does not correspond to the order in which they are summed up. This means we renumber the data in comparison with their natural indexing. While this is not a problem in the off-line approach, it may cause some difficulties when trying to construct an on-line (adaptive) version of the method.

However, it is possible to reformulate this method in a recursive manner, i.e. to adapt it to account for incoming data *on-line*. The core issue is to decide how to define the summation pattern similar to the one given by (9). The most natural way is to subdivide the region in which $\tilde{f}$ is non-zero into an increasing number of areas along with incoming data. Thus, we start with

2^N subregions obtained from the initial values $\lambda_0 = (y_{t_0}, y_{t_0-1}, \ldots, y_{t_0-n+1}, u_{t_0}, u_{t_0-1}, \ldots, u_{t_0-m+1})$, then after the arrival of the first measured value of the function $\tilde{f}$ the number of subdivisions increases to 3^N, etc. At each step we are able to evaluate the approximate value of (9) in the form

$$\tilde{F}_i(\omega_1, \omega_2) = \sum_{t=1}^{i} \tilde{f}(y_{t-1}, u_{t-1}) \times \mathrm{e}^{-j(\omega_1 y_{t-1} + \omega_2 u_{t-1}} A_t, \qquad (11)$$

with $i = 1, 2, \ldots, T$, where A_t's are computed on-line. In general, we set $A_t = (b-a)(c-d)/(i+1)$. In the first instance these are only two subregions along each axis obtained from λ_0. In the simplest case they might be of the interval form $\{[a, y_{t_0}], [y_{t_0}, b]\}$, $\{[a, y_{t_0-1}], [y_{t_0-1}, b]\}$, ..., $\{[a, y_{t_0-n+1}], [y_{t_0-n+1}, b]\}$, $\{[c, u_{t_0}], [u_{t_0}, d]\}$, $\{[c, u_{t_0-1}], [u_{t_0-1}, d]\}$, ... $\{[c, u_{t_0-m+1}], [u_{t_0-m+1}, d]\}$. Next, in each of these pairs of subintervals one of them (it depends on whether the value of the incoming sample lies in the first or the second subinterval of the pair) is further subdivided giving the triple of subintervals along each axis, and so on. This proceeds as long as new measurements arrive. Finally, after all T samples have arrived, we obtain the same $\tilde{F}_T(\omega_1, \omega_2)$ as with (9).

Some other interesting details of the method have also been investigated. The problem of determining a function of compact support from the values of its Fourier transform on a finite segment is linked to the problem of *analytic continuation*, which is an ill-posed problem in Hadamard's sense. The solution to such a problem is not unique. However, it is possible to find an approximate solution with a small error by universal methods of regularisation (Tikhonov and Arsenin, 1977). The N-D Paley-Wiener theory (Żbikowski and Dzieliński, 1996) is also of relevance here.

3. Applicability of Approximate NARX Models

In this section we discuss the adequacy of an approximate (due to modelling errors) NARX model (1) of the real plant for control purposes. In other words, we want to know if an inaccurate NARX representation (1) of the real NARX system would reflect well the system's behaviour when influenced by the same control signal.

Consider a function constituting a bound on the norm of the modelling error, i.e. with the difference between f in (1) and g being the right-hand side (RHS) of the NARX representation of the real plant. This function should bound the norm uniformly in u (for all admissible control signals u). The question is what this tells us about the error between y and z being the real plant output. If the error is small, then applying a control signal to the approximate model would cause a similar behaviour of the real system. This also applies to the Bounded Input Bounded Output (BIBO) stability analysis, because if we prove stability for the model (1), then it will hold for the real plant, provided the difference between y and z is bounded.

The approach to the problem is based on the finite difference inequalities.

Proposition 1. *The controlled difference equation (1) is equivalent to the following n-th order controlled finite difference equation:*

$$\Delta^{(n)} y(k) = \bar{f}(Y(k), U(k)), \qquad (12)$$

where

$$Y(k) = \big(\Delta^{(n-1)} y(k), \ldots, \Delta y(k), y(k)\big),$$
$$U(k) = \big(\Delta^{(n-1)} u(k), \ldots, \Delta u(k), u(k)\big). \qquad (13)$$

From now on $\bar{f}$ denotes the right-hand side (RHS) of the finite difference equation corresponding to (1) and, similarly, $\bar{g}$ for the real plant.

By Proposition 1, we may consider the model-plant correspondence and the BIBO stability in the framework of (controlled) finite difference equations.

Lemma 1. *Let the function $W\colon \mathbb{R}^n \times I_{k_0} \to \mathbb{R}$ be continuous, non-negative, monotonically increasing on $\mathbb{R}^n$ for each $k \in I_{k_0}$, and let $r\colon I_{k_0} \to \mathbb{R}$ be the solution of*

$$\Delta^{(n)} r(k) = W(R(k), k),$$
$$\Delta^{(i)} r(k_0) = \bar{r}_i \quad \textit{for} \quad i = 0, 1, \ldots, n-1, \qquad (14)$$

where

$$R(k) = \big(\Delta^{(n-1)} r(k), \ldots, \Delta r(k), r(k)\big). \qquad (15)$$

With the notation as in (13), consider two n-th order finite difference equations:

$$\Delta^{(n)} z(k) = \bar{g}(Z(k), U(k)),$$
$$\Delta^{(i)} z(k_0) = \bar{z}_i \quad \textit{for} \quad i = 0, 1, \ldots, n-1 \qquad (16)$$

for the true NARX description of the plant, where

$$Z(k) = \big(\Delta^{(n-1)} z(k), \ldots, \Delta z(k), z(k)\big) \qquad (17)$$

and

$$\Delta^{(n)} y(k) = \bar{f}(Y(k), U(k)),$$
$$\Delta^{(i)} y(k_0) = \bar{y}_i \quad \textit{for} \quad i = 0, 1, \ldots, n-1, \qquad (18)$$

stand for the approximate NARX model (1) of the plant. Here $\bar{f}\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *and* $\bar{g}\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *are assumed to be continuous and satisfy*

$$\begin{aligned}\|\bar{f}(Y(k),U(k)) - \bar{g}(Z(k),U(k))\| \\ \leq W\big(\|\Delta^{(n-1)}y(k) - \Delta^{(n-1)}z(k)\|, \\ \ldots, \|y(k)-z(k)\|, k\big)\end{aligned} \tag{19}$$

uniformly with respect to $u \in \mathcal{U}$ *for all* $k \geq k_0$*, where* $\mathcal{U}$ *is the set of admissible control signals, defined on* I_{k_0}*. Finally, let* $y\colon I_{k_0} \to \mathbb{R}$ *and* $z\colon I_{k_0} \to \mathbb{R}$ *be any solutions of (18) and (16), respectively, such that*

$$\|\bar{y}_i - \bar{z}_i\| \leq \bar{r}_i \quad \textit{for} \quad i = 0,1,\ldots,n-1. \tag{20}$$

Then

$$\|y(k) - z(k)\| \leq r(k) \ \textit{ for all } \ k \geq k_0. \tag{21}$$

Remark 1. The continuity of $W\colon \mathbb{R}^n \times I_{k_0} \to \mathbb{R}$ is to be understood by interpreting W as a restriction of a function continuous on $\mathbb{R}^n \times \mathbb{R}$. Alternatively, we may require that W is continuous on $\mathbb{R}^n$ for each fixed $k \in I_{k_0}$.

Remark 2. Note that we assume the same order, n, of the model (18) and the plant (16).

Remark 3. For BIBO stability considerations the set of admissible control signals $\mathcal{U}$ is the set of bounded functions $u\colon I_{k_0} \to \mathbb{R}$. Because of the form of (18), $\mathcal{U}$ describes not only the constraints on u, but also on its finite differences $\Delta^{(n-1)}u(k), \ldots, \Delta u(k)$.

Lemma 1 has important consequences for modelling and control of NARX systems. Solutions of (14) determine the discrepancy (see (20) and (21)) between the model (18) and the real plant (16) under the action of the same control signal. Thus, if we can find W satisfying (19) with $\bar{r}_i$, $i = 0,1,\ldots,n-1$, satisfying (20) and such that solutions of (14) are bounded, then the discrepancy is also bounded.

This is of particular interest for the neural modelling of NARX systems (Żbikowski and Dzieliński, 1996; Dzieliński and Żbikowski, 1995), where (18) is a neural approximation of the real plant. If the discrepancy is small, then the controller designed for the *approximate* NARX model should perform well for the *real* NARX plant. In practice, this equivalence may be provable only for a subset of admissible controls $\mathcal{U}$, because Lemma 1 gives only sufficient conditions.

Lemma 1 and the developments above could be applied to the BIBO stability analysis of the real plant in the closed-loop context. Recall that we do not know g of the real plant, or—equivalently—$\bar{g}$ of (16), but we have f of (1), or—equivalently—$\bar{f}$ of (18). Thus, the design of a control law must be based on $\bar{f}$, but the control signal will be applied to the real plant (16). The fundamental requirement is that this approach will lead to a BIBO stable closed-loop system. The closed-loop system is the real plant (16) with a controller designed for its approximate model (18). The stability in this context is the BIBO stability meaning that the controller generates bounded inputs resulting in bounded outputs of (16). In this sense we can talk about a closed-loop BIBO stability.

The main result in this area is based on one of the comparison theorems given by Pachpatte (1970, Thm. 5).

Theorem 1. *Let the functions* $\hat{f}_1\colon \mathbb{R}^n \times I_{k_0} \to \mathbb{R}$ *and* $\hat{f}_2\colon \mathbb{R}^n \times I_{k_0} \to \mathbb{R}$ *be continuous, non-negative and monotonically increasing on* $\mathbb{R}^n$ *for each* $k \in I_{k_0}$*. Moreover, let* $\hat{f}_1$ *and* $\hat{f}_2$ *satisfy the inequalities*

$$\hat{f}_1(Y(k),k) \leq \Delta^{(n)}y(k) \leq \hat{f}_2(Y(k),k) \tag{22}$$

for all $k \geq k_0$*, where* $\Delta^{(n)}y(k)$ *is as in (12). Let* $v(k)$ *and* $w(k)$ *be the solutions of*

$$\begin{aligned}\Delta^{(n)}v(k) &= \hat{f}_1(\Delta^{(n-1)}v(k),\ldots,v(k),k), \\ \Delta^{(i)}v(k_0) &= v_0^i, \quad \textit{for} \quad i = 0,1,\ldots,n-1\end{aligned} \tag{23}$$

and

$$\begin{aligned}\Delta^{(n)}w(k) &= \hat{f}_2(\Delta^{(n-1)}w(k),\ldots,w(k),k), \\ \Delta^{(i)}w(k_0) &= w_0^i, \quad \textit{for} \quad i = 0,1,\ldots,n-1,\end{aligned} \tag{24}$$

respectively, such that

$$v_0^i \leq \Delta^{(i)}y(k_0) \leq w_0^i \quad \textit{for} \quad i = 0,1,\ldots,n-1. \tag{25}$$

Then

$$v(k) \leq y(k) \leq w(k) \tag{26}$$

for all $k \geq k_0$.

Thus, if we are able to find two functions $\hat{f}_1$ and $\hat{f}_2$ such that (22) holds, it means (by (12)) that

$$\hat{f}_1(Y(k),k) \leq \hat{f}(Y(k),k) \leq \hat{f}_2(Y(k),k) \tag{27}$$

for all $k \geq k_0$ and for a given $u \in \mathcal{U}$, say $u(k) \equiv \vartheta(k)$. Additionally, let $\hat{f}_1$ and $\hat{f}_2$ be such that (23) and (24) have bounded solutions (see, e.g., (Dzieliński, 1999) for a criterion of the boundedness of the solutions), let and u be bounded. Then, from (26), the solution of (12) corresponding to $u(k) \equiv \vartheta(k)$ is also bounded. If this can be shown for all $u \in \mathcal{U}$, then the system described by (12) is BIBO stable and, by Proposition 1, so is (1).

The above can be applied in the closed-loop context, where u is generated by a control law. However, it should be borne in mind that the controller design is possible only on the basis of $\bar{f}$ in (18), while u will be applied to (16).

In order to obtain a stable closed-loop system, the following procedure of *BIBO redesign* can be devised: First, the model-plant equivalence must be established, i.e. a set $\mathcal{U}$ of admissible, bounded inputs must be found for which Lemma 1 holds. Thus, based on an *a-priori* estimate of the modelling error, a function W satisfying (19) should be constructed, so that (21) is satisfied with r bounded. The set $\mathcal{U}$ for which these hold is then the starting point for the second step of BIBO redesign, since it ensures that bounded $\|\bar{f}(Y(k),U(k))-\bar{g}(Z(k),U(k))\|$ results in bounded $\|y(k)-z(k)\|$ for all $k \geq k_0$. Now, given a reference signal $\xi\colon I_{k_0} \to \mathbb{R}$, a control law ϕ, $\phi(Y(k),\xi(k)) = u(k)$ with $u \in \mathcal{U}$, must be designed, so that (27) holds with $\hat{f}_1$, $\hat{f}_2$ of Theorem 1 satisfying the condition of the boundedness of solutions.

4. Conclusions

A new version of the method for identification of non-linear systems given as NARX models was described. It is based on a harmonically limited N-D Fourier transform, which enables reconstruction of the right-hand side of the NARX equation (1) in the multi-dimensional frequency domain via feedforward neural networks. It entails a novel method for the approximate interpolation of a non-linear function from a finite set of its irregular samples. The novelty of the presented method lies in its adaptive character, i.e. the possibility of accounting for the incoming data online.

The neural models usually suffer from a certain degree of inaccuracy. The results given in the paper allow us to check whether the inaccurate model is a sufficiently good approximation of the real plant, i.e. whether the difference between the model's and real plant's outputs is bounded.

This result can also be important for checking the BIBO stability of the model and for designing the stable closed-loop system.

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