Appl. Math. and Comp. Sci., 1994, vol.4, No.4, 507–515

EXTENSIONS OF THE CAYLEY-HAMILTON THEOREM FOR 2-D CONTINUOUS-DISCRETE LINEAR SYSTEMS

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The Cayley-Hamilton theorem is extended for new classes of 2-D continuous-discrete linear systems.

1. Introduction

Recently a new class of two-dimensional (2-D) continuous-discrete linear system has been introduced (Kaczorek, 1994a; 1994b). Using the Weierstrass decomposition in (Kaczorek, 1994a) the general response formula for regular 2-D continuous-discrete linear systems was derived. In (Kaczorek, 1994b) the general response formula was derived and the necessary and sufficient conditions for the reachability and controllability of the standard 2-D continuous-discrete linear systems were established.

The classical Cayley-Hamilton theorem was extended for regular and singular 2-D and n-D linear systems. (Ciftcibasi and Yuksel, 1982; Chang and Chen, 1992; Kaczorek, 1994a; 1994b; Smart and Barnett, 1986; Warwick, 1983). In (Chang and Chen, 1992) the generalized Cayley-Hamilton theorem for standard pencils was presented. In this paper the Cayley-Hamilton theorem will be extended for new classes of 2-D continuous-discrete linear systems.

2. Models of 2-D Continuous-Discrete Linear Systems

Let $\mathbb{R}_+$ be the set of nonnegative real numbers and $\mathbb{Z}_+$ be the set of nonnegative integers. Consider a 2-D continuous-discrete linear system described by the equations

$$E\dot{x}(t,k+1) = Ax(t,k+1) + Bx(t,k) + Cu(t,k), \qquad t \in \mathbb{R}_+,\ k \in \mathbb{Z}_+ \tag{1a}$$

$$y(t,k) = Dx(t,k), \qquad t \in \mathbb{R}_+,\ k \in \mathbb{Z}_+ \tag{1b}$$

where $\dot{x}(t,k) = \dfrac{\partial x(t,k)}{\partial t}$, $x(t,k) \in \mathbb{R}^n$ is the semistate vector, $u(t,k) \in \mathbb{R}^m$ is the input vector, $y(t,k) \in \mathbb{R}^p$ is the output vector, $E \in \mathbb{R}^{q\times n}$, $A \in \mathbb{R}^{q\times n}$, $B \in \mathbb{R}^{q\times n}$, $C \in \mathbb{R}^{q\times n}$, $D \in \mathbb{R}^{q\times n}$ and $\mathbb{R}^{p\times q}$ is the set of $p \times q$ real matrices.

If $q \neq n$ or $q = n$ but $\det E = 0$, system (1) will be called singular. System (1) will be called regular if $q = n$, $\det E = 0$ and

$$\det[Es - A] \neq 0 \text{ for some } \mathbb{C} \text{ (the field of complex numbers)} \tag{2}$$

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If $q = n$ and $\det E \neq 0$, then premultiplying (1a) by E^{-1} we obtain

$$\dot{x}(t, k+1) = A'x(t, k+1) + B'x(t, k) + C'u(t, k) \tag{1'a}$$

where $A' := E^{-1}A$, $B' := E^{-1}B$, $C' := E^{-1}C$.

A 2-D system described by (1'a) and (1b) will be called standard. Boundary conditions for (1a) (or(1'a)) are given by

$$x(t, 0) = x_1(t), \quad t \in \mathbb{R}_+ \quad \text{and} \quad x(0, k) = x_2(k), \quad k \in \mathbb{Z}_+ \tag{3}$$

where $x_1(t)$ and $x_2(k)$ are known and $x_1(0) = x_2(0)$.

The boundary conditions (3) are called admissible for (1a) if equation (1a) has a solution $x(t, k)$ for (3) and a given $u(t, k)$. The set of admissible boundary conditions (3) is determined by the matrices A, B, C, E and the given input $u(t, k)$.

3. Solution and General Response Formula

Consider the standard system described by (1'a) and (1b) with boundary conditions (3).

Theorem 1. *The solution $x(t, k)$ to (1'a) with (3) is given by*

$$\begin{aligned} x(t,k) &= e^{At}x_2(k) + \sum_{i=1}^{k-1} \int_0^t e^{A(t-\tau_1)}B \int_0^{\tau_1} e^{A(\tau_1-\tau_2)} \\ &\quad \cdots B \int_0^{\tau_{k-i-1}} e^{A(\tau_{k-i-1}-\tau_{k-i})}Be^{A\tau_{k-i}}x_2(i)\,\mathrm{d}\tau_{k-i}\,\mathrm{d}\tau_{k-i-1}\cdots \mathrm{d}\tau_1 \\ &\quad + \int_0^t e^{A(t-\tau_1)}B \int_0^{\tau_1} e^{A(\tau_1-\tau_2)}B \int_0^{\tau_2} e^{A(\tau_2-\tau_3)} \\ &\quad \cdots B \int_0^{\tau_{k-1}} e^{A(\tau_{k-1}-\tau_k)}Bx_1(\tau_k)\,\mathrm{d}\tau_k\,\mathrm{d}\tau_{k-1}\cdots \mathrm{d}\tau_1 \\ &\quad + \sum_{i=0}^{k-1} \int_0^t e^{A(t-\tau_1)}B \int_0^{\tau_1} e^{A(\tau_1-\tau_2)}B \int_0^{\tau_2} e^{A(\tau_2-\tau_3)} \\ &\quad \cdots B \int_0^{\tau_{k-i-1}} e^{A(\tau_{k-i-1}-\tau_{k-i})}Bu(\tau_{k-i}, i)\,\mathrm{d}\tau_{k-i}\,\mathrm{d}\tau_{k-i-1}\cdots \mathrm{d}\tau_1 \end{aligned} \tag{4}$$

for $t \in \mathbb{R}_+$, $k \in \mathbb{Z}_+$ (by definition $\tau_0 = t$).

Theorem 1 can be proved by induction or by checking that (4) satisfies (1'a) and (3) (Kaczorek, 1994b). In particular case for $A = 0$ we have the following.

Corollary 1. *The solution $x(t, k)$ to the equation*

$$\dot{x}(t, k+1) = Bx(t, k) + Cu(t, k), \qquad t \in \mathbb{R}_+, \quad k \in \mathbb{Z}_+ \tag{5}$$

with boundary conditions (3) is given by

$$
\begin{aligned}
x(t,k) &= B^k \int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_2} x_1(\tau_1)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \cdots \mathrm{d}\tau_k \\
&+\sum_{i=0}^{k-1}\left[\frac{(Bt)^i}{i!}x_2(k-i)+B^{k-i-1}C\int_0^t\int_0^{\tau_k}\cdots\int_0^{\tau_{i+2}}u(\tau_{i+1})\,\mathrm{d}\tau_{i+1}\cdots\mathrm{d}\tau_k\right]
\end{aligned}
\tag{6}
$$

for $t \in \mathbb{R}_+,\ k \in \mathbb{Z}_+$.

The general response formula for the standard system (1′a), (1b) can be obtained by substitution of (4) into (1b).
The general response formula for the regular system (1) with was derived in (Kaczorek, 1994a).

4. Cayley-Hamilton Theorem for 2-D Continuous-Discrete Linear Systems

4.1. Standard Systems

It is assumed that for (1′a) the following condition holds

$$\det[I_n sz - A'z - B'] \neq 0 \ \text{ for some } \ (s,z) \in \mathbb{C}\times\mathbb{C} \tag{7}$$

where I_n is the $n \times n$ identity matrix.
If condition (7) is satisfied, then we may write

$$[I_n sz - A'z - B']^{-1} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} T_{ij} s^{-(i+1)} z^{-(j+1)} \tag{8}$$

where T_{ij} are some real matrices called the transition matrices of (1′a).

Equating the matrix coefficients at the same powers of s and z of the equality.

$$
\begin{aligned}
&[I_n sz - A'z - B']\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} T_{ij} s^{-(i+1)} z^{-(j+1)}\right) \\
&= \left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} T_{ij} s^{-(i+1)} z^{-(j+1)}\right)[I_n sz - A'z - B'] = I_n
\end{aligned}
\tag{9}
$$

we obtain

$$
T_{ij} = \begin{cases} I_n & \text{for } i=j=0 \\ A'T_{i-1,j}+B'T_{i-1,j-1}=T_{i-1,j}A'+T_{i-1,j-1}B' & \text{for } i\geq 0,\ j\geq 0 \\ 0 \ \text{ the zero matrix} & \text{for } i<0 \text{ or/and } j<0 \end{cases}
\tag{10}
$$

Lemma. *The transition matrices* T_{ij} *satisfy the relation*

$$T_{ij} = \begin{cases} A'^i & \text{for } i \geq 0 \text{ and } j = 0 \\ 0 & \text{for } j > i \geq 1 \\ B'^i & \text{for } i = j \geq 1 \end{cases} \tag{11}$$

Proof. From (10) we have for $i = j = 0$, $T_{00} = I_n = A'^0$ and for $i > 0$, $j = 0$

$$T_{i0} = A'T_{i-1,0} + B'T_{i-1,-1} = A'T_{i-1,0} = A'^2 T_{i-2,0} = \cdots = A'^i T_{00} = A'^i, \quad i = 1, 2, \ldots$$

Similarily we have from (10) for $i = 0$, $j > 0$

$$T_{0j} = A'T_{-1,j} + B'T_{-1,j-1} = 0$$

and

$$T_{1j} = A'T_{0j} + B'T_{0,j-1} = 0 \quad \text{for} \quad j > 1$$

Assuming that $T_{kl} = 0$ for $l > k$ we shall show that $T_{k+1,l+1} = 0$. Really from (10) we have

$$T_{k+1,l+1} = A'T_{k,l+1} + B'T_{k,l} = 0 \quad \text{for} \quad l > k$$

Therefore, by induction $T_{ij} = 0$ for $j > i \geq 1$.

From (10) for $i = j \geq 1$ we have

$$T_{ii} = A'T_{i-1,i} + B'T_{i-1,i-1} = B'T_{i-1,i-1} = B'^2 T_{i-2,i-2} = \cdots = B'T_{00} = B'^i$$

since $T_{i-1,i} = 0$ for $i \geq 1$. ■

Theorem 2. *Let*

$$\det[I_n sz - A'z - B'] := \sum_{i=0}^{n} \sum_{j=0}^{n} d_{ij} s^i z^j \quad (d_{nn} = 1) \tag{12}$$

Then the transition matrices T_{ij} *of (1′a) satisfy the equations*

$$\sum_{i=0}^{n} \sum_{j=0}^{n} d_{ij} T_{i+k-1,j+l-1} = 0 \quad \text{for} \quad k, l = 1, 2, \ldots \tag{13}$$

Proof. Let $[I_n sz - A'z - B']_{ad}$ be the adjoint matrix of $[I_n sz - A'z - B']$ and

$$[I_n sz - A'z - B']_{ad} := \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} H_{ij} s^i z^j \tag{14}$$

Taking into account that

$$[I_n sz - A'z - B']_{ad} = [I_n sz - A'z - B']^{-1} \det[I_n sz - A'z - B']$$

and using (12), (8) and (14) we may write

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} H_{ij} s^i z^j = \left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} T_{ij} s^{-(i+1)} z^{-(j+1)}\right]\left[\sum_{i=0}^{n}\sum_{j=0}^{n} d_{ij} s^i z^j\right]$$
$$= I_n s^{n-1} z^{n-1} + \cdots + \left(\sum_{i=0}^{n-k-1}\sum_{j=0}^{n-l-1} T_{ij} d_{i+k+1,j+l+1}\right) s^k z^l$$
$$+ \cdots + \left(\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} T_{ij} d_{i+1,j+1}\right) s^0 z^0 \tag{15}$$
$$+ \cdots + \left(\sum_{i=0}^{n+k-1}\sum_{j=0}^{n+l-1} T_{ij} d_{i-k+1,j-l+1}\right) s^{-k} z^{-l} + \cdots$$

The comparison of the matrix coefficients at the same powers of s and z yields

$$\sum_{i=0}^{n+k-1}\sum_{j=0}^{n+l-1} T_{ij} d_{i-k+1,j-l+1} = 0 \quad \text{for} \quad k,l = 1,2,\ldots \tag{16}$$

Note that (16) is equivalent to (13). ■

Taking into account (11) we may write (13) in the form

$$d_{00} I_n + \sum_{i=1}^{n} d_{i0} A'^{i+k-1} + \sum_{i=1}^{n} d_{ii} B'^{i+k-1} + \sum_{i=2}^{n}\sum_{j=1}^{i-1} d_{ij} T_{i+k-1,j+l-1} = 0 \tag{17}$$

for $i+k>1,\ j+l>1,\ k,l=1,2,\ldots$.

Therefore, we have the following important

Corollary 2. *The transition matrices T_{ij} of (1′a) satisfy equations (17). In particular case for $k=l=1$ from (17) we have*

$$\sum_{i=0}^{n} d_{i0} A'^i + \sum_{i=1}^{n} d_{ii} B'^i + \sum_{i=2}^{n}\sum_{j=1}^{i-1} d_{ij} T_{ij} = 0 \tag{18}$$

Example. Consider system (1′a) with

$$A' = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \qquad B' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

In this case

$$\det[I_n sz - A'z - B'] = \begin{vmatrix} sz - z, & z-1 \\ -1 & sz - z \end{vmatrix} = s^2 z^2 - 2sz^2 + z^2 + z - 1$$

and

$$d_{00} = -1, \quad d_{01} = 1, \quad d_{02} = 1$$
$$d_{10} = 0, \quad d_{11} = 0, \quad d_{12} = -2$$
$$d_{20} = 0, \quad d_{21} = 0, \quad d_{22} = 1$$

Using (10) and (11) we obtain

$$T_{00} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad T_{01} = T_{02} = T_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$T_{10} = A' = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad T_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$T_{20} = A'^2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \quad T_{21} = A'T_{11} + B'T_{10} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$
$$T_{22} = B'^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T_{31} = A'T_{21} + B'T_{20} = \begin{bmatrix} -3 & 4 \\ 3 & -3 \end{bmatrix}$$

From (18) we have for $k = l = 1$

$$d_{00}I_2 + d_{10}A' + d_{20}A'^2 + d_{11}B' + d_{22}B'^2 + d_{21}T_{21}$$
$$= -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and for $k = 2$, $l = 1$ from (17) we obtain

$$d_{00}I_2 + d_{10}A'^2 + d_{20}A'^3 + d_{11}B'^2 + d_{22}B'^3 + d_{21}T_{31}$$
$$= -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} -3 & 4 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Similarily it can be checked that (17) is satisfied for $k \geq 2$ and $l \geq 2$.

4.2. Singular Systems

The above considerations can be extended for the singular system (1) satisfying the condition

$$\det[Esz - Az - B] \neq 0 \quad \text{for some} \quad (s, z) \in \mathbb{C} \times \mathbb{C} \tag{19}$$

In this case instead of (8) and (12) we have

$$[Esz - Az - B]^{-1} = \sum_{i=-n_1}^{\infty} \sum_{j=-n_2}^{\infty} T'_{ij} s^{-(i+1)} z^{-(j+1)} \tag{20}$$

and

$$\det[Esz - Az - B] := \sum_{i=0}^{r_1}\sum_{j=0}^{r_2} d'_{ij} s^i z^j \tag{21}$$

respectively, where $r_1 \le rank\, E$, $r_2 \le \max(rank\, E, rank\, A)$.

In general case n_1 and n_2 may be infinite. If $\det A \neq 0$ then $n_2 = 0$ and if $d'_{r_1 r_2} \neq 0$ then n_1 and n_2 are finite. Equating the matrix coefficients at the same powers of s and z of the equality

$$\begin{aligned}[Esz - Az - B]\left(\sum_{i=-n_1}^{\infty}\sum_{j=-n_2}^{\infty} T'_{ij} s^{-(i+1)} z^{-(j+1)}\right) \\ = \left(\sum_{i=-n_1}^{\infty}\sum_{j=-n_2}^{\infty} T'_{ij} s^{-(i+1)} z^{-(j+1)}\right)[Esz - Az - B] = I_n\end{aligned} \tag{22}$$

we obtain

$$ET'_{ij} = \begin{cases} AT'_{-1,0} + BT'_{-1,-1} + I_n & \text{for } i = j = 0 \\ AT'_{i-1,j} + BT'_{i-1,j-1} & \text{for } i \neq 0 \text{ or/and } j \neq 0 \\ 0 & \text{for } i < -n_1 \text{ or/and } j < -n_2 \end{cases} \tag{23a}$$

and

$$T'_{ij}E = \begin{cases} T'_{-1,0}A + T'_{-1,-1}B + I_n & \text{for } i = j = 0 \\ T'_{i-1,j}A + T'_{i-1,j-1}B & \text{for } i \neq 0 \text{ or/and } j \neq 0 \\ 0 & \text{for } i < -n_1 \text{ or/and } j < -n_2 \end{cases} \tag{23b}$$

Note that (10) is a particular case of (23) for $E = I_n$

Let

$$[Esz - Az - B]_{ad} := \sum_{i=0}^{m_1}\sum_{j=0}^{m_2} H'_{ij} s^i z^j \tag{24}$$

where $m_1 \le r_1$ and $m_2 \le r_2$.

Using (20), (21) and (24) in a similar way as Theorem 2, the following theorem can be proved.

Theorem 3. *The transition matrices T'_{ij} of (1a) satisfy the equations*

$$\sum_{i=0}^{r_1}\sum_{j=0}^{r_2} d'_{ij} T_{i+k-1,j+l-1} = 0 \text{ for } n_1 + r_1 - 1 > m_1,\quad n_2 + r_2 - 1 > m_2 \tag{25}$$

and $k > 0$, $l > 0$.

Note that (13) is a particular case of (25) for $n_1 = n_2 = 0$, $r_1 = r_2 = n$ and $m_1 = m_2 = n - 1$.

4.3. Particular Case

Consider equation (1a) for $A = 0$, i.e.

$$E\dot{x}(t, k+1) = Bx(t,k) + Cu(t,k)$$

with the commutative pair (E, B)

$$EB = BE \tag{26a}$$

and

$$\det[Es - B] \neq 0 \quad \text{for some} \quad s \in \mathbb{C} \tag{26b}$$

Theorem 4. *If (26) holds, then*

$$\sum_{i=0}^{n} a_i B^i E^{n-i} = 0 \tag{27}$$

where a_i are the coefficients of the characteristic polynomial

$$\det[Es - B] = \sum_{i=0}^{n} a_i s^i \tag{28}$$

Proof. From (26a) it follows that

$$B(Es - B) = (Es - B)B, \qquad E(Es - B) = (Es - B)E$$

and

$$(Es - B)^{-1}B = B(Es - B)^{-1}, \quad (Es - B)^{-1}E = E(Es - B)^{-1} \tag{29}$$

Using (29) it is easy to show that

$$\begin{aligned} \Big[(Es - B)^{-1}B\Big]^k &= B^k(Es - B)^{-k}, \\ \Big[(Es - B)^{-1}E\Big]^k &= (Es - B)^{-k}E^k, \end{aligned} \qquad k = 1, 2, \ldots \tag{30}$$

and

$$(Es - B)^{-n}E^{n-i} = E^{n-i}(Es - B)^{-n}, \qquad i = 0, 1, \ldots \tag{31}$$

In (Ciftcibasi and Yuksel, 1982) it was shown that if

$$\overline{E} = (Es - B)^{-1}E, \qquad \overline{B} = (Es - B)^{-1}B \tag{32}$$

then

$$\sum_{i=0}^{n} a_i \overline{B}^i \overline{E}^{n-i} = 0 \tag{33}$$

Using (33), (32), (30) and (31) we may write

$$\sum_{i=0}^{n} a_i \overline{B}^i \overline{E}^{n-i} = \sum_{i=0}^{n} a_i \Big[(Es-B)^{-1}B\Big]^i \Big[(Es-B)^{-1}E\Big]^{n-i}$$

$$= \sum_{i=0}^{n} a_i B^i (Es-B)^{-i}(Es-B)^{i-n} E^{n-i}\overline{E}^{n-i} \tag{34}$$

$$= \sum_{i=0}^{n} a_i B^i E^{n-i}(Es-B)^{-n} = 0$$

Postmultiplication of (34) by $(Es-B)^n$ yields (27). ■

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Received: July 27, 1994