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CONTROLLABILITY AND MINIMUM ENERGY CONTROL OF 2-D CONTINUOUS-DISCRETE LINEAR SYSTEMS

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A general 2-D continuous-discrete model of linear systems and its particular cases the models of Fornasini-Marchesini type and Attasi type are introduced. A general 2-D continuous-discrete model of Roesser type is also introduced. A solution and the general response formula to the regular general 2-D model are derived. The necessary and sufficient conditions for the local reachability and the local controllability of the regular general 2-D model are established. The minimum energy control of the regular general 2-D model is solved.

1. Introduction

The most popular models of two-dimensional (2-D) linear systems are the state space models introduced by Roesser (1975), Fornasini-Marchesini (1976; 1978) and Kurek (1985). The models have been extended in (Kaczorek, 1988b; 1990; Gregor, 1992) for singular (implicit) linear discrete systems. A review of singular 2-D linear discrete systems has been given in (Kaczorek, 1993c; Lewis, 1992). Continuous 2-D models of linear and non-linear systems have been considered in (Bergman *et al.*, 1989; Idczak and Walczak, 1992; Walczak, 1988). Recently in (Kaczorek, 1994a) a singular 2-D continuous-discrete model of linear systems has been introduced. In 2-D continuous-discrete systems one independent variable is continuous and the second variable is discrete. Such continuous-discrete models appear for example in the iterative learning control synthesis (Kurek and Zaremba, 1993). Another example of such systems are the repetitive processes (Rogers and Owens, 1992).

In this paper a general 2-D continuous-discrete model and a general Roesser type model will be introduced. The general response formula to the regular general 2-D model will be derived and the necessary and sufficient conditions for the local reachability and the local controllability will be established. The minimum energy control problem for the regular general model will be solved.

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2. Models and Relations Between Them

Consider a general 2-D continuous-discrete model of linear systems described by the equations

$$\begin{aligned} E\dot{x}(t,k+1) &= Ax(t,k+1) + Bx(t,k) + C\dot{x}(t,k) \\ &\quad + D_0u(t,k) + D_1\dot{u}(t,k) + D_2u(t,k+1) \\ &\quad \text{for } t \in \mathbb{R}_+, \ k \in \mathbb{Z}_+ \end{aligned} \tag{1a}$$

$$y(t,k) = Fx(t,k) + Gu(t,k) \tag{1b}$$

where $\dot{x}(t,k) := \dfrac{\partial x(t,k)}{\partial t}$, $x(t,k) \in \mathbb{R}^n$, is the semistate vector, $u(t,k) \in \mathbb{R}^m$ is the input vector, $y(t,k) \in \mathbb{R}^p$ is the output vector $E \in \mathbb{R}^{q\times n}$, $A \in \mathbb{R}^{q\times n}$, $B \in \mathbb{R}^{q\times m}$, $C \in \mathbb{R}^{q\times n}$, $D_i \in \mathbb{R}^{q\times m}$, $i = 0,1,2$, $F \in \mathbb{R}^{p\times n}$, $G \in \mathbb{R}^{p\times m}$, and $\mathbb{R}^{q\times n}$ is the set of real matrices, $\mathbb{R}_+$ and $\mathbb{Z}_+$ is the set of non-negative real numbers and integers, respectively.

If $q \neq n$ or $\det E = 0$ when $q = n$, then model (1) is called singular (or implicit). If $n = q$ and $\det E = 0$ but

$$\det[Es - A] \neq 0 \text{ for some } s \in \mathbb{C} \text{ (the field of complex numbers)} \tag{2}$$

then model (1) is called regular.
If $q = n$ and $\det E \neq 0$, then premultiplying (1a) by E^{-1} we obtain

$$\begin{aligned} \dot{x}(t,k+1) &= A'x(t,k+1) + B'x(t,k) + C'\dot{x}(t,k) \\ &\quad + D_0'u(t,k) + D_1'\dot{u}(t,k) + D_2'u(t,k+1) \end{aligned} \tag{1c}$$

where $A' := E^{-1}A$, $B' := E^{-1}B$, $C' := E^{-1}C$, $D_i' := E^{-1}D_i$, $i = 0,1,2$.
The model described by (1c) and (1b) is called standard.

Particular cases of model (1) are:

1) for $D_1 = D_2 = 0$ the first model of Fornasini-Marchesini type
2) for $B = 0$ and $D_0 = 0$ the second model of Fornasini-Marchesini type
3) for $B = -CA$ and $D_1 = D_2 = 0$ the model of Attasi type

Boundary conditions for (1a) (and its particular cases) are given by

$$x(t,0) = \widehat{x}_1(t),\ t \in \mathbb{R}_+ \quad \text{and} \quad x(0,k) = \widehat{x}_2(k),\ k \in \mathbb{Z}_+ \tag{3}$$

where $\widehat{x}_1(t)$ and $\widehat{x}_2(k)$ are known.

Next, consider a general 2-D continuous-discrete model of Roesser type described by the equations

$$\overline{E}\begin{bmatrix} \dot{\overline{x}}_1(t,k) \\ \overline{x}_2(t,k+1) \end{bmatrix} = \overline{A}\begin{bmatrix} \overline{x}_1(t,k) \\ \overline{x}_2(t,k) \end{bmatrix} + \overline{B}u(t,k) \qquad t \in \mathbb{R}_+, \ k \in \mathbb{Z}_+ \tag{4a}$$

$$y(t,k) = \overline{C}\begin{bmatrix} \overline{x}_1(t,k) \\ \overline{x}_2(t,k) \end{bmatrix} + \overline{D}u(t,k) \tag{4b}$$

where $\dot{\overline{x}}_1 := \dfrac{\partial \overline{x}_1(t,k)}{\partial t}$, $\overline{x}_1(t,k) \in \mathbb{R}^{n_1}$ and $\overline{x}_2(t,k) \in \mathbb{R}^{n_2}$ are the semistate vectors, $u(t,k)$ and $y(t,k)$ are the same as for (1), $\overline{E} \in \mathbb{R}^{q\times n}$, $n := n_1 + n_2$, $\overline{A} \in \mathbb{R}^{q\times n}$, $\overline{B} \in \mathbb{R}^{q\times m}$, $\overline{C} \in \mathbb{R}^{p\times n}$ and $\overline{D} \in \mathbb{R}^{p\times m}$.

If $q \neq n$ or $\det \overline{E} = 0$, then model (4) is called singular (implicit). If $q = n$ and $\det \overline{E} \neq 0$, then premultiplying (4a) by $\overline{E}^{-1}$ we obtain the standard model with $\overline{E} = I_n$, where I_n is the $n \times n$ identity matrix. Boundary conditions for (4a) are given by

$$\overline{x}_1(0,k) = \widehat{x}_1(k),\ k \in \mathbb{Z}_+ \quad \text{and} \quad \overline{x}_2(t,0) = \widehat{x}_2(t) \tag{5}$$

where $\widehat{x}_1(k)$ and $\widehat{x}_2(t)$ are known. If some or all entries of (1) and (4) depend on t and k, then they are called the models with variable coefficients.

Defining

$$\overline{x}_1(t,k) := Ex(t,k+1) - Cx(t,k), \quad \overline{x}_2(t,k) := x(t,k)$$

we may write (1) in the form

$$\begin{bmatrix} I & -A \\ 0 & E \end{bmatrix}\begin{bmatrix} \dot{\overline{x}}_1(t,k) \\ \overline{x}_2(t,k+1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & B \\ I & C \end{bmatrix}\begin{bmatrix} \overline{x}_1(t,k) \\ \overline{x}_2(t,k) \end{bmatrix} + \begin{bmatrix} D_0 & D_1 & D_2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u(t,k) \\ \dot{u}(t,k) \\ u(t,k+1) \end{bmatrix}$$

Therefore, model (1) is a particular case of model (4).

Let

$$\overline{E} = [E_1, E_2] = [E_1\, 0] + [0\, E_2], \quad E_1 \in \mathbb{R}^{q\times n_1}, \ E_2 \in \mathbb{R}^{q\times n_2} \tag{6a}$$

and

$$x(t,k) := \begin{bmatrix} \overline{x}_1(t,k) \\ \overline{x}_2(t,k) \end{bmatrix} \tag{6b}$$

Using (6) we may write (4a) in the form

$$[E_1\, 0]\dot{x}(t,k) + [0\, E_2]x(t,k+1) = \overline{A}x(t,k) + \overline{B}u(t,k)$$

Therefore, model (4) is a particular case of model (1). Others well-known relations between 2-D discrete models (Kaczorek, 1993c) can also be extended for the 2-D continuous-discrete models.

3. Solutions to the Models

To simplify the notation we write (1a) in the form

$$E\dot{x}(t,k+1) = Ax(t,k+1) + Bx(t,k) + C\dot{x}(t,k) + f(t,k) \tag{7}$$

where $f(t,k) = D_0u(t,k) + D_1\dot{u}(t,k) + D_2u(t,k+1)$

In this section we shall find the solution $x(t,k)$ to the regular model (7) with boundary conditions (3).

If condition (2) is satisfied, then there exist non-singular matrices $P, Q \in \mathbb{R}^{n\times n}$ such that (Gantmacher, 1959; Gregor, 1992; Kaczorek, 1994a)

$$P[Es-A]Q = \begin{bmatrix} I_{n_1} - sA_1, & 0 \\ 0, & Ns - I_{n_2} \end{bmatrix} \tag{8}$$

where n_1 is the degree of $\det[Es-A]$, $n_2 := n - n_1$, $A_1 \in \mathbb{R}^{n_1\times n_1}$, $N \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix with index $\nu (N^{\nu-1} \neq 0$ and $N^\nu = 0)$.

It is assumed that $\widehat{x}_1(t)$ and $u(t,k)$ are $(\nu+1)$ – times differentiable with respect to t. Premultiplying (7) by P, introducing the new vector

$$Q^{-1}x(t,k) := \begin{bmatrix} x_1(t,k) \\ x_2(t,k) \end{bmatrix}, \quad x_1(t,k) \in \mathbb{R}^{n_1}, \quad x_2(t,k) \in \mathbb{R}^{n_2}$$

and using (8) we obtain

$$PEQQ^{-1}\dot{x}(t,k+1) = PAQQ^{-1}x(t,k+1) + PBQQ^{-1}x(t,k) + PCQQ^{-1}\dot{x}(t,k) + Pf(t,k)$$

and

$$\begin{aligned} \dot{x}_1(t,k+1) &= A_1x_1(t,k+1) + B_{11}x_1(t,k) + B_{12}x_2(t,k) \\ &\quad + C_{11}\dot{x}_1(t,k) + C_{12}\dot{x}_2(t,k) + f_1(t,k) \end{aligned} \tag{9a}$$

$$\begin{aligned} N\dot{x}_2(t,k+1) &= x_2(t,k+1) + B_{21}x_1(t,k) + B_{22}x_2(t,k) + \\ &\quad C_{21}\dot{x}_1(t,k) + C_{22}\dot{x}_2(t,k) + f_2(t,k) \\ &\quad \text{for } t \in \mathbb{R}_+, \ k \in \mathbb{Z}_+ \end{aligned} \tag{9b}$$

where

$$PBQ = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad PCQ = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \quad Pf(t,k) = \begin{bmatrix} f_1(t,k) \\ f_2(t,k) \end{bmatrix}$$

The submatrices B_{ij}, C_{ij}, $ij = 1,2$ and the vectors $f_1(t,k)$, $f_2(t,k)$ have dimensions compatible with the dimensions of x_1 and x_2, respectively.

Knowing (3) we can find the boundary conditions for (9) in the form

$$\begin{bmatrix} x_{10}(t) \\ x_{20}(t) \end{bmatrix} = Q^{-1}\widehat{x}_1(t), \ t \in \mathbb{R}_+, \qquad \begin{bmatrix} x'_{10}(k) \\ x'_{20}(k) \end{bmatrix} = Q^{-1}\widehat{x}_2(k), \ k \in \mathbb{Z}_+ \tag{10}$$

Note that equations (9) are coupled by the matrices B_{12}, B_{21}, C_{12} and C_{21} and if at least one of the matrices is non-zero then the equations cannot be solved independently.

To find $x_1(t,k)$ and $x_2(t,k)$ the solutions to (9) with (10) let us consider the equations for $k=0$

$$\dot{x}_1(t,1) = A_1 x_1(t,1) + F_{10}(t) \tag{11a}$$

$$N\dot{x}_2(t,1) = x_2(t,1) + F_{20}(t) \tag{11b}$$

where

$$F_{10}(t) := B_{11}x_{10}(t) + B_{12}x_{20}(t) + C_{11}\dot{x}_{10}(t) + C_{12}\dot{x}_{20}(t) + f_1(t,0) \tag{12a}$$

$$F_{20}(t) := B_{21}x_{10}(t) + B_{22}x_{20}(t) + C_{21}\dot{x}_{10}(t) + C_{22}\dot{x}_{20}(t) + f_2(t,0) \tag{12b}$$

are known for given (10) and $f_1(t,k)$ and $f_2(t,k)$.

The solution $x_1(t,1)$ to (11a) has the form (Kaczorek, 1993c; Klamka, 1991)

$$x_1(t,1) = e^{A_1 t}x'_{10}(1) + \int_0^t e^{A_1(t-\tau)}F_{10}(\tau)\,\mathrm{d}\tau \tag{13}$$

Premultiplying (11b) successively by $N, N^2, \ldots, N^{\nu-1}$ and differentiating with respect to t we obtain

$$\begin{aligned} &N\dot{x}_2(t,1) - x_2(t,1) = F_{20}(t) \\ &N^2 x_2^{(2)}(t,1) - N\dot{x}_2(t,1) = N\dot{F}_{20}(t) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &N^\nu x_2^{(\nu)}(t,1) - N^{\nu-1}x_2^{(\nu-1)}(t,1) = N^{\nu-1}F_{20}^{(\nu-1)}(t) \end{aligned} \tag{14}$$

where $x_2^{(i)}(t,1)$ $\left(F_{20}^{(i)}(t)\right)$ denotes the i-th order derivative of $x_2(t,1)$ $\left(F_{20}(t)\right)$.

Adding equations (14) and taking into account that $N^\nu = 0$ we obtain the solution of (11b) in the form

$$x_2(t,1) = -\sum_{i=0}^{\nu-1} N^i F_{20}^{(i)}(t) \tag{15}$$

Substituting $k=1$ into (9) and using (13) and (15) we obtain

$$\begin{aligned} \dot{x}_1(t,2) &= A_1x_1(t,2) + B_{11}x_1(t,1) + B_{12}x_2(t,1) + C_{11}\dot{x}_1(t,1) + C_{12}\dot{x}_2(t,1) \\ &+f_1(t,1) = A_1x_1(t,2) + B_{11}\left[e^{A_1t}x'_{10}(1) + \int_0^t e^{A_1(t-\tau)}F_{10}(\tau)\,\mathrm{d}\tau\right] \\ &+B_{12}\left[-\sum_{i=0}^{\nu-1}N^iF_{20}^{(i)}(t)\right] + C_{11}\left[A_1e^{A_1t}x'_{10}(1) + F_{10}(t)\right. \\ &\left.+A_1\int_0^t e^{A_1(t-\tau)}F_{10}(\tau)\,\mathrm{d}\tau\right] + C_{12}\left[-\sum_{i=0}^{\nu-1}N^iF_{20}^{(i+1)}(t)\right] + f_1(t,1) \\ &= A_1x_1(t,2) + F_{11}(t) \end{aligned} \tag{16a}$$

and

$$N\dot{x}_2(t,2) = x_2(t,2) + B_{21}x_1(t,1) + B_{22}x_2(t,1) + C_{21}\dot{x}_1(t,1) + C_{22}\dot{x}_2(t,1)$$
$$+f_2(t,1) = x_2(t,2) + B_{21}\Big[e^{A_1t}x'_{10}(1) + \int_0^t e^{A_1(t-\tau)}F_{10}(\tau)\,\mathrm{d}\tau\Big]$$
$$+B_{22}\Big[-\sum_{i=0}^{\nu-1}N^iF_{20}^{(i)}(t)\Big] + C_{21}\Big[A_1e^{A_1t}x'_{10}(1) + F_{10}(t) \tag{16b}$$
$$+A_1\int_0^t e^{A_1(t-\tau)}F_{10}(\tau)\,\mathrm{d}\tau\Big] + C_{22}\Big[-\sum_{i=0}^{\nu-1}N^iF_{20}^{(i+1)}(t)\Big] + f_2(t,1)$$
$$= x_2(t,2) + F_{21}(t)$$

where

$$F_{11}(t) := \overline{A}_1\Big[e^{A_1t}x'_{10}(1) + \int_0^t e^{A_1(t-\tau)}F_{10}(\tau)\,\mathrm{d}\tau\Big] + C_{11}F_{10}(t)$$
$$-\sum_{i=0}^{\nu-1}\Big[B_{12}N^iF_{20}^{(i)}(t) + C_{12}N^iF_{20}^{(i+1)}(t)\Big] + f_1(t,1)$$

$$F_{21}(t) := \overline{A}_2\Big[e^{A_1t}x'_{10}(1) + \int_0^t e^{A_1(t-\tau)}F_{10}(\tau)\,\mathrm{d}\tau\Big] + C_{21}F_{10}(t)$$
$$-\sum_{i=0}^{\nu-1}\Big[B_{22}N^iF_{20}^{(i)}(t) + C_{22}N^iF_{20}^{(i+1)}(t)\Big] + f_2(t,1)$$

$$\overline{A}_1 := B_{11} + C_{11}A_1, \qquad \overline{A}_2 := B_{21} + C_{21}A_1$$

The solutions of (16) have the form

$$x_1(t,2) = e^{A_1t}x'_{10}(2) + \int_0^t e^{A_1(t-\tau)}F_{11}(\tau)\,\mathrm{d}\tau \tag{17a}$$

$$x_2(t,2) = -\sum_{i=0}^{\nu-1}N^iF_{21}^{(i)}(t) \tag{17b}$$

Similarly, substituting $k = 2$ into (9) and using (17) we obtain

$$\dot{x}_1(t,3) = A_1x_1(t,3) + B_{11}x_1(t,2) + B_{12}x_2(t,2) + C_{11}\dot{x}_1(t,2) + C_{12}\dot{x}_2(t,1)$$
$$+f_1(t,2) = A_1x_1(t,3) + B_{11}\Big[e^{A_1t}x'_{10}(2) + \int_0^t e^{A_1(t-\tau)}F_{11}(\tau)\,\mathrm{d}\tau\Big]$$
$$+B_{12}\Big[-\sum_{i=0}^{\nu-1}N^iF_{21}^{(i)}(t)\Big] + C_{11}\Big[A_1e^{A_1t}x'_{10}(2) + F_{11}(t) \tag{18a}$$
$$+A_1\int_0^t e^{A_1(t-\tau)}F_{11}(\tau)\,\mathrm{d}\tau\Big] + C_{12}\Big[-\sum_{i=0}^{\nu-1}N^iF_{21}^{(i+1)}(t)\Big] + f_1(t,2)$$
$$= A_1x_1(t,3) + F_{12}(t)$$

and

$$N\dot{x}_2(t,3) = x_2(t,3) + B_{21}x_1(t,2) + B_{22}x_2(t,2) + C_{21}\dot{x}_1(t,2) + C_{22}\dot{x}_2(t,2)$$
$$+f_2(t,2) = x_2(t,3) + B_{21}\left[e^{A_1t}x'_{10}(2) + \int_0^t e^{A_1(t-\tau)}F_{11}(\tau)\,\mathrm{d}\tau\right]$$
$$+B_{22}\left[-\sum_{i=0}^{\nu-1}N^iF_{21}^{(i)}(t)\right] + C_{21}\left[A_1e^{A_1t}x'_{10}(2) + F_{11}(t)\right. \tag{18b}$$
$$\left.+A_1\int_0^t e^{A_1(t-\tau)}F_{11}(\tau)\,\mathrm{d}\tau\right] + C_{22}\left[-\sum_{i=0}^{\nu-1}N^iF_{21}^{(i+1)}(t)\right] + f_2(t,2)$$
$$= x_2(t,3) + F_{22}(t)$$

where

$$F_{12}(t) := \overline{A}_1\left[e^{A_1t}x'_{10}(2) + \int_0^t e^{A_1(t-\tau)}F_{11}(\tau)\,\mathrm{d}\tau\right] + C_{11}F_{11}(t)$$
$$-\sum_{i=0}^{\nu-1}\left[B_{12}N^iF_{21}^{(i)}(t) + C_{12}N^iF_{21}^{(i+1)}(t)\right] + f_1(t,2)$$

$$F_{22}(t) := \overline{A}_2\left[e^{A_1t}x'_{10}(2) + \int_0^t e^{A_1(t-\tau)}F_{11}(\tau)\,\mathrm{d}\tau\right] + C_{21}F_{11}(t)$$
$$-\sum_{i=0}^{\nu-1}\left[B_{22}N^iF_{21}^{(i)}(t) + C_{22}N^iF_{21}^{(i+1)}(t)\right] + f_2(t,2)$$

Continuning this procedure after k steps we obtain

$$x_1(t,k) = e^{A_1t}x'_{10}(k) + \int_0^t e^{A_1(t-\tau)}F_{1,k-1}(\tau)\,\mathrm{d}\tau \tag{19a}$$
$$t \in \mathbb{R}_+,\quad k \in \mathbb{Z}_+$$

$$x_2(t,k) = -\sum_{i=0}^{\nu-1}N^iF_{2,k-1}^{(i)}(t) \tag{19b}$$

where $F_{1,k-1}(t)$ and $F_{2,k-1}(t)$ satisfy the equations

$$F_{1,k}(t) := \overline{A}_1\left[e^{A_1t}x'_{10}(k) + \int_0^t e^{A_1(t-\tau)}F_{1,k-1}(\tau)\,\mathrm{d}\tau\right] + C_{11}F_{1,k-1}(t)$$
$$-\sum_{i=0}^{\nu-1}\left[B_{12}N^iF_{2,k-1}^{(i)}(t) + C_{12}N^iF_{2,k-1}^{(i+1)}(t)\right] + f_1(t,k) \tag{20a}$$

$$F_{2,k}(t) := \overline{A}_2\left[e^{A_1t}x'_{10}(k) + \int_0^t e^{A_1(t-\tau)}F_{1,k-1}(\tau)\,\mathrm{d}\tau\right] + C_{21}F_{1,k-1}(t)$$
$$-\sum_{i=0}^{\nu-1}\left[B_{22}N^iF_{2,k-1}^{(i)}(t) + C_{22}N^iF_{2,k-1}^{(i+1)}(t)\right] + f_2(t,k) \tag{20b}$$

Let $P_t := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ be a linear operator (map) defined as follows

$$\begin{aligned} P_{11}F_1(t) &:= \overline{A}_1 \int_0^t e^{A_1(t-\tau)} F_1(\tau)\, \mathrm{d}\tau + C_{11}F_1(t) \\ P_{12}F_2(t) &:= -\sum_{i=0}^{\nu-1} \left[B_{12}N^i F_2^{(i)}(t) + C_{12}N^i F_2^{(i+1)}(t) \right] \\ P_{21}F_1(t) &:= \overline{A}_2 \int_0^t e^{A_1(t-\tau)} F_1(\tau)\, \mathrm{d}\tau + C_{21}F_1(t) \\ P_{22}F_2(t) &:= -\sum_{i=0}^{\nu-1} \left[B_{22}N^i F_2^{(i)}(t) + C_{22}N^i F_2^{(i+1)}(t) \right] \end{aligned} \tag{21}$$

and $P^k := P \circ P \circ \ldots \circ P$ is the k multiple composition of P ($P^0 := I$ the identity).

Using (21) and

$$h_1(t,k) := \overline{A}_1 e^{A_1 t} x'_{10}(k) + f_1(t,k)$$
$$h_2(t,k) := \overline{A}_2 e^{A_1 t} x'_{10}(k) + f_2(t,k)$$

we may write equations (20) as

$$\begin{bmatrix} F_{1,k}(t) \\ F_{2,k}(t) \end{bmatrix} = P_t \begin{bmatrix} F_{1,k-1}(t) \\ F_{2,k-1}(t) \end{bmatrix} + \begin{bmatrix} h_1(t,k) \\ h_2(t,k) \end{bmatrix}, \qquad t \in \mathbb{R}_+, \quad k \in \mathbb{Z}_+ \tag{22}$$

It is easy to show that the solution to (22) has the form

$$\begin{bmatrix} F_{1,k}(t) \\ F_{2,k}(t) \end{bmatrix} = P_t^k \begin{bmatrix} F_{10}(t) \\ F_{20}(t) \end{bmatrix} + \sum_{i=0}^{k-1} P_t^{k-i-1} \begin{bmatrix} h_1(t,i+1) \\ h_2(t,i+1) \end{bmatrix} \tag{23}$$

where $F_{10}(t)$, $F_{20}(t)$ are defined by (12).
Therefore, the following theorem has been proved

Theorem 1. *The solution $x(t,k)$ to equation (7) with (3) has the form*

$$x(t,k) = Q \begin{bmatrix} e^{A_1 t} x'_{10}(k) + \int_0^t e^{A_1(t-\tau)} F_{1,k-1}(\tau)\, \mathrm{d}\tau \\ -\sum_{i=0}^{\nu-1} N^i F_{2,k-1}^{(i)}(t) \end{bmatrix}, \qquad t \in \mathbb{R}_+, \quad k \in \mathbb{Z}_+ \tag{24}$$

where $F_{1,k}(t)$, $F_{2,k}(t)$ are given by (23).

Substitution of (24) into (1b) yields the general response formula

$$y(t,k) = CQ \begin{bmatrix} e^{A_1 t} x'_{10}(k) + \int_0^t e^{A_1(t-\tau)} F_{1,k-1}(\tau)\, \mathrm{d}\tau \\ -\sum_{i=0}^{\nu-1} N^i F_{2,k-1}^{(i)}(t) \end{bmatrix} + Du(t,k), \qquad t \in \mathbb{R}_+, \quad k \in \mathbb{Z}_+ \tag{25}$$

Using (25) we may find $y(t,k)$ for given $u(t,k)$ and boundary conditions (3). In particular case for $E = I_n (N = 0,\ x_2 = 0)$ we have.

Theorem 2. *The solution $x(t,k)$ to the equation*

$$\dot{x}(t,k+1) = Ax(t,k+1) + Bx(t,k) + C\dot{x}(t,k) + f(t,k) \tag{26}$$

with boundary conditions (3) has the form

$$x(t,k) = e^{At}\widehat{x}_2(k) + \int_0^t e^{A(t-\tau)} F_{k-1}(\tau)\,\mathrm{d}\tau, \quad t \in \mathbb{R}_+, \quad k \in \mathbb{Z}_+ \tag{27}$$

where $F_{k-1}(t)$ is given by

$$F_{k-1}(t) = P_t^{k-1}\Big[B\widehat{x}_1(t) + C\dot{\widehat{x}}_1(t) + f(t,0)\Big] + \sum_{i=0}^{k-2} P_t^{k-i-2}\Big[(B+CA)e^{At}\widehat{x}_2(i+1) + f(t,i+1)\Big] \tag{28}$$

and P_t is an operator defined as follows

$$P_t F(t) := (B+CA)\int_0^t e^{A(t-\tau)} F(\tau)\,\mathrm{d}\tau + CF(t) \tag{29}$$

4. Local Reachability and Local Controllability

The local controllability of 2-D discrete and continuous linear and non-linear systems has been considered in many papers (Bergman, *et al.*, 1989; Fornasini and Marchesini, 1976; 1978; Idczak and Walczak, 1992; Kaczorek, 1988a; 1990; Kaczorek and Klamka, 1986; 1987; Klamka, 1983; Roesser, 1975) and books (Kaczorek, 1993c; Klamka, 1991). In this section the necessary and sufficient conditions for the local reachability and local controllability of 2-D continuous-discrete regular systems will be established.

Consider the regular system described by the equation

$$E\dot{x}(t,k+1) = Ax(t,k+1) + Bx(t,k) + C\dot{x}(t,k) + Du(t,k) \tag{30}$$

which is a particular case of (1a) for $D = D_0$, $D_1 = D_2 = 0$ and of (7) for $f(t,k) = Du(t,k)$.

Definition 1. System (30) is called locally reachable in the rectangle

$$[h,r] := \Big\{(t,k) \in \mathbb{R}_+ \times \mathbb{Z}_+ : 0 \le t \le h,\ 0 < k \le r\Big\} \tag{31}$$

if for any boundary conditions (3) with $\widehat{x}_1(t)$, $0 \le t \le h$, $\widehat{x}_2(k)$, $1 \le k \le r$ and every vector $x_f \in \mathbb{R}^n$ there exists $u(t,k)$ for $0 \le t \le h$, $0 \le k \le r-1$ such that $x(h,r) = x_f$.

Using

$$P_t^i := \begin{bmatrix} P_{1,t}^i \\ P_{2,t}^i \end{bmatrix}, \quad i \in \mathbb{Z}_+; \qquad F_0(t) := \begin{bmatrix} F_{10}(t) \\ F_{20}(t) \end{bmatrix}$$

$$h(t,k) := \begin{bmatrix} h_1(t,k) \\ h_2(t,k) \end{bmatrix}, \quad t \in \mathbb{R}_+, \ k \in \mathbb{Z}_+$$

we may write (23) as

$$\begin{aligned} F_{1,k}(t) &= P_{1,t}^k F_0(t) + \sum_{i=0}^{k-1} P_{1,t}^{k-i-1} h(t, i+1) \\ F_{2,k}(t) &= P_{2,t}^k F_0(t) + \sum_{i=0}^{k-1} P_{2,t}^{k-i-1} h(t, i+1) \end{aligned} \tag{32}$$

From (24) we have

$$x(t,k) = Q_1 \left[e^{A_1 t} x'_{10}(k) + \int_0^t e^{A_1(t-\tau)} F_{1,k-1}(\tau)\, \mathrm{d}\tau \right] - \sum_{i=0}^{\nu-1} Q_2 N^i F_{2,k-1}^{(i)}(t) \tag{33}$$

where $Q = [Q_1, Q_2]$.
Substitution of (32) into (33) yields

$$\begin{aligned} x(t,k) &= Q_1 e^{A_1 t} x'_{10}(k) + \int_0^t Q_1 e^{A_1(t-\tau)} P_{1,\tau}^{k-1} F_0(\tau)\, \mathrm{d}\tau \\ &+ \sum_{i=0}^{k-2} \int_0^t Q_1 e^{A_1(t-\tau)} P_{1,\tau}^{k-i-2} h(\tau, i+1)\, \mathrm{d}\tau - \sum_{i=0}^{\nu-1} Q_2 N^i \left[P_{2,t}^{k-1} F_0(t) \right]^{(i)} \\ &- \sum_{i=0}^{\nu-1} \sum_{j=0}^{k-2} Q_2 N^i \left[P_{2,t}^{k-j-2} h(t, j+1) \right]^{(i)} \end{aligned} \tag{34}$$

From (12) for $f(t,k) = Du(t,k)$ we obtain

$$F_0(t) = \widehat{B} \widehat{x}_1(t) + \widehat{C} \dot{\widehat{x}}_1(t) + \widehat{D} u(t,k)$$

and

$$h(t,k) = H e^{A_1 t} x'_{10}(k) + \widehat{D} u(t,k) \tag{35}$$

where $\widehat{B} := PB$, $\widehat{C} := PC$, $\widehat{D} := PD$, $H = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} + \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} A_1$.

Let assume that

$$u(t,k) := u_k \quad \text{for} \quad 0 \le t \le h, \ \ 0 \le k \le r$$

where u_k is independent of t. In this case formula (34) for (30) takes the form

$$\begin{aligned} x(t,k) &= x_{bc}(t,k) + \int_0^t Q_1 e^{A_1(t-\tau)} P_{1,\tau}^{k-1} \widehat{D}\, \mathrm{d}\tau\, u_0 \\ &+ \sum_{i=0}^{k-2} \int_0^t Q_1 e^{A_1(t-\tau)} P_{1,\tau}^{k-i-2} \widehat{D}\, \mathrm{d}\tau\, u_{i+1} - \sum_{i=0}^{\nu-1} Q_2 N^i \left[P_{2,t}^{k-1} \widehat{D} \right]^{(i)} u_0 \\ &- \sum_{i=0}^{\nu-1} \sum_{j=0}^{k-2} Q_2 N^i \left[P_{2,t}^{k-j-2} \widehat{D} \right]^{(i)} u_{j+1} \end{aligned} \tag{36}$$

where

$$\begin{aligned} x_{bc}(t,k) := & Q_1 e^{A_1 t} x'_{10}(k) + \int_0^t Q_1 e^{A_1(t-\tau)} P_{1,\tau} \left(\widehat{B} \widehat{x}_1(\tau) + \widehat{C} \dot{\widehat{x}}_1(\tau) \right) \mathrm{d}\tau \\ & + \sum_{i=1}^{k-2} \int_0^t Q_1 e^{A_1(t-\tau)} P_{1,\tau}^{k-i-2} H e^{A_1 \tau} x'_{10}(i+1) \, \mathrm{d}\tau \\ & - \sum_{i=0}^{\nu-1} Q_2 N^i \left[P_{2,t}^{k-1} \left(\widehat{B} x_1(t) + \widehat{C} \dot{\widehat{x}}_1(t) \right) \right]^{(i)} \\ & - \sum_{i=0}^{\nu-1} \sum_{j=0}^{k-2} Q_2 N^i \left[P_{2,t}^{k-j-2} H e^{A_1 t} \right]^{(i)} x'_{10}(j+1) \end{aligned} \tag{37}$$

Theorem 3. *System (30) is locally reachable in rectangle (31) if and only if*

$$rank[R_0, R_1, \ldots, R_{r-1}] = n \tag{38}$$

or

$$rank[V_0, V_1, \ldots, V_{r-1}] = n \tag{39}$$

where

$$R_{r-i-1} = R(h) := \int_0^h Q_1 e^{A_1(h-\tau)} P_{1,\tau}^i \widehat{D} \, \mathrm{d}\tau - \sum_{j=0}^{\nu-1} Q_2 N^j \left[P_{2,h}^i \widehat{D} \right]^{(j)} \qquad i = 0, 1, \ldots, r-1 \tag{40}$$

$V_i = V_i(h) := R_i R_i^T$ *and* T *denotes the transposition.*

Proof. Using (36) for $t = h$, $k = r$, (37) and $x(h,r) = x_f$ we obtain

$$x_f - x_{bc}(h,r) = \begin{bmatrix} R_0, & R_1, & \ldots, & R_{r-1} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{r-1} \end{bmatrix} \tag{41}$$

From (41) it follows that for any boundary conditions (3) and every vector x_f there exists a sequence $u_0, u_1, \ldots, u_{r-1}$ if and only if (38) holds. The equivalence of conditions (38) and (39) can be shown in a similar way as for 1-D case (Klamka, 1991). ■

In particular case for (26) with $f(t,k) = Du(t,k)$ we obtain.

Theorem 4. *System (26) with* $f(t,k) = Du(t,k)$ *is locally reachable in rectangle (31) if and only if*

$$rank[R_0, R_1, \ldots, R_{r-1}] = n \tag{42}$$

or

$$rank[V_0, V_1, \dots, V_{r-1}] = n \tag{43}$$

where

$$R_i := \int_0^h Q_1 e^{A(h-\tau)} P_\tau^i D \, d\tau \tag{44}$$

$$i = 0, 1, \dots, r-1$$

$$V_i = R_i R_i^T$$

and the operator P_t is defined by (29).

Example. Consider system (26) with

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$f(t,k) = Du(t,k) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t,k)$$

To check the reachability of the system in rectangle (31) with $h = 1$, $r = 2$ using (44) we calculate

$$R_0 = \int_0^h e^{A(h-\tau)} P_\tau^0 D \, d\tau = \int_0^1 e^{A\tau} \, d\tau \, D = A^{-1}[e^A - I]B = \begin{bmatrix} e-1 \\ 0 \end{bmatrix}$$

$$R_1 = \int_0^h e^{A(h-\tau)} P_\tau D \, d\tau = \int_0^h e^{A_2(h-\tau)} \left[(B + CA) \int_0^\tau e^{A(h-\tau_1)} \, d\tau_1 + C \right] d\tau \, D$$

$$= \begin{bmatrix} e \\ \frac{1}{2}(e^2+1) - e \end{bmatrix}$$

From (42) for $r = 2$ we have

$$rank[R_0, R_1] = rank \begin{bmatrix} e-1, & e \\ 0, & \frac{1}{2}(e^2+1) - e \end{bmatrix} = 2$$

Therefore, by Theorem 4 the system is locally reachable in the rectangle (Bergman *et al.*, 1989; Fornasini and Marchesini, 1976).

To find a sequence $\{u_0, u_1\}$ which transfers the system from the boundary conditions

$$\widehat{x}_1(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}_+, \qquad \widehat{x}_2(k) = \begin{bmatrix} k \\ 1 \end{bmatrix}, \quad k \in \mathbb{Z}_+$$

to the desired final state $x_f(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ at the point (1,2) we use the formula

$$x_f - e^{Ah}\widehat{x}_2(r) - \int_0^h e^{A(h-\tau)}\Big[P_\tau\Big(B\widehat{x}_1(\tau) + C\dot{\widehat{x}}_1(\tau)\Big) + (B+CA)e^{A\tau}\widehat{x}_2(1)\Big]\,d\tau$$

$$= [R_0, R_1]\begin{bmatrix} u_1 \\ u_0 \end{bmatrix}$$

Taking into account that

$$x_f - e^{A}\widehat{x}_2(2) = \begin{bmatrix} 1-2e \\ 1-e^2 \end{bmatrix}, \qquad (B+CA)e^{A\tau}\widehat{x}_2(1) = \begin{bmatrix} e^\tau + e^{2\tau} \\ e^\tau + 2e^{2\tau} \end{bmatrix}$$

$$P_\tau\Big(B\widehat{x}_1(\tau) + C\dot{\widehat{x}}_1(\tau)\Big) = \begin{bmatrix} e^\tau \\ e^\tau - 1 \end{bmatrix}$$

$$\int_0^1 e^{A(1-\tau)}\Big[P_\tau\Big(B\widehat{x}_1(\tau) + C\dot{\widehat{x}}_1(\tau)\Big) + (B+CA)e^{A\tau}\widehat{x}_2(1)\Big]\,d\tau$$

$$= \begin{bmatrix} e^2 + e \\ \frac{7}{2}e^2 - 2e + \frac{1}{2} \end{bmatrix}$$

we obtain the equation

$$\begin{bmatrix} e-1, & e \\ 0, & \frac{1}{2}(e^2+1) - e \end{bmatrix}\begin{bmatrix} u_1 \\ u_0 \end{bmatrix} = \begin{bmatrix} -e^2 - 3e + 1 \\ -\frac{9}{2}e^2 + 2e + \frac{1}{2} \end{bmatrix}$$

and

$$\begin{bmatrix} u_1 \\ u_0 \end{bmatrix} = \begin{bmatrix} \dfrac{(e+1)(1-7e+9e^2-e^3)}{(e-1)^3} \\[2ex] \dfrac{1+4e-9e^2}{(e-1)^2} \end{bmatrix}$$

Definition 2. System (30) is called locally controllable in rectangle (31) if for any boundary conditions (3) with $\widehat{x}_1(t),\ 0 \le t \le h,\ \widehat{x}_2(k),\ 1 \le k \le r$ there exists $u(t,k)$ for $0 \le t \le h,\ 0 \le k \le r-1$ such that $x(h,r) = 0$.

Theorem 5. *System (30) is locally controllable in rectangle (31) if and only if condition (38) or (39) is satisfied.*

Proof. Using (36) for $t = h,\ k = r$, (37) and $x(h,r) = 0$ we obtain (41) for $x_f = 0$. From (37) for $t = h,\ k = r$ it follows that for any boundary conditions $Q_1 e^{A_1 h} x'_{10}(r)$

is any n-dimensional vector. Therefore, there exists a sequence $u_0, u_1, \ldots, u_{r-1}$ satisfying (41) for $x_f = 0$ if and only if (38) holds. ∎

From theorem 4 and 5 we have the following important

Corollary. *For the 2-D continuous-discrete system (30) the local controllability is equivalent to its local reachability.*

5. Minimum Energy Control

The minimum energy control problem for 2-D discrete linear systems has been considered in many papers (Kaczorek, 1988a; 1990; Kaczorek and Klamka, 1986; 1987; Klamka, 1983; 1991; 1993). In this section the problem will be extended for regular 2-D continuous-discrete linear systems.

Consider system (30) and the performance index

$$I(u) := \sum_{i=0}^{r-1} u_i^T Q u_i \tag{45}$$

where Q is the $m \times m$ symmetric and positive definite weighting matrix.

The minimum energy control problem for system (30) can be stated as follows. Given the matrices E, A, B, C, D of (30), the weighting matrix Q, the numbers h, r and the boundary conditions $\widehat{x}_1(t)$, $0 \le t \le h$, $\widehat{x}_2(k)$, $1 \le k \le r$, find a sequence $u_0, u_1, \ldots, u_{r-1}$ which transfers the system to the desired final state $x_f, x(h,r) = x_f$, and minimizes the performance index (45).

To solve the problem we define the matrix

$$W_Q := \sum_{i=0}^{r-1} R_i Q^{-1} R_i^T \tag{46}$$

where R_i is defined by (40). It is easy to show that the matrix (46) is non-singular if and only if system (30) is reachable in (31). Thus, we may define the input vector

$$\widehat{u}_k := Q^{-1} R_k^T W_Q^{-1} (x_f - x_{bc}), \qquad k = 0, 1, \ldots, r-1 \tag{47}$$

where x_{bc} is given by (37) for $t = h$, $k = r$.

Theorem 6. *Let us assume that*

i) *system (30) is reachable in rectangle (31),*

ii) $\overline{u}_k$ *is any input defined for* $k \in [0, r-1]$ *which transfer system (30) to* x_f.

Then input (47) accomplishes the same task and

$$I(\widehat{u}) \le I(\overline{u}) \tag{48}$$

Moreover, the minimum value of (45) is given by

$$I(\widehat{u}) = (x_f - x_{bc})^T W_Q^{-1} (x_f - x_{bc}) \tag{49}$$

Proof. First we shall show that input (47) provides $x(h,r) = x_f$. Using (36) for $t = h$, $k = r$, (40), (47) and (46) we obtain

$$x(h,r) = x_{bc} + \sum_{i=0}^{r-1} R_i \widehat{u}_i = x_{bc} + \sum_{i=0}^{r-1} R_i Q^{-1} R_i^T W_Q^{-1}(x_f - x_{bc}) = x_f$$

Since $\overline{u}_i$ and $\widehat{u}_i$ transfer system (30) to the same x_f then

$$\sum_{i=0}^{r-1} R_i \overline{u}_i = \sum_{i=0}^{r-1} R_i \widehat{u}_i$$

and

$$\sum_{i=0}^{r-1} R_i[\overline{u}_i - \widehat{u}_i] = 0 \tag{50}$$

From (50) and (47) it follows that

$$\sum_{i=0}^{r-1} [\overline{u}_i - \widehat{u}_i]^T R_i^T W_Q^{-1}(x_f - x_{bx}) = \sum_{i=0}^{r-1} [\overline{u}_i - \widehat{u}_i]^T Q \widehat{u}_i = 0 \tag{51}$$

Using (51) it is easy to show that

$$\sum_{i=0}^{r-1} \overline{u}_i^T Q \overline{u}_i = \sum_{i=0}^{r-1} \widehat{u}_i^T Q \widehat{u}_i + \sum_{i=0}^{r-1} [\overline{u}_i - \widehat{u}_i]^T Q [\overline{u}_i - \widehat{u}_i] \tag{52}$$

Inequality (48) holds since the last term in (52) is always non-negative.

To obtain the minimum value of (45) we substitute (47) into (45).

$$I(\widehat{u}) = \sum_{i=0}^{r-1} \widehat{u}_i^T Q \widehat{u}_i = \sum_{i=0}^{r-1} \Big[Q^{-1} R_i^T W_Q^{-1}(x_f - x_{bc})\Big]^T Q \Big[Q^{-1} R_i^T W_Q^{-1}(x_f - x_{bc}\Big]$$

$$= \sum_{i=0}^{r-1} (x_f - x_{bc})^T W_Q^{-1} R_i Q^{-1} R_i^T W_Q^{-1}(x_f - x_{bc}) = (x_f - x_{bc})^T W_Q^{-1}(x_f - x_{bc})$$

■

In particular case for $E = I_n$ we obtain the results presented in (Kaczorek, 1994b).

6. Concluding Remarks

The general 2-D continuous-discrete model (1) for linear systems and the general 2-D Roesser type model (4) for linear systems have been introduced. The solution (24) to the regular model (7) with boundary conditions (3) and the general response formula (25) have been derived. The necessary and sufficient conditions for the local reachability (Theorem 3) and the local controllability (Theorem 4) of the regular 2-D model (30) have been established. It has been shown that for the regular 2-D continuous-discrete linear systems the local controllability is equivalent to their local reachability. The minimum energy control problem for the regular 2-D continuous-discrete linear system (30) has been solved (theorem 6). An extension of Theorems 1, 3, 4, 5 and 6 for the regular model (1) is straightforward. An extension of the above considerations for singular model (1), which does not satisfy condition (2), is not easy and it will be considered in subsequent paper.

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