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ANALYSIS AND APPLICATION OF OUTPUT ROBUSTNESS FOR NON-LINEAR SYSTEMS

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This paper studies the problem of robustness for non-linear control systems using the differential geometric method. First of all, the paper proposes the concepts of output robustness and feedback output robustness, and gives the sufficient and necessary conditions for them. Furthermore, the paper studies the application of them to the robust disturbance decoupling, robust input-output decoupling and robust asymptotic tracking, such that the main robustness problems in non-linear control systems can be dealt with by a unified method.

1. Introduction

With the development of differential geometric methods, there are many achievements made in the field of non-linear control system theory, such as controllability, linearization, stabilization and decoupling etc. (Isidori, 1989). Many of them have been successfully applied in aircraft or robot technology.

However, for well known reasons, they must have uncertain terms in modelling concrete control systems. The existence of those will affect seriously the control performance, so the study of robustness is of a great importance not only in theory but also in practice.

There have been many papers studying the robustness for non-linear control systems (Barmish *et al.*, 1983; Chen and Leitmann, 1987; Liu Jing-Sin and Yuan King, 1991; Misawa, 1992; Teh-Lu *et al.*, 1991). Misawa (1992) gives a better review. From this paper and the references therein, we can see that there exist various study methods, but no unified approach has been arrived at yet.

This paper tries to give a more unified method of study of the robustness for non-linear control systems. It first proposes the concepts of output robustness and feedback output robustness, and gives the sufficient and necessary conditions of them. Secondly, it deals with the application of them to the problems of robust disturbance decoupling, robust input-output (IO for simplicity) decoupling and robust asymptotic tracking, respectively.

The meaning of the notations through this paper is as follows: M, N are n-dimensional, l-dimensional smooth manifolds, respectively, T_M is tangent bundle on M, $C^\infty(M,N)$ means all smooth maps from M to N, span$\{\ \}$ expresses the distribution generated by all smooth vectors in $\{\ \}$, $\mathrm{d}h$ is the reduced generator of h, $\ker \mathrm{d}h$ represents the kernel of $\mathrm{d}h$, R^m the m-dimensional Euclidean space. If

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f_1, f_2 belong to the distribution $\Delta, [f_1, f_2]$ represents Lie bracket, then the $[f, \Delta]$ means $\{[f, X] : X \in \Delta\}$.

The paper is organized as follows: Section 1 is introduction, Section 2 is preliminary, Section 3 deals with the output robustness and feedback output robustness, Section 4 studies the application to robust disturbance decoupling, robust IO decoupling and robust asymptotic tracking, respectively, Section 5 is conclusion.

2. Preliminary

Consider the following affine non-linear system

$$\dot{x} = f(x) + G(x)u \tag{1a}$$

$$y = h(x) \tag{1b}$$

where $x \in M$, $f(x) \in T_M$, $G(x) = \big(g_1(x), g_2(x), \cdots, g_m(x)\big)$, $g_i(x) \in T_M$, $i = 1, 2, ..., m$, $h = (h_1, h_2, \cdots, h_l) \in C^\infty(M, N)$, $u = (u_1, u_2, \cdots, u_m)^T$ is control vector.

Definition 1. A distribution $\Delta \subset T_M$ is said to be (f, g)-invariant on some open set $U \subset M$, if

$$[f, \Delta] \subset \Delta + G \tag{2a}$$

$$[g_i, \Delta] \subset \Delta + G \tag{2b}$$

where $G = \text{span}\{g_1(x), g_2(x), \cdots, g_m(x)\}$. If for each $x \in U$, there exists a neighborhood U_0 of x with the property that Δ is (f, g)-invariant on U_0, Δ is said to be locally $[f, g]$-invariant.

Definition 2. A distribution Δ is said to be controlled invariant on U if there exists a feedback pair (α, β) defined on U such that Δ satisfied

$$[\overline{f}, \Delta] \subset \Delta \tag{3a}$$

$$[\overline{g}_i, \Delta] \subset \Delta \tag{3b}$$

where $\overline{f} = f + G\alpha$, $\overline{g}_i = (G\beta)_i$, $(G\beta)_i$ is the i-th column of the matrix $G\beta$. If for each $x \in U$ there exists a neighborhood U_0 of x with the properties (3a)–(3b) on U_0, Δ is said to be locally controlled invariant.

The relation between Definition 1 and Definition 2 is as follows:

Lemma. (Isidori, 1989) *Let Δ be an involutive distribution. Suppose that Δ, G, and $\Delta + G$ are non-singular on U, then Δ is controlled invariant on U if and only if Δ is (f, g)-invariant on U.*

3. The Analysis of Output Robustness

This section deals with the problem of output robustness. We first give the related definition.

Consider the following system with uncertain terms

$$\dot{x} = f(x) + \Delta f(x) + \Big(G(x) + \Delta G(x)\Big)u \tag{4a}$$

$$y = h(x) \tag{4b}$$

where f, G, h, u are the same as in system (1), $\Delta f \in T_M$, $\Delta G = (\Delta g_1, \Delta g_2, \cdots, \Delta g_m)$, $\Delta g_i \in T_M$, $i = 1, 2, \ldots, m$.

System (1) is said to be the nominal system of system (4).

Definition 3. Suppose the solution of system (4a) is $x(t, x_0, u)$ for initial value $x_0 \in U$. If for any different vectors $\Delta f_1(x), \Delta f_2(x)$ and different matrices $\Delta G_1(x)$, $\Delta G_2(x)$, the corresponding solutions $x_1(t)$, $x_2(t)$ satisfy

$$h\Big(x_1(t)\Big) = h\Big(x_2(t)\Big), \tag{5}$$

then system (4) is said to be output robust on U. In addition, if system (4) is output robust on some neighborhood of $x_0 \in M$, it is said to be locally output robust.

Remark 1. The meaning of Definition 3 is that although system (4) has uncertain terms, its variance only affects the internal structure of those but not the output, this is to say that system (4) is robust with respect to output.

Theorem 1. *System (4) is output robust on some neighborhood U of $x \in M$ if and only if there exists a non-singular involutive distribution Δ on U, such that*

$$[f, \Delta] \subset \Delta, \qquad [g_i, \Delta] \subset \Delta, \quad i = 1, 2, \cdots, m \tag{6a}$$

$$\Delta f + \mathrm{span}\{\Delta g_1, \Delta g_2, \cdots, \Delta g_m\} \subset \Delta \tag{6b}$$

$$\Delta \subset \ker dh \tag{6c}$$

Proof. Since Δ is non-singular involutive, from Frobenius theorem, there must exist a diffeomorphic coordinate transformation, $z = \psi(x)$ such that

$$\Delta = \mathrm{span}\left\{\frac{\partial}{\partial z_1^2}, \frac{\partial}{\partial z_2^2}, \cdots, \frac{\partial}{\partial z_k^2}\right\},$$

where $z = \Big((z_1)^T, (z_2)^T\Big)^T = \Big(z_1^1, z_2^1, \cdots, z_{n-k}^1, z_1^2, \cdots, z_k^2\Big)^T$, $k = \dim \Delta$ on U.

Substitute $z = \psi(x)$ into system (4), without loss of generality, we still denote the resulting system as follows

$$\dot{z} = f(z) + \Delta f(z) + \Big(G(z) + \Delta G(z)\Big)u \tag{7a}$$

$$y = h(z) \tag{7b}$$

From (6a), we have

$$\frac{\partial}{\partial z_i^2} f \in \Delta, \qquad \frac{\partial}{\partial z_i^2} g_j \in \Delta, \quad i = 1, 2, \cdots, k; \ j = 1, 2, \cdots, m$$

so f, g have the following forms

$$f(z) = \begin{pmatrix} f^1(z^1) \\ f^2(z) \end{pmatrix}_{|k}, \qquad g_i(z) = \begin{pmatrix} g_i^1(z^1) \\ g_i^2(z) \end{pmatrix}_{|k}$$

From (6b), we can get

$$\Delta f(z) = \begin{pmatrix} 0 \\ \Delta f^2(z) \end{pmatrix}_{|k}, \qquad \Delta g_i(z) = \begin{pmatrix} 0 \\ \Delta g_i^2(z) \end{pmatrix}_{|k}$$

Condition (6c) yields

$$h(z) = h(z^1)$$

So system (7) can be expressed

$$\dot{z}_1 = f^1(z^1) + G^1(z^1)u \tag{8a}$$

$$\dot{z}_2 = f^2(z) + \Delta f^2(z) + \left(G^2(z) + \Delta G^2(z)\right)u \tag{8b}$$

$$y = h(z^1) \tag{8c}$$

where $G^1(z^1) = \left(g_1^1(z^1), \cdots, g_m^1(z^1)\right)$, $\Delta G^2(z) = \left(\Delta g_1^2(z^2), \cdots, \Delta g_m^2(z^2)\right)$.

That is to say that system (4) is output robust.

If system (4) is output robust, then there must exist a diffeomorphic transformation $z = \psi(x)$ such that system (4) is changed into the form of system (8).

Set

$$\Delta = \text{span}\left\{ \frac{\partial}{\partial z_1^2}, \frac{\partial}{\partial z_2^2}, \cdots, \frac{\partial}{\partial z_k^2} \right\}$$

formula (6a), (6b) can be easily verified. ■

Theorem 1 gives the sufficient and necessary condition for output robustness. But many systems do not satisfy this condition, so we must use feedback making up those systems into output robust.

Definition 4. For system (4), if there exist a feedback pair $\left(\alpha(x), \beta(x)\right)$ and a diffeomorphic transformation $z = \psi(x)$ such that the resulting system is output robust on some neighborhood U of $x \in M$, then system (4) is said to be locally feedback output robust. In addition, if $U = M$, system (4) is said to be feedback output robust.

Theorem 2. *System (4) is locally feedback output robust on some neighborhood of* $x \in M$ *if and only if there exists a controlled invariant distribution* Δ *on* U *such that*

$$\Delta f + \operatorname{span}\{\Delta g_1, \Delta g_2, \cdots, \Delta g_m\} \subset \Delta \tag{9a}$$

$$\Delta \subset \ker \mathrm{d}h \tag{9b}$$

Proof. Because Δ is involutive, there exists a diffeomorphic transformation $z = \psi(x)$ such that

$$\Delta = \operatorname{span}\left\{ \frac{\partial}{\partial z_1^2}, \frac{\partial}{\partial z_1^2}, \cdots, \frac{\partial}{\partial z_1^2} \right\}$$

where $k = \dim \Delta$ on U, $z = \left((z^1)^T, (z^2)^T \right)^T = \left(z_1^1, z_2^1, \cdots, z_{n-k}^1, z_1^2, \cdots, z_k^2 \right)^T$.

Since Δ is controlled invariant on U, so there exists a feedback pair $\left(\alpha(x), \beta(x) \right)$ such that through the transformation $z = \psi(x)$, $f + G\alpha$, $(G\beta)_j$ have the following forms respectively,

$$f(z) + G(z)\alpha(z) = \begin{pmatrix} \overline{f}^1(z^1) \\ \overline{f}^2(z) \end{pmatrix}_{|k}$$

$$\left(G(z)\beta(z) \right)_j = \begin{pmatrix} \overline{g}_i^1(z^1) \\ \overline{g}_j^2(z) \end{pmatrix}_{|k}, \quad j = 1, 2, \cdots, m$$

where $\left(G(z)\beta(z) \right)_j$ is the j-th column of the matrix $G(z)\beta(z)$.

From (8a),(8b), we have

$$\Delta f(z) + \Delta G(z)\alpha(z) = \begin{pmatrix} 0 \\ \Delta \overline{f}^2(z) \end{pmatrix}_{|k}$$

$$\left(\Delta G(z)\beta(z) \right)_j = \begin{pmatrix} 0 \\ \Delta \overline{g}_j^2(z) \end{pmatrix}_{|k}, \qquad h(z) = h(z^1)$$

so the system resulting from (4) can be expressed as

$$\dot{z}^1 = \overline{f}^1(z^1) + \overline{G}^1(z^1)\nu \tag{10a}$$

$$\dot{z}^2 = \overline{f}^2(z) + \Delta \overline{f}^2(z) + \left(\overline{G}^2(z) + \Delta \overline{G}^2(z) \right)\nu \tag{10b}$$

$$y = h(z^1) \tag{10c}$$

this means that system (4) is feedback output robust on U.

If system (4) is feedback output robust, there must exist a feedback pair $\big(\alpha(x),\beta(x)\big)$ and a transformation such that the resulted system has the form of system (10).

Set

$$\Delta = \operatorname{span}\left\{\frac{\partial}{\partial z_1^2},\frac{\partial}{\partial z_2^2},\cdots,\frac{\partial}{\partial z_k^2}\right\}$$

It is easily verified that (9b) holds and

$$\Delta f(z) + \Delta G(z)\alpha(z) \in \Delta$$

$$\Big(\Delta G(z)\beta(z)\Big)_j \in \Delta$$

From the invertibility of $\beta(x)$, we have

$$\operatorname{span}\{\Delta g_1, \Delta g_2, \cdots, \Delta g_m\} \subset \Delta$$

so $\Delta f(z) \in \Delta$, and (9a) holds. ■

We have the following remarks for Theorem 1 and Theorem 2.

Remark 2. In Theorem 1, Δ is first the (f,g)-invariant distribution included in $\ker dh$, so Δ is included in the unobservable distribution Δ^* of the nominal system (1) of system (4), that is to say that if (6b) holds, then we have

$$\Delta f + \operatorname{span}\{\Delta g_1, \Delta g_2, \cdots, \Delta g_m\} \subset \Delta^* \tag{11}$$

Vice versa, if (11) holds, from Theorem 1, system (4) is output robust. So we get the following corollary;

Corollary 1. *System (4) is output robust if and only if (11) holds.*

From Corollary 1 we naturally established the following definition.

Definition 5. If the unobservable distribution Δ of the nominal system (1) of system (4) satisfies (11), then Δ is said to be output robust distribution.

Remark 3. In Theorem 2, Δ is a controlled invariant distribution included in $\ker dh$, so Δ is included in the biggest controlled distribution Δ^{**} included in $\ker dh$. Hence, if (8a) holds, we have

$$\Delta f + \operatorname{span}\{\Delta g_1, \Delta g_2, \cdots, \Delta g_m\} \subset \Delta^{**} \tag{12}$$

On the other hand, if (12) holds, system (4) is feedback output robust. So we have another corollary;

Corollary 2. *Suppose that Δ is the biggest (f,g)-invariant distribution included in* $\ker dh$, *G/Δ^{**} is non-singular at $x \in M$, then system (4) is feedback output robust if and only if (12) holds.*

Proof. It is easily verified from the Lemma in Section 2. ■

Definition 6. The biggest (f, g)-invariant distribution in $\ker dh$ with property (12) is said to be feedback output robust distribution.

4. The Application of Feedback Output Robustness

This section studies the application of feedback output robustness to the robust disturbance decoupling, robust IO decoupling and asymptotic tracking, respectively.

4.1. Robust Disturbance Decoupling

Consider the following system with uncertain terms

$$\dot{x} = f(x) + \Delta f(x) + \Big(G(x) + \Delta G(x)\Big)u + \Big(P(x) + \Delta P(x)\Big)\omega \tag{13a}$$

$$y = h(x) \tag{13b}$$

and its nominal system

$$\dot{x} = f(x) + G(x)u + P(x)\omega \tag{14a}$$

$$y = h(x) \tag{14b}$$

where ω is disturbance, $P(x) = \Big(p_1(x), p_2(x), \cdots, p_l(x)\Big)$, $\Delta P(x) = \Big(\Delta p_1(x), \cdots, \Delta p_l(x)\Big)$, $p_i(x) \in T_M$, $\Delta p_i(x) \in T_M$, $i = 1, 2, \cdots, m$; f, G, h, u are the same as in system (4).

Definition 7. For system (13),(14), if there exists a feedback pair $\Big(\alpha(x), \beta(x)\Big)$ on the neighborhood U of $x_0 \in M$ with the property that the resulting closed loop systems of (13), (14) are disturbance decoupling respectively, and for the initial value x_0, the outputs of systems (13), (14) are equal on U, then system (13) is said to be locally robust disturbance decoupling. In addition, if $U = M$, system (13) is said to be disturbance decoupling on M.

Theorem 3. *Suppose that Δ is (f, g)-invariant distribution on some neighborhood U of $x \in M$, G/Δ is non-singular on U, then system (13) is robust disturbance decoupling on U if and only if Δ is a feedback output robust distribution, and*

$$p_i \in \Delta, \qquad \Delta p_i \in \Delta, \quad i = 1, 2, \cdots, l \tag{15}$$

Proof. Because of the involution of Δ, we can get a diffeomorphic transformation $z = \psi(x)$ such that

$$\Delta = \operatorname{span}\left\{\frac{\partial}{\partial z_1^2}, \frac{\partial}{\partial z_2^2}, \cdots, \frac{\partial}{\partial z_k^2}\right\}$$

where $z = \Big((z_1^1)^T, (z_2^2)^T\Big) = \Big(z_1^1, \cdots, z_{n-k}^1, z_1^2, \cdots, z_k^2\Big)^T$, $k = \dim \Delta$ on U.

From (15), $p_i(x)$, $\Delta p_i(x)$ can be expressed as follows;

$$p_i(z) = \begin{pmatrix} 0 \\ p_i^2(z) \end{pmatrix}_{|k}, \qquad \Delta p_i(z) = \begin{pmatrix} 0 \\ \Delta p_i^2(z) \end{pmatrix}_{|k}, \qquad i = 1, 2, \cdots, l$$

so we deduce from the proof of Theorem 2 that there exists a feedback pair (α, β) such that the resulting systems of (13), (14) expressed respectively as

$$\dot{z}^1 = \overline{f}^1(z^1) + \overline{G}^1(z^1)\nu \tag{16a}$$

$$\dot{z}^2 = \overline{f}^2(z) + \Delta\overline{f}^2(z) + \left(\overline{G}^2(z) + \Delta\overline{G}^2(z)\right)\nu + \left(P^2(z) + \Delta P^2(z)\right)\omega \tag{16b}$$

$$y = h(z^1) \tag{16c}$$

and

$$\dot{z}^1 = \overline{f}^1(z^1) + \overline{G}^1(z^1)\nu \tag{17a}$$

$$\dot{z}^2 = \overline{f}^2(z) + \overline{G}^2(z)\nu + \overline{P}^2(z)\omega \tag{17b}$$

$$y = h(z^1) \tag{17c}$$

this means that system (13) is robust disturbance decoupling.

If system (13) is robust disturbance decoupling on U, then it can be changed into the form of (15).

Set

$$\Delta = \text{span}\left\{\frac{\partial}{\partial z_1^2}, \frac{\partial}{\partial z_2^2}, \cdots, \frac{\partial}{\partial z_k^2}\right\},$$

formula (15) can be easily verified. ■

4.2. Robust Input-Output Decoupling

Consider the following system with uncertain terms

$$\dot{x} = f(x) + \Delta f(x) + \left(G(x) + \Delta G(x)\right)u \tag{18a}$$

$$y = h(x) \tag{18b}$$

and its nominal system

$$\dot{x} = f(x) + G(x)u \tag{19a}$$

$$y = h(x) \tag{19b}$$

where f, G, Δf, ΔG are the same as in system (4), $h(x) \in C^\infty(M, M)$, that means that systems (18), (19) are square systems, respectively.

Definition 8. For systems (18), (19), if there exists a feedback law $u = \alpha(x) + \beta(x)\nu$ on some neighborhood U of $x \in M$ such that systems (18), (19) are IO decoupling respectively, and the outputs of (18), (19) are equal on U, then system (18) is said to be locally robust IO decoupling. If, in addition, $U = M$, system (18) is said to be robust IO decoupling.

Define the relative degree d_i of the output y_i and decoupling matrix as follows,

$$d_i = \min\{k; L_{g_j} L_f^{k-1} h_i(x) \neq 0, \quad \text{for some } j\}$$

$$A(x) = \left(L_{g_j} L_f^{d_i-1} h_i\right)_{m\times m}$$

Set

$$N = \bigcap_{i=1}^{m} N_i, \qquad N_i = \bigcap_{j=1}^{d_i} \ker \mathrm{d} L_f^{j-1} h_i$$

Theorem 4. *System (18) is robust IO decoupling on the neighborhood U of $x \in M$ if and only if $A(x)$ is invertible and N is a feedback output robust distribution.*

Proof. If the conditions in Theorem 4 is satisfied, then we have

$$\begin{aligned} \dot{y}_i &= L_f h_i \\ &\vdots \\ y_i^{d_i} &= L_f^{d_i} h_i + \sum_{j=1}^{m} u_j L_{g_i} L_f^{d_i-1} h_i, \quad i = 1, 2, \cdots, m \end{aligned}$$

Denote

$$b(x) = \left(L_f^{d_1} h_1, L_f^{d_2} h_2, \cdots, L_f^{d_m} h_m\right)^T$$

$$u = A^{-1}(x)\Big(-b(x) + \nu\Big), \qquad \nu = (\nu_1, \nu_2, \cdots, \nu_m)^T$$

then we have

$$\begin{aligned} \dot{y}_i &= L_f h_i \\ &\vdots \\ y_i^{d_i} &= \nu_i, \qquad i = 1, 2, \cdots, m \end{aligned}$$

It is easily deduced that for (18) we still get the formula above. So system (18) is robust IO decoupling.

The property of system (18) being IO robust yields that $A(x)$ is invertible. The IO robust of system (18) means that N is a output robust distribution. ■

Remark 4. Theorem 4 is equivalent to the main theorem in (Liu Jing-Sin and Yuan King, 1991). We only give it another expression here.

4.3. Robust Asymptotic Tracking

We only consider the following SISO system

$$\dot{x} = f(x) + \Delta f(x) + \Big(g(x) + \Delta g(x)\Big)u \tag{20a}$$

$$y = h(x) \tag{20b}$$

and its nominal system

$$\dot{x} = f(x) + g(x)u \tag{21a}$$

$$y = h(x) \tag{21b}$$

for simplicity, where f, g, Δf, Δg are all belong to T_M, $h(x) \in C^\infty(M, R^1)$.

Definition 9. For any initial value $x_0 \in M$, if there exists a feedback pair (α, β) such that the outputs of systems (20), (21) asymptotically converge to a prescribed reference function $y_R(t)$ respectively, then system (20) is said to be robust asymptotically tracking the function $y_R(t)$.

Theorem 5. *Suppose the relative degree of system (21) is d, $N = \cap_{i=1}^{d} \ker \mathrm{d}L_f^{i-1}h$ is a feedback output robust distribution, then the reference function $y_R(t)$ with the r-th derivative can be robust asymptotically tracked by the output of system (20).*

Proof. From the condition of the Theorem, we can get

$$\begin{aligned} \dot{y} &= L_f h \\ &\vdots \\ y^d &= L_f^d h + uL_gL_f^{d-1}h \end{aligned} \tag{22}$$

Set $z_1 = h$, $z_2 = L_f h, \cdots, z_d = L_f^{d-1}h$. The involution of span$\{g\}$ implies that there must exist $z_{d+1} = \psi_{d+1}(x), \cdots, z_n = \psi_n(x)$ such that $z = \psi(x)$ is a diffeomorphism with the property $L_g\psi_i(x) = 0$, $i = k, k+1, \cdots, n$, where $z \doteq (z_1, \cdots, z_n)^T$, $\psi = (h, \cdots, L_f^{d-1}h, \psi_{d+1}, \cdots, \psi_n)^T$.

Denote $\xi = (z_1, \cdots, z_d)^T$, $\eta = (z_{d+1}, \cdots, z_n)^T$. System (20) can be expressed by

$$\begin{aligned} \dot{\xi}_i &= \xi_{i+1}, \qquad i = 1, 2, \cdots, d-1 \\ \dot{\xi}_d &= b(\xi, \eta) + a(\xi, \eta)u \\ \dot{\eta} &= q(\xi, \eta) + \Delta q(\xi, \eta) + q_1(\xi, \eta)u \\ y &= \xi_1 \end{aligned} \tag{23}$$

where Δq, q_1, q are all $(n-d)$ dimensional smooth vector functions, $\xi_i = \delta_i$, $i = 1, 2, \ldots, d$.

If we take

$$u = a^{-1}(\xi,\eta)\Big(- b(\xi,\eta) + y_R^d(t)\Big) - \sum_{i=1}^{d} c_{i-1}\Big(\xi_i - y^{i-1}(t)\Big)$$

where c_i are the Hurwitz polynomial coefficients, $c_0 + c_1 s + \cdots + c_{d-1}s^{d-1} + s^d$, and denote $e(t) = y(t) - y_R(t)$, then the following equation can be obtained, $c_0 + c_1 e(t) + \cdots + c_{d-1}e^{d-1}(t) + e^d(t) = 0$.

It is easily verified that $y(t)$ asymptotically converge to $y_R(t)$ as t tends to infinite. At the same time, the output of system (21) also asymptotically converge to $y_R(t)$. The proof is completed. ■

Remark 5. This theorem can be compared with the paper (Teh-Lu *et al.*, 1991) which studied the problem of robust output tracking via a Lyapunov-based approach. The main assumption there is the mismatching condition, that is that there exist smooth functions $\delta_1(x)$, $\delta_2(x)$ and smooth vector functions f_1, g_1 such that the uncertainties Δf, Δg satisfy

$$\Delta f(x) = g(x)\delta_1(x) + f_1(x) \tag{24a}$$

$$\Delta g(x) = g(x)\delta_2(x) + g_1(x) \tag{24b}$$

with f_1, g_1 satisfying some properties. The tracking error in (Teh-Lu *et al.*, 1991) only remains bounded but not converges to zero. If $f_1(x) = g_1(x) = 0$, it is easy to verify that system (20) with (24a), (24b) satisfies the condition of Theorem 5. So the tracking error converges to zero.

5. Conclusion

This paper has studied the problem of robustness for non-linear control systems. The sufficient and necessary conditions of output robustness and feedback output robustness are given. From Theorem 1 and Theorem 2, we can conclude that the so called output robustness and feedback output robustness are in fact included the uncertain terms. Furthermore, the paper has studied the application of feedback output robustness to the problems of robust disturbance decoupling, robust IO decoupling and robust asymptotic tracking, respectively. Therefore, the main robust problems in non-linear control systems can be studied by a unified method.

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