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MULTILAYER FUZZY CONTROL OF MULTIVARIABLE SYSTEMS BY PASSIVE DECOMPOSITION

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The paper considers the problem of multilayer fuzzy control of multivariable systems. Some relevant definitions and theorems are given. Multilayer fuzzy control algorithms, based on passive decomposition of the original system into layers, are presented and illustrated by numerical examples. The algorithms use a subset of state variables, leading to a unilayer control solution by taking into account implicitly the other variables. It is shown that the number of measured variables and fuzzy relations is significantly reduced and thus the real time measurement and control implementation are facilitated.

1. Introduction

Multivariable fuzzy control systems have met with great interest recently (Baboshin and Naryshkin, 1990; Gegov and Frank, 1994; Gupta *et al.*, 1986). This is due to the inherent complexity of most real control processes which are characterized by the simultaneous presence of many state variables (multivariability) and model uncertainty (fuzziness). However, mostly empirical and heuristic techniques have been used in this field up to now which is due to the fact that fuzzy control theory has been developed only for simple low-order systems and not for multivariable and large scale ones. Similar conclusions are presented in (Gegov, 1994; Palm *et al.*, 1993; Titli, 1992; Zimmermann, 1991) where the necessity of investigations in this direction is pointed out.

One of the most important problems in multivariable fuzzy control systems is the computational complexity of the corresponding control algorithms. Many investigations have been made with the aim to reduce this complexity by decomposing the fuzzy control rule basis into separate layers and thus to reduce the number of rules (Burke and Rattan, 1993; De Silva and MacFarlane, 1989; Koczy and Hirota, 1993; Raju *et al.*, 1992; Sustal, 1993). However, no reduction in the number of *on-line* computations is achieved in this case. On the contrary, this number is usually increased because of the necessity to calculate the additionally introduced intermediate variables which do not have any physical meaning. In this sense, all these works are more concerned with *off-line* relational identification than with the real-time fuzzy control. More specifically, these works consider systems, described by a large number of linguistic rules and a small number of state variables, taking into account the fact that the amount of computations by identification of fuzzy relations is proportional to the number of rules.

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This paper uses a multilayer approach which is different from that presented above. It is assumed that there exists partial information about the system which is expressed in a small number of linguistic rules. At the same time, the system is a multivariable one and is characterized by a relatively large number of state variables. Moreover, all the variables in the separate layers are supposed to have physical meaning and are the same as those in the original non-layered presentation of the system. In this case, no additional intermediate variables are introduced and the corresponding *on-line* computations by real-time control may be significantly reduced. The reduction results from the fact that the amount of these computations is proportional to the number of state variables.

The paper is structured as follows: Section 2 contains the problem statement, Section 3 introduces some definitions and theorems, Section 4 presents multilayer control algorithms, Section 5 illustrates the theoretical results by numerical examples, and Section 6 gives some analysis of the results.

2. Problem Statement

It is known that a multivariable system can be controlled by the following linguistic rules:

$$\begin{array}{l} \text{If } x_{1(1)} \text{ and} \dots \text{and } x_{n(1)} \text{ Then } u_{1(1)} \text{ and} \dots \text{and } u_{m(1)} \\ \vdots \\ \text{If } x_{1(h)} \text{ and} \dots \text{and } x_{n(h)} \text{ Then } u_{1(h)} \text{ and} \dots \text{and } u_{m(h)} \end{array} \tag{1}$$

where $x_{j(s)}$, $j = 1, \dots, n$ and $u_{i(s)}$, $i = 1, \dots, m$ are respectively the j-th input (state) and the i-th output (control) fuzzy variables in the s-th rule, $s = 1, \dots, h$ (Gupta *et al.*, 1986). Both $x \in E^n$ and $u \in E^m$ are defined in universal sets X and U of equal power f, i.e. $X, U \in E^f$, where E is a vector space.

It is also known that the multivariable system (1) can be represented approximately by m single-output systems in accordance with the so-called "decompositional inference" (Gupta *et al.*, 1986; Kosko, 1992; Lee, 1990). In this case, the 'if' parts in (1) are repeated for each output variable u_i, $i = 1, \dots, m$ and the following control law is obtained:

$$u_i = \bigcap_{j=1}^{n} (x_j \circ R_{ji}), \qquad i = 1, \dots, m \tag{2}$$

The symbol '$\circ$' in (2) denotes the max-min composition and $R_{ij} \in E^{f.f}$, $j = 1, \dots, n$, $i = 1, \dots, m$ are two-dimensional fuzzy relations, calculated by

$$R_{ji} = \bigcup_{s=1}^{h} (x_{j(s)} \cap u_{i(s)}), \qquad j = 1, \dots, n, \ \ i = 1, \dots, m \tag{3}$$

where the symbols '$\cup$' and '$\cap$' stand for the max and min operator, respectively. The symbol '.' in the notation $f.f$ separates the dimensions of the vector space E and the quantity in the outer brackets of (3) is the Cartesian product of the variables x and u.

It should be pointed out that the representation (2)–(3) is not unique, i.e. there are many ways of defining R_{ji}, $j = 1,\ldots,n$, $i = 1,\ldots,m$ and u, $i = 1,\ldots,m$ (Driankov *et al.*, 1993; Harris *et al.*, 1993; Pedrycz, 1993). However, the above max-min compositional inference seems to be most widely used at present as shown in (Gupta *et al.*, 1986), and for this reason it is considered here. In this sense, the results in the paper are valid only for this inference but can be extended easily for other types of inferences.

The control law (2) can be represented by

$$\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}^T = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^T * \begin{bmatrix} R_{11} & \ldots & R_{n1} \\ \vdots & & \vdots \\ R_{n1} & \ldots & R_{nm} \end{bmatrix} \tag{4}$$

where $*$ is the $(\circ, \cap)$ operator. This law is shown schematically in Fig. 1.

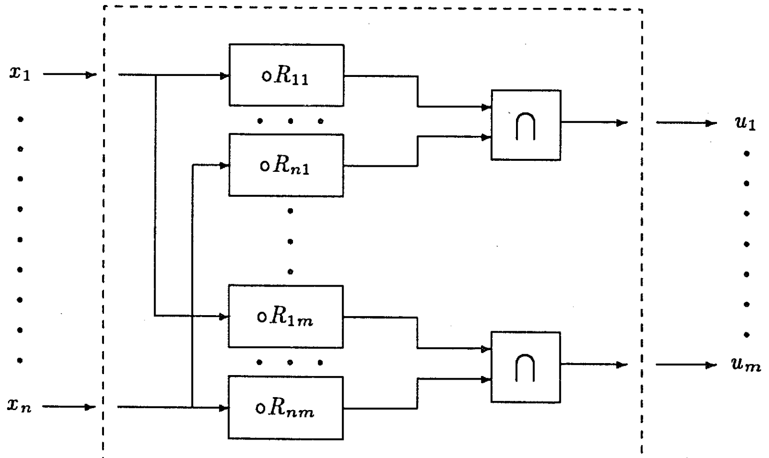

Fig. 1. Multivariable system by fuzzy control.

The detailed presentation of (4) is given by

$$\begin{bmatrix} u_i^1 \\ \vdots \\ u_i^f \end{bmatrix}^T = \bigcap_{j=1}^{n} \begin{bmatrix} x_j^1 \\ \vdots \\ x_j^f \end{bmatrix}^T \circ \begin{bmatrix} r_{ji}^{11} & \ldots & r_{ji}^{1f} \\ \vdots & & \vdots \\ r_{ji}^{f1} & \ldots & r_{ji}^{ff} \end{bmatrix}, \qquad i = 1,\ldots,m \tag{5}$$

where the upper index $t = 1,\ldots,f$ stands for the respective element of the universal set and r_{ji}^{st}, $j = 1,\ldots,n$, $i = 1,\ldots,m$, $s,t = 1,\ldots,f$ are elements of the fuzzy relation R_{ji}.

The development of (5) for a given element of the universal set u_i^t, $i = 1,\dots,m$, $t = 1,\dots,f$ leads to the following expression:

$$u_i^t = \bigcap_{j=1}^{n} \left[\bigcup_{s=1}^{f} \left(x_j^s \cap r_{ji}^{st} \right) \right] \tag{6}$$

It is evident from the control law (4) that the number of fuzzy relations and *on-line* computations can be enormous which could cause difficulties in the real-time measurement and control implementation. This number is proportional to the dimension of the state vector n and the power of the universal set f. For multivariable systems, n may be a large number while f is chosen subjectively but in any case it should not be very small if a satisfactory accuracy of the computations is required. Therefore, it would be reasonable to find a suitable decomposed form of the control law that would reduce the computational complexity of the problem. One possible way in this direction is to find the conditions under which the control law (4) may be passively decomposed into separate layers. This approach is analogous to the one presented in (Gegov and Frank, 1994) but the difference here consists in the decomposition which is temporal and not spatial. It is expressed by the term "passive decomposition" that the fuzzy relations in the original control law (4) are not influenced during the decomposition.

3. Definitions and Theorems

For the purpose of multilayer control, two general types of decomposed subsystems (layers) are considered: single-input single-output (SISO) and non-single-input single-output (NSISO). NSISO may be multiple-input single-output, single-input multiple-output or multiple-input multiple-output subsystems (Jamshidi, 1983). The number of layers will be denoted by N where $N = n$ for SISO layers and $N < n$ for NSISO layers. In both cases the inputs of each layer are outputs of the next layer. The only exceptions in this context are the inputs of the first layer and the outputs of the last layer which are not outputs and inputs of other layers.

The mathematical notation in this section is presented in a way which gives the minimal necessary information to understand the idea of the proposed multilayer approach.

Definition 1. The fuzzy relation (relations) R_{ki}, $k \in [1,n]$, $i = 1,m$ is (are) dominant with respect to the control variable u_i, $i = 1,m$ in the control law (2) if the effect of the other fuzzy relations R_{ji}, $j = 1,n$, $j \neq k$ in the same i-th column of the corresponding relational block matrix in (4) are negligible with respect to u_i.

The notation $k = 1,n$ in the above definition means that k takes all the values in the range of the integers $1,2,\dots,n-1,n$. On the contrary, the notation $k \in [1,n]$ will mean further in the text that k can take one or more, but not all, values in this range.

The neglecting of the effect of one or more relations on a given control variable in Definition 1 is based on the observation that some intermediate max-min operations may not influence the final solution. Thus, the fuzzy control law may be significantly simplified without any serious consequences if these computations are omitted.

3.1. SISO Layers

Let system (4) be decomposed into N SISO layers with respect to each control variable u_i, $i = \overline{1, m}$ as follows:

$$\begin{aligned} x_2 &= x_1 \circ D_{1,2} \\ &\vdots \\ x_N &= x_{N-1} \circ D_{N-1,N} \\ u_i &= x_N \circ D_{N,i} \end{aligned} \tag{7}$$

where $D_{j-1,j} \in E^{f \cdot f}$, $j = 2, \dots, N$ and $D_{N,i} \in E^{f \cdot f}$. This multilayer presentation is shown schematically in Fig. 2.

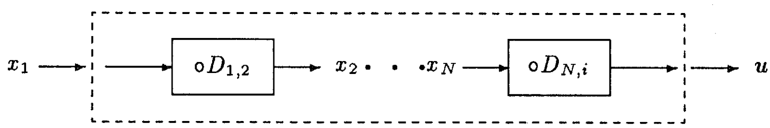

Fig. 2. Multivariable system by multilayer fuzzy control: the SISO case.

The brief form of the first $N - 1$ equations in (7) is given by

$$x_j = x_{j-1} \circ D_{j-1,j}, \qquad j = \overline{2, N}, \quad i = \overline{1, m} \tag{8}$$

where the corresponding relations are calculated by

$$D_{j-1,j} = \bigcup_{k=1}^{h} \left(x_{j-1(k)} \cap x_{j(k)} \right), \qquad i = \overline{1, m} \tag{9}$$

on the basis of the 'if' parts of the linguistic control rules (1). These relations show how the separate state variables of the system are interrelated and the way of their determination is in accordance with the basic notion of fuzzy relations, i.e. how to reflect the mutual connection between an arbitrary couple of physical variables. The relation in the last equation of (7) is calculated by (3).

Definition 2. The control law (2) is partially decomposable into N SISO layers in the form (7) if the fuzzy membership values of at least one control variable u_i, $i \in [1, m]$ in both cases overlap in a linguistic sense.

The overlap in the above definition means that one and the same linguistic value of the control variable is inferred in both cases in spite of the possible difference in the corresponding fuzzy membership values. This consideration of the system on a linguistic level is more flexible than the relational consideration which usually imposes some restrictions.

Proposition 1. *The control law (2) is partially decomposable into* N *SISO layers in the form (7) if there exists at least one fuzzy relation* R_{ji}, $j = 1,n$, $i \in [1,m]$ *which is dominant with respect to at least one control variable* u_i, $i \in [1,m]$.

The assumption of decomposability into layers in the above proposition follows from the notion of dominance of fuzzy relations with respect to a given control variable in accordance with Definition 1. In this case, the effect of the other fuzzy relations can be considered as negligible in the sense of the same definition.

The order of state variables in the corresponding layers in (7) is conditional, i.e. the lower indices of these variables correspond to the indices of the corresponding layers and not to the original indices in the linguistic control rules (1). Therefore there may exist different permutations of these variables on the basis of the reindexation used.

Let a given permutation of the state variables in (7) be denoted by P_x^v, $v \in [1, N!]$ where $N! = 1 \cdot 2 \cdots (N-1) \cdot N$ is the number of all permutations. The original index of the input state variable of the first layer in the v-th permutation will be denoted by v_1.

Theorem 1. *The control law (2) of the multivariable system (1) is partially decomposable into* N *SISO layers in the form (7) if all the state variables are represented by normal fuzzy sets and the following conditions hold for all the elements of the universal set* u_i^t, $t = 1,f$ *of at least one control variable* u_i, $i \in [1,m]$ *and for at least one permutation* P_x^v, $v \in [1, N!]$:

$$r^{st}_{v_1 i},\ v_1 \in [1,N],\ s = 1,f \le r^{st}_{ji}, \quad j = 1,N,\ j \neq v_1,\ s = 1,f \tag{10}$$

$$R_{v_1 i} = \left(\mathop{\circ}_{j=2}^{N} D_{j-1,j} \right) \circ D_{N,i}, \qquad v_1 \in [1,N] \tag{11}$$

Proof. It is supposed that conditions (10)–(11) hold. In accordance with Definition 2, it will be necessary to compare the fuzzy membership values of the control variables in (2) and (7). For this purpose, all intermediate state variables x_j, $j = 1,N$ in (7) are successively substituted and the following expression is obtained

$$u_i = x_1 \circ D_{1,2} \circ \ldots \circ D_{N-1,N} \circ D_{N,i} = x_1 \circ D_{1,i} = x_1 * D_{1,i}, \quad i = 1,m \tag{12}$$

where the operators '$\circ$' and '$*$' are equivalent because of the block-wise influence of the latter. This expression is an equivalent unilayer form of (7) and is shown schematically in Fig. 3.

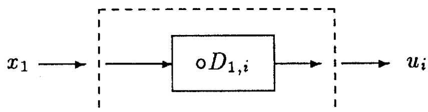

Fig. 3. Equivalent unilayer presentation: the SISO case.

Expression (12) shows the relation between x_1 and u_i, $i = 1,m$ as all other state variables are taken into account implicitly by $D_{J-1,J}$, $j = 2,N$ and $D_{N,i}$.

The index 1 of the state variable is conditional and denotes the first element in the considered permutation P_x^v, $v \in [1, N!]$ of x_j, $j = \overline{1, N}$. Therefore this index can be substituted by a more general index v_1. It is proved in (Gegov, 1994) that when condition (10) holds, the relation $R_{v_1 i}$, $v_1 \in [1, N]$, $i = \overline{1, m}$ is dominant with respect to u_i, $i = \overline{1, m}$ in the control law (2) in accordance with Definition 1. At the same time, this relation is equal to the relation $D_{1,i}$ in (12) when condition (11) holds. It is evident in this case that the fuzzy membership values of the control variables in (2) and (7) will overlap in a linguistic sense according to Definition 2 and therefore the proof is completed.

Definition 3. The control law (2) is fully decomposable into N SISO layers in the form (7) if the fuzzy membership values of all the control variables u_i, $i = \overline{1, m}$ in both cases overlap in a linguistic sense.

Proposition 2. *The control law (2) is fully decomposable into N SISO layers in the form (7) if there exist m fuzzy relations R_{ji}, $j = \overline{1, n}$, $i = \overline{1, m}$ such that each of them is dominant with respect to each control variable u_i, $i = \overline{1, m}$.*

The notion of full decomposability in the above definition and proposition is concerned with all control variables. Therefore it is a more desirable option than the partial decomposability which is concerned with at least one but not all such variables.

Theorem 2. *The control law (2) of the multivariable system (1) is fully decomposable into N SISO layers in the form (7) if all the state variables are represented by normal fuzzy sets and conditions (10)–(11) hold for all the elements of the universal set u_i^t, $t = \overline{1, f}$ of all control variables u_i, $i = \overline{1, m}$ and for at least one permutation P_x^v, $v \in [1, N!]$.*

The proof of Theorem 2 is analogous to the proof of Theorem 1 and for this reason it is omitted.

3.2. NSISO Layers

Let system (4) be decomposed into N NSISO layers with respect to each control variable u_i, $i = \overline{1, m}$ as follows:

$$
\begin{aligned}
x_{21} &= \left(x_{11} \circ D_{1,1,2,1}\right) \cap \ldots \cap \left(x_{1n_1} \circ D_{1,n_1,2,1}\right) \\
&\vdots \\
x_{2n_2} &= \left(x_{11} \circ D_{1,1,2,n_2}\right) \cap \ldots \cap \left(x_{1n_1} \circ D_{1,n_1,2,n_2}\right) \\
&\vdots \\
x_{N1} &= \left(x_{N-1,1} \circ D_{N-1,1,N,1}\right) \cap \ldots \cap \left(x_{N-1,n_{N-1}} \circ D_{N-1,n_{N-1},N,1}\right) \\
&\vdots \\
x_{Nn_N} &= \left(x_{N-1,1} \circ D_{N-1,1,N,n_N}\right) \cap \ldots \cap \left(x_{N-1,n_{N-1}} \circ D_{N-1,n_{N-1},N,n_N}\right) \\
&\vdots \\
u_i &= \left(x_{N1} \circ D_{N,1,i}\right) \cap \ldots \cap \left(x_{Nn_N} \circ D_{N,n_N,i}\right)
\end{aligned}
\tag{13}
$$

where $D_{j-1,s,j,q} \in E^{f \cdot f}$, $j = 2, N$, $s = 1, n_{j-1}$, $q = 1, n_j$, $D_{N,q,i} \in E^{f \cdot f}$, $q = 1, n_N$. The first lower index of each state variable x denotes the respective layer and the second index — the input state variable of the layer. This multilayer presentation is shown schematically in Fig. 4.

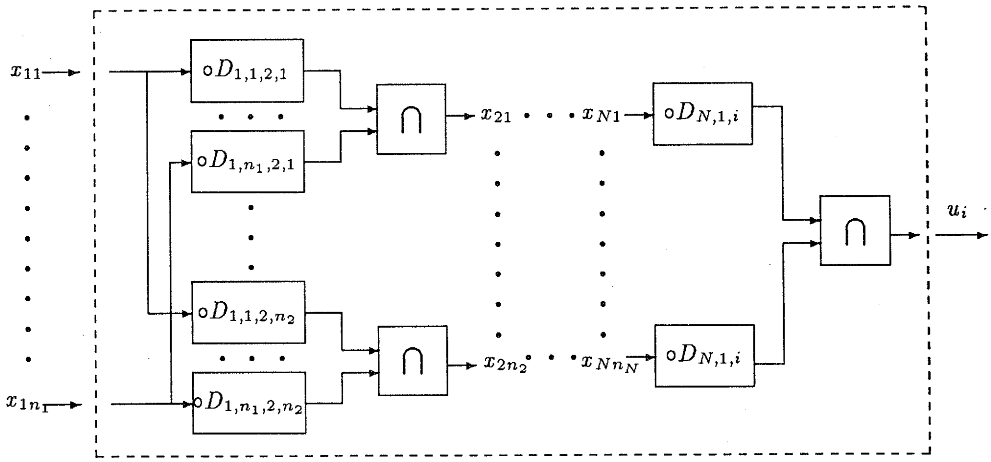

Fig. 4. Multivariable system by multilayer fuzzy control: the NSISO case.

The following notation and substitutions are introduced for further considerations:

$$\sum_{j=1}^{N} n_j = n,\ x_j = \left[x_{j1}, \ldots, x_{jn_j}\right], \quad R_{ji} = \left[R_{j1i}, \ldots, R_{jn_ji}\right]^T, \quad j = 1, N \tag{14}$$

The brief form of (13) is given by

$$\begin{aligned} x_{jq} &= \bigcap_{s=1}^{n_j} \left(x_{j-1s} \circ D_{j-1,s,j,q}\right), & j = 2, N,\ \ q = 1, n_j \\ u_i &= \bigcap_{s=1}^{n_N} \left(x_{Ns} \circ D_{N,s,i}\right), & i = 1, m \end{aligned} \tag{15}$$

where the quantities $D_{j-1,s,j,q}$, $j = 2, N$, $s = 1, n_{j-1}$, $q = 1, n_j$ are calculated as

$$D_{j-1,s,j,q} = \bigcup_{k=1}^{h} \left(x_{j-1s(k)} \cap x_{j\,q(k)}\right), \quad i = 1, m \tag{16}$$

on the basis of the 'if' parts of the linguistic control rules (1) and the relations $D_{N,s,i}$, $s = 1, n_N$ are calculated based on (3).

Definition 4. The control law (2) is partially decomposable into N NSISO layers in the form (13) if the fuzzy membership values of at least one control variable u_i, $i \in [1,m]$ in both cases overlap in a linguistic sense.

Proposition 3. *The control law (2) is partially decomposable into N NSISO layers in the form (13) if there exists at least one set of 'q' fuzzy relations R_{jqi}, $j = 1, N$, $q = 1, n_j$, $i \in [1,m]$ which is dominant with respect to at least one control variable u_i, $i \in [1,m]$.*

The order of the state variables in the corresponding layers in (13) is also conditional. Therefore there may exist different permutations of these variables on the basis of the reindexation used.

It is known from combinatorics that the number of n_k-combinations from among all n state variables in (13), corresponding to the k-th layer, $k = 1, N$, is calculated according to

$$C_x \binom{n}{n_k} = \frac{n!}{n_k!(n-n_k)!} \tag{17}$$

A given permutation of all the state variables with respect to all the layers will be evidently a function of n_k-combinations of these layers and could be denoted by P_x^w, $w \in [1, z]$. In this case, the number of all the permutations z is calculated through the formula

$$z = \frac{n!}{n_1! \dots n_N!} \tag{18}$$

The multilayer presentation (13) can be developed in the following recursive form:

Layer N

$$u_i = x_N * D_{N,i} \tag{19}$$

where $x_N = [x_{N1}, \dots, x_{Nn_N}] \in E^{fn_N}$, $D_{N,i} = [D_{N,1,i}, \dots, D_{N,n_N,i}]^T \in E^{fn_N \cdot f}$.

Layer $N-1$

$$\begin{aligned} x_{N1} &= x_{N-1} * D_{N-1,N,1} \\ &\vdots \\ x_{Nn_N} &= x_{N-1} * D_{N-1,N,n_N} \end{aligned} \tag{20}$$

where $x_{N-1} = [x_{N-11}, \dots, x_{N-1n_{N-1}}] \in E^{fn_{N-1}}$,

$D_{N-1,N,1} = [D_{N-1,1,N,1}, \dots, D_{N-1,n_{N-1},N,1}]^T \in E^{fn_{N-1} \cdot f}$,

$D_{N-1,N,n_N} = [D_{N-1,1,N,n_N}, \dots, D_{N-1,n_{N-1},N,n_N}]^T \in E^{fn_{N-1} \cdot f}$,

$\vdots$

Layer 1

$$\begin{aligned} x_{21} &= x_1 * D_{1,2,1} \\ &\vdots \\ x_{2n_2} &= x_1 * D_{1,2,n_2} \end{aligned} \tag{21}$$

where $x_1 = [x_{11}, \dots, x_{1n_1}] \in E^{fn_1}$, $D_{1,2,1} = [D_{1,1,2,1}, \dots, D_{1,n_1,2,1}]^T \in E^{fn_1 \cdot f}$, $D_{1,2,n_2} = [D_{1,1,2,n_2}, \dots, D_{1,n_1,2,n_2}]^T \in E^{fn_1 \cdot f}$.

Further, the input state variables of the N-th NSISO layer $x_{N1}, \dots, x_{Nn_N}$ are expressed by (20) and substituted in (19) as follows:

$$u_i = \left[x_{N-1} * D_{N-1,N,1}, \dots, x_{N-1} * D_{N-1,N,n_N}\right] * D_{N,i} \tag{22}$$

On the basis of the commutativity of the operator '$*$' and its block-wise influence, the following substitutions can be made:

$$\begin{aligned} D_{N-1,N,k} * D_{N,k,i} &= \left[D_{N-1,1,N,k}, \dots, D_{N-1,n_{N-1},N,k}\right]^T \circ D_{N,k,i} \\ = C^i_{N-1,N,k} &= \left[C^i_{N-1,1,N,k}, \dots, C_{N-1,n_{N-1},N,k}\right]^T \in E^{fn_{N-1} \cdot f}, \quad k = 1, n \end{aligned} \tag{23}$$

Therefore the control variable u_i, $i = 1, m$ can be expressed by

$$u_i = \left[x_{N-1}, \dots, x_{N-1}\right] * \left[C^i_{N-1,N,1}, \dots, C^i_{N-1,N,n_N}\right]^T \tag{24}$$

where the input vector of state variables of the $(N-1)$-th layer is taken n_N times.

By successive substitution of the input vectors of state variables of the other NSISO layers, the following expression is obtained:

$$u_i \doteq \left[x_1, \dots, x_1\right] * \left[C^i_{1,N,1}, \dots, C^i_{1,N,n_2 \dots n_N}\right]^T, \qquad i = 1, m \tag{25}$$

Analogously to the SISO case, the control variable u_i, $i = 1, m$ is obtained as an explicit function of the input vector of state variables of the first layer. However, this vector is taken $n_2 \cdots n_N$ times which is due to the implicit accountancy of the state vectors of the other layers. The notation $n_2 \cdots n_N$ here means that the dimensions of the input vectors of all layers from the 2-nd to the n-th one are multiplied by one another.

Theorem 3. *The control law (2) of the multivariable system (1) is partially decomposable into N NSISO layers in the form (13) if all the state variables are represented by normal fuzzy sets and the following conditions hold for all the elements of the universal set u^t_i, $t = 1, f$ of at least one control variable u_i, $i \in [1, m]$ and for at least one permutation P^w_x, $w \in [1, z]$:*

$$r^{st}_{w_1 q_i}, q = 1, n_{w_1}, \quad s = 1, f \leq r^{st}_{jqi}, \quad j = 1, N, \quad j \neq w_1, \quad q = 1, n_j, \quad s = 1, f \tag{26}$$

$$c^{ist}_{w_1 q w_N p}, q = 1, n_{w_1}, \;\; w_1, w_N \in [1, N], \;\; p \in [1, n_2 \cdots n_N], \; s = 1, f \leq c^{ist}_{w_1 q w_N j}, \quad q = 1, n_{w_1}, \;\; w_1, w_N \in [1, N], \;\; j = 1, n_2 \ldots n_N, \;\; j \neq p, \; s = 1, f \tag{27}$$

$$R_{w_1,q,i} = C^i_{w_1,q,w_N,p}, \quad w_1, w_N \in [1, N], \;\; q = 1, n_{w_1}, \;\; p \in [1, n_2 \cdots n] \tag{28}$$

Proof. The four lower indices of the relations $C^i_{w_1,q,w_N,p}$, $w_1, w_N \in [1, N]$, $q = 1, n_{w_1}$, $p \in [1, n_2 \cdots n_N]$, $i = 1, m$ in (28) have the following meanings: w_1 denotes the first layer in the permutation for which the decomposability conditions hold, q is the index of the corresponding state variable in this layer, w_N denotes the last layer in the same permutation and p stands for the set of 'q' relations which are dominant with respect to the control variable u_i. The upper index i is the index of this control variable.

It is supposed that conditions (26)–(28) hold. In accordance with Definition 4, it will be necessary to compare the fuzzy membership values of control variables in (2) and (25). For this purpose, the control law (25) is presented in the following way:

$$u_i = \bigcap_{p=1}^{n_2 \cdots n_N} \left[\bigcap_{q=1}^{n_1} \left(x^T_{1q} \circ C_{1,q,N,p} \right) \right], \qquad i = 1, m \tag{29}$$

where $x_1 = [x_{11}, \ldots, x_{1n_1}]$ is the input vector of state variables of the first layer in the considered permutation. It is proved in (Gegov, 1994) that when conditions (26)–(27) hold, the relations $R_{v_1,q,i}$ and $C^i_{w_1,q,w_N,p}$ $w_1, w_N \in [1, N]$, $q = 1, n_{w_1}$, $p \in [1, n_2 \cdots n_N]$, $i = 1, m$ are dominant with respect to u_i, $i \in [1, m]$ in the control laws (2) and (25) in accordance with Definition 1. In this case, the control law (29) is reduced to the expression

$$u_i = \bigcap_{q=1}^{n_1} \left(x_{1q} \circ C^i_{1,q,N,p} \right), \quad p \in [1, n_2 \cdots n_N], \quad i = 1, m \tag{30}$$

which is an equivalent unilayer form of (13) and is shown schematically in Fig. 5.

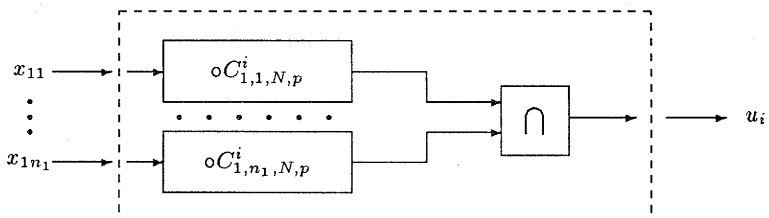

Fig. 5. Equivalent unilayer presentation: the NSISO case.

At the same time, the relations R_{w_1qi} and $C^i_{w_1,q,w_N,p}, w_1, w_N \in [1,N]$, $q = 1, n_{w_1}$, $p \in [1, n_2 \cdots n_N]$, $i = 1, m$ are equal when condition (28) holds. It is evident in this case that the fuzzy membership values of control variables in (2) and (25) will overlap in a linguistic sense according to Definition 4 and therefore the proof is completed. ■

Definition 5. The control law (2) is fully decomposable into N NSISO layers in the form (13) if the fuzzy membership values of all the control variables u_i, $i = 1, m$ in both cases overlap in a linguistic sense.

Proposition 4. *The control law (2) is fully decomposable into N NSISO layers in the form (13) if there exist 'm' sets of 'q' fuzzy relations R_{jqi}, $j = 1, N$, $q = 1, n_j$, $i = 1, m$ such that each of them is dominant with respect to each control variable u_i, $i = 1, m$.*

Theorem 4. *The control law (2) of the multivariable system (1) is fully decomposable into N NSISO layers in the form (13) if all the state variables are represented by normal fuzzy sets and conditions (26)–(28) hold for all the elements of the universal set u_i^t, $t = 1, f$ of all the control variables u_i, $i = 1, m$ and for at least one permutation P_x^w, $w \in [1, z]$.*

The proof of Theorem 4 is similar to the proof of Theorem 3 and for this reason it is omitted.

4. Control Algorithms

On the basis of the theoretical results of Section 3, two control algorithms are presented below. The first algorithm refers to systems decomposed into SISO layers, and the other — to systems, decomposed into NSISO layers. The algorithms can be implemented by a fully decentralized computational structure where each unit calculates only the i-th control variable u_i, $i = 1, m$. Two stages are distinguished: *off-line* and *on-line*.

Both algorithms require an extensive search with respect to permutations of state variables. However, this is carried out in the *off-line* stage and therefore is not a great disadvantage. If the search turns out to be unsuccessful after testing all the permutations, the original control law is applied in the *on-line* stage instead of the multilayer one.

4.1. SISO Layers

The block-diagram of the control algorithm is presented in Fig. 6.

4.2. NSISO Layers

The block-diagram of the control algorithm is presented in Fig. 7.

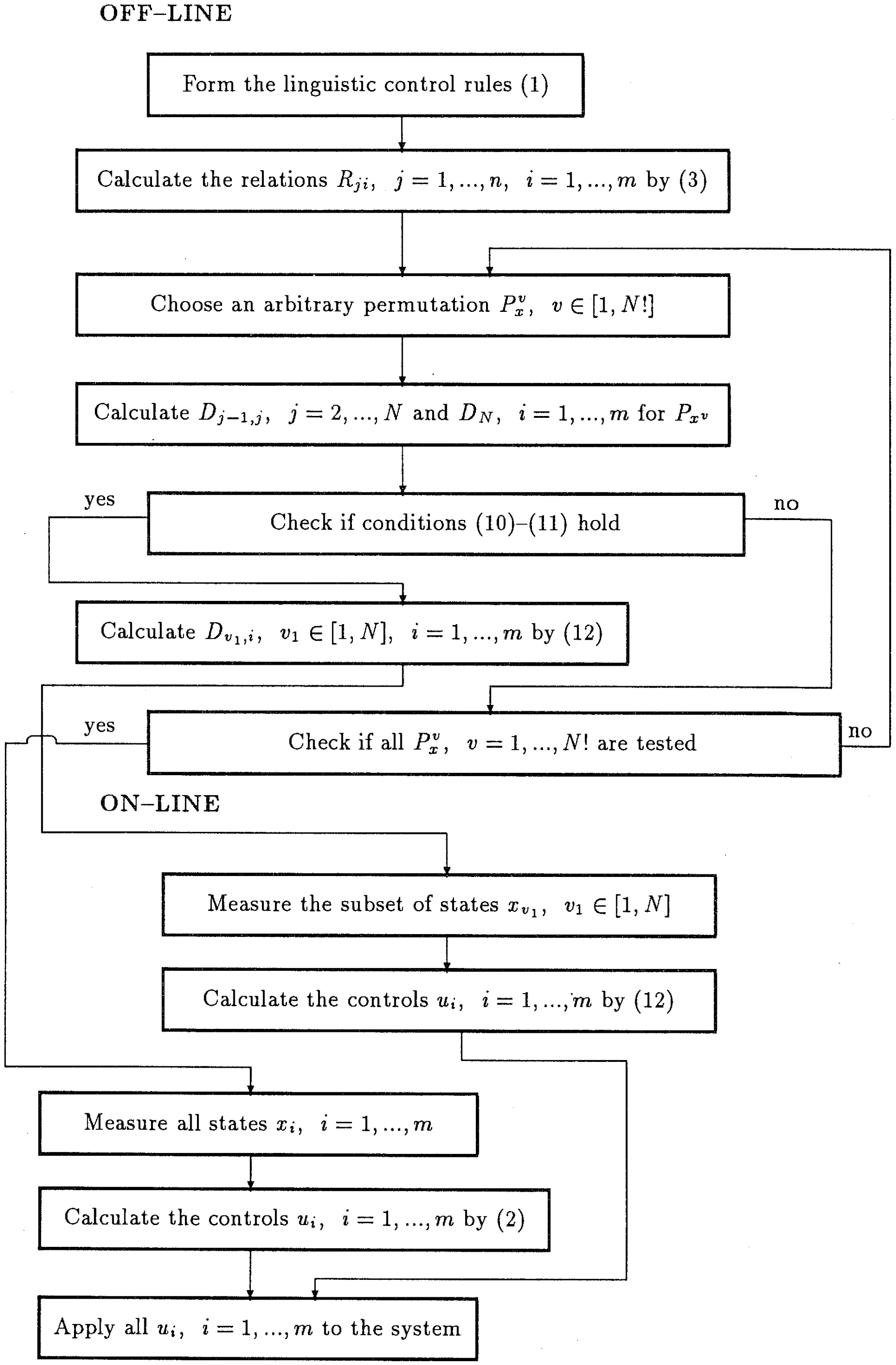

Fig. 6. Block-diagram of the control algorithm: the SISO case.

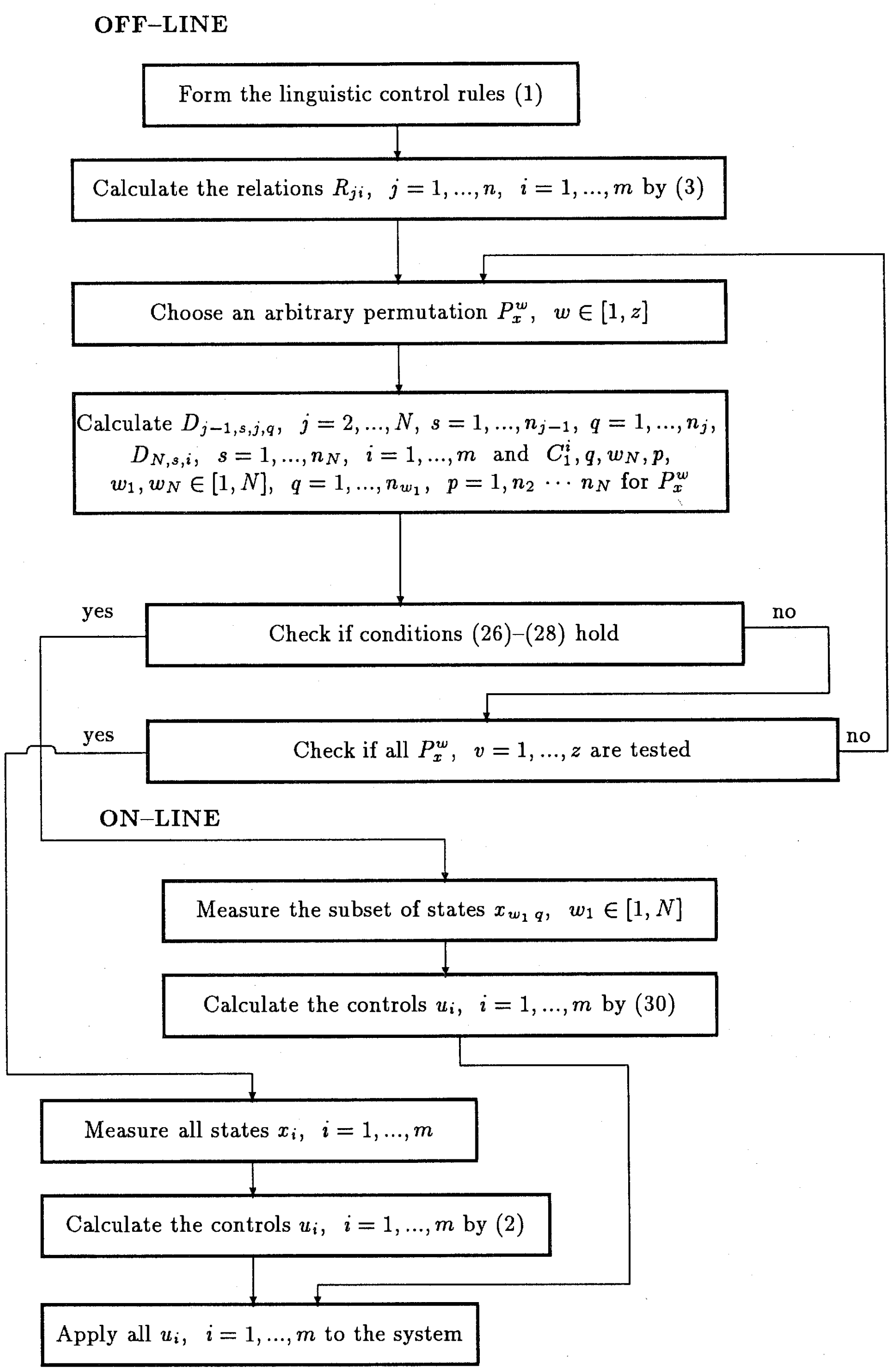

Fig. 7. Block-diagram of the control algorithm: the NSISO case.

5. Numerical Examples

Two numerical examples are presented below for illustration of the theoretical results from the preceding sections. The first example refers to systems decomposed into SISO layers, whereas the other one — to systems decomposed into NSISO layers. The linguistic control rules in both examples can be observed in multi-tank systems, where the state variables are the liquid levels in each tank and the control variables are the inflow rates of the liquid in the tanks. These rules are similar to those presented in (Gupta *et al.*, 1986) where the task of the control system is to maintain the liquid at a desired level in spite of the presence of disturbing leaks.

The control system usually involves an operator who has an assigned control goal and performs visual observation of the state and control variables. In addition to that, it evaluates these variables intuitively and alters manually the system control actions in order to achieve the assigned control goal.

Example 1. A two-tank system shown in Fig. 8 is considered. One of the tanks can be filled with liquid through a separate inflow channel while the other tank is only interconnected. Therefore the corresponding fuzzy control system has two inputs and one output. The state and control variables can take the following linguistic values: S – small, M - - medium, B – big. These values are presented by:

$$S = [1.0, 0.5, 0.5], \quad M = [0.5, 1.0, 0.5], \quad B = [0.5, 0.5, 1.0] \tag{31}$$

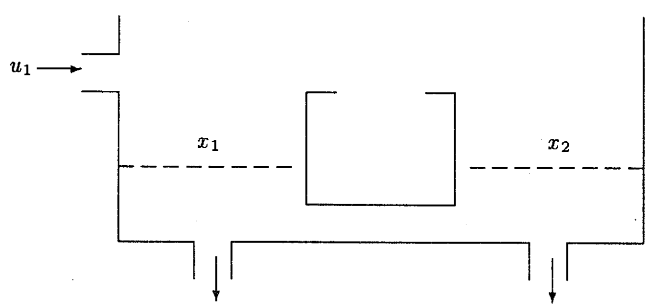

Fig. 8. A two-tank fuzzy control system: the SISO case.

The system is described by the following linguistic rules:

$$\begin{array}{llllll} \text{If} & x_{1(1)} = M & \text{and} & x_{2(1)} = M & \text{Then} & u_{1(1)} = B \\ \text{If} & x_{1(2)} = B & \text{and} & x_{2(2)} = B & \text{Then} & u_{1(2)} = M \end{array} \tag{32}$$

The corresponding fuzzy relations in the control law (2) are calculated via (3) as follows:

$$R_{11} = R_{21} = \left(M \cap B\right) \cup \left(B \cap M\right) = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1.0 \\ 0.5 & 1.0 & 0.5 \end{bmatrix} \tag{33}$$

The system should be decomposed into two SISO layers, each of them with one input state variable. In this case, the correspodding relations are calculated as follows:

$$D_{1,2} = \left(M \cap M\right) \cup \left(B \cap B\right) = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 1.0 & 0.5 \\ 0.5 & 0.5 & 1.0 \end{bmatrix} \tag{34}$$

$$D_{2,1} = \left(M \cap B\right) \cup \left(B \cap M\right) = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1.0 \\ 0.5 & 1.0 & 0.5 \end{bmatrix}$$

$$D_{1,1} = D_{1,2} \circ D_{2,1} = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1.0 \\ 0.5 & 1.0 & 0.5 \end{bmatrix} \tag{35}$$

It is evident from (33) and (35) that conditions (10)–(11) hold and therefore system (32) may be fully decomposed into two SISO subsystems.

To verify the above result, it is supposed that the linguistic values of the state variables are $x_1 = M$ and $x_2 = M$. The calculation of the linguistic values of the control variable u_1 by the original and multilayer control laws (2) and (12) leads to the following overlapping results:

$$u_1 = \left(M \circ R_{11}\right) \cap \left(M \circ R_{21}\right) = \left[0.5, 0.5, 1.0\right] = B \tag{36}$$

$$u_1 = M \circ D_{1,1} = \left[0.5, 0.5, 1.0\right] = B \tag{37}$$

Example 2. A four-tank system shown in Fig. 9 is considered. One of the tanks can be filled with liquid through a separate inflow channel while the other three tanks are only interconnected. Therefore the corresponding fuzzy control system has four inputs and one output. The state and control variables can take the same linguistic values as in Example 1 and are also expressed by (31).

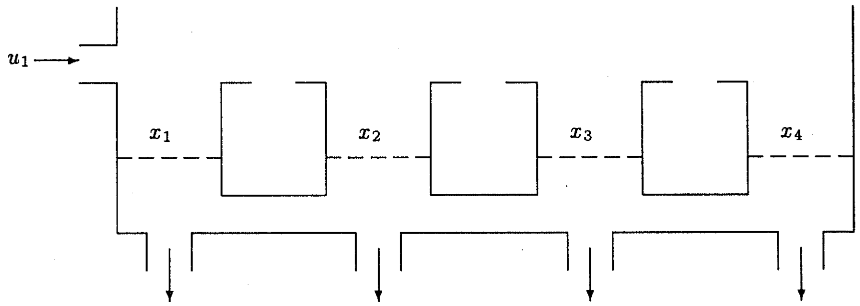

Fig. 9. A four-tank fuzzy control system: the NSISO case.

The system is described by the following linguistic rules:

$$\begin{array}{l} \text{If } x_{1(1)} = S \text{ and } x_{2(1)} = S \text{ and } x_{3(1)} = S \text{ and } x_{4(1)} = S \text{ Then } u_{1(1)} = M \\ \text{If } x_{1(2)} = M \text{ and } x_{2(2)} = M \text{ and } x_{3(2)} = M \text{ and } x_{4(2)} = M \text{ Then } u_{1(2)} = S \end{array} \tag{38}$$

The corresponding fuzzy relations in the control law (2) are calculated via (3) as follows:

$$R_{11} = R_{21} = R_{31} = R_{41} = \left(S \cap M\right) \cup \left(M \cap S\right) = \begin{bmatrix} 0.5 & 1.0 & 0.5 \\ 1.0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{bmatrix} \tag{39}$$

The system should be decomposed into two NSISO layers, each of them with two input state variables. Thus the following substitutions are made:

$$x_{11} = x_1, \quad x_{12} = x_2, \quad x_{21} = x_3, \quad x_{22} = x_4$$

$$R_{111} = R_{11}, \quad R_{121} = R_{21}, \quad R_{211} = R_{31}, \quad R_{221} = R_{41} \tag{40}$$

The corresponding relations are calculated as follows:

$$D_{1,1,2,1} = D_{1,2,2,1} = D_{1,1,2,2} = D_{1,2,2,2} = \left(S \cap S\right) \cup \left(M \cap M\right) = \begin{bmatrix} 1.0 & 0.5 & 0.5 \\ 0.5 & 1.0 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{bmatrix} \tag{41}$$

$$D_{2,1,1} = D_{2,2,1} = \left(S \cap M\right) \cup \left(M \cap S\right) = \begin{bmatrix} 0.5 & 1.0 & 0.5 \\ 1.0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{bmatrix} \tag{42}$$

Further, the following notation and transformations are introduced:

$$x_1 = \left[x_{11}, x_{12}\right], \qquad x_2 = \left[x_{21}, x_{22}\right]$$

$$u_1 = x_2 * \left[D_{2,1,1}, D_{2,2,1}\right]^T = \left[x_{21}, x_{22}\right] * \left[D_{2,1,1}, D_{2,2,1}\right]^T$$

$$x_{21} = x_1 * \left[D_{1,1,2,1}, D_{1,2,2,1}\right]^T = \left[x_{11}, x_{12}\right] * \left[D_{1,1,2,1}, D_{1,2,2,1}\right]^T$$

$$x_{22} = x_1 * \left[D_{1,1,2,2}, D_{1,2,2,2}\right]^T = \left[x_{11}, x_{12}\right] * \left[D_{1,1,2,2}, D_{1,2,2,2}\right]^T$$

$$u_1 = \left(\left[x_{11}, x_{12}\right] * \left[D_{1,1,2,1}, D_{1,2,2,1}\right]^T, \left[x_{11}, x_{12}\right] * \left[D_{1,1,2,2}, D_{1,2,2,2}\right]^T\right) * \left[D_{2,1,1}, D_{2,2,1}\right]^T$$

$$D_{1,1,2,1} \circ D_{2,1,1} = C^1_{1,1,2,1}, D_{1,2,2,1} \circ D_{2,1,1} = C^1_{1,2,2,1}$$

$$D_{1,1,2,2} \circ D_{2,2,1} = C^1_{1,1,2,2}, D_{1,2,2,2} \circ D_{2,2,1} = C^1_{1,2,2,2} \tag{43}$$

The above relations and the corresponding multilayer control law are obtained in the following form:

$$C^1_{1,1,2,1} = C^1_{1,2,2,1} = C^1_{1,1,2,2} = C^1_{1,2,2,2} = \begin{bmatrix} 0.5 & 1.0 & 0.5 \\ 1.0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{bmatrix} \tag{44}$$

$$u_1 = \left[x_{11}, x_{12}, x_{11}, x_{12}\right] * \left[C^1_{1,1,2,1}, C^1_{1,2,2,1}, C^1_{1,1,2,2}, C^1_{1,2,2,2}\right]^T \tag{45}$$

It is evident from (39) and (44) that conditions (26)–(28) hold and therefore system (38) may be fully decomposed into two NSISO subsystems.

To verify the above result, it is supposed that the linguistic values of the state variables are $x_i = M,\ i = 1, 4$. The calculation of the linguistic values of the control variable u_1 by the original and multilayer control laws (2) and (25) leads to the following overlapping results:

$$u_1 = \left(M \circ R_{111}\right) \cap \left(M \circ R_{121}\right) \cap \left(M \circ R_{211}\right) \cap \left(M \circ R_{221}\right)$$

$$= [1.0, 0.5, 0.5] = S \tag{46}$$

$$u_1 = \left(M \circ C^1_{1,1,2,1}\right) \cap \left(M \circ C^1_{1,2,2,1}\right) = [1.0, 0.5, 0.5] = S \tag{47}$$

6. Analysis of Results

The proposed method of multilayer fuzzy control by passive decomposition reduces the number of fuzzy relations in the *on-line* control laws. For systems fully decomposable into N SISO layers, there is only one relation for the calculation of each control variable. At the same time, the corresponding number of relations in the original control law (2) is n. Analogously, for systems fully decomposable into N NSISO layers, there are n_1 relations for the calculation of each control variable and this number is usually much smaller than n.

The reduction is based on the use of a subset of state variables, leading to a unilayer solution by taking into account implicitly the other variables. As a result, the amount of *on-line* computations is reduced. Another advantage of the method is the reduction of *on-line* measurements due to the partial use of state variables. Therefore the method facilitates the real-time measurement and control implementation.

The method is suitable for multivariable systems and is based on partial or full decomposition of the system into layers. Such decomposition can be achieved only if appropriate conditions are fulfilled. However, it may be possible to expand the range of fulfilment of these conditions by a suitable choice of the fuzzy membership functions by fuzzification.

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