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A NOTE ON APPLICATIONS OF INTERPOLATION THEORY TO CONTROL PROBLEMS OF INFINITE-DIMENSIONAL SYSTEMS†‡

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In this paper we shall give two examples of how interpolation theory plays an important role in control theory. The first one is proving the necessary and sufficient conditions for admissibility of a one-dimensional input vector for a diagonal semigroup. The other problem is showing lack of optimizability for the well-known Zabczyk example (Zabczyk, 1975).

1. Introduction

Complex analysis has played an important role in control theory since its inception. For instance, complex analysis appears the Nyquist stability test, Bode diagrams, and more recently, $\boldsymbol{H}_\infty$–control. These applications of complex analysis are in the frequency domain. In this paper, we focus on the use of complex analysis, in particular interpolation theory, in questions concerning time-domain properties. Roughly speaking, interpolation theory deals with the following two questions:

1. Given two sequences of complex numbers, a_n and b_n, when does there exist a holomorphic function with certain growth properties such that $f(a_n) = b_n$?

2. Given a complex valued function with certain growth properties and a set of numbers in its domain, what is the image of this set of numbers?

The fact that interpolation problems are closely related to moment problems make them useful in control theory, in particular in solving controllability questions — see e.g. (Fattorini, 1975; Parks, 1973; Russell, 1967; 1978). In this paper, the focus is on the use of interpolation problems in proving admissibility and lack of optimizability.

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2. Admissible Input Operators

In this section, we reprove a well-known result from (Ho and Russell, 1983; Weiss, 1988) in a simple way, by using strong results from interpolation theory. We remark that a different, but also simple, proof was given recently by Grabowski (1995).

Consider the abstract equation

$$\dot{z}(t) = Az(t) + bu(t), \qquad z(0) = z_0 \tag{1}$$

where A is an infinite matrix given by

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & & \\ 0 & \lambda_2 & 0 & & \\ \vdots & 0 & \lambda_3 & 0 & \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix} \tag{2}$$

with $\mathrm{Re}(\lambda_n) < -\varepsilon < 0$ and b is an infinite vector

$$b = \begin{pmatrix} b_1 & b_2 & b_3 & \cdots \end{pmatrix}^T \tag{3}$$

System (1) is then equivalent to

$$\dot{z}_n(t) = \lambda_n z_n(t) + b_n u(t) \tag{4}$$

for all n. It can be shown that A generates a C_0-semigroup on ℓ_2, the space of all square summable sequences (see e.g. Curtain and Zwart, 1995, Chapter 2), so the solution of (1) is, for each $t > 0$, an element of ℓ_2, provided that $u \equiv 0$. If this is true for any locally square integrable input function u, then b is said to be *admissible* (Weiss, 1988). The following definition states this precisely.

Definition 1. Let $T(t)$ be the C_0-semigroup generated by A. The input operator b is admissible for $T(t)$ if there exists a $t_0 > 0$ such that

$$\int_0^{t_0} T(t_0 - \tau) b u(\tau)\, \mathrm{d}\tau \tag{5}$$

is an element of ℓ_2 for all $u \in \boldsymbol{L}_2(0, t_0)$.

Since $T(t)x_0 \in \ell_2$ for every $t \geq 0$ and $x_0 \in \ell_2$, we have that if the system is admissible, there is at least one t for which the solution of (1) lies in ℓ_2. In the next lemma it is shown that admissibility implies that (1) has a solution which lies in ℓ_2 for all $t > 0$ and all locally square integrable u. Furthermore, we shall show that we can take t_0 equal to infinity.

Lemma 1. *Consider system (1) with assumptions (2) and (3).*
1. *b is admissible for $T(t)$ if and only if*

$$\int_0^{t} T(t - \tau) b u(\tau)\, \mathrm{d}\tau \in \ell_2 \tag{6}$$

for all $u \in \boldsymbol{L}_2(0, t)$ and all $t > 0$.

2. *b is admissible for $T(t)$ if and only if*

$$\int_0^\infty T(\tau)bu(\tau)\,\mathrm{d}\tau \in \ell_2 \tag{7}$$

for all $u \in \boldsymbol{L}_2(0,\infty)$.

Proof. The proof of the first part can be found in (Weiss, 1989). To prove the second part, let us assume that $u \in \boldsymbol{L}_2(0,\infty)$, so from the first part we know that the sequence $\{z_n\}$, $z_n := \int_0^n T(\tau)bu(\tau)\,\mathrm{d}\tau$, is in ℓ_2. If $M > N$, then it follows easily from the semigroup property of $T(t)$ that

$$z_M - z_N = \sum_{k=0}^{M-N-1} T(N+k) \int_0^1 T(\tau)bu(\tau+N+k)\,\mathrm{d}\tau$$

Since the system is exponentially stable and b is admissible, it follows from this equality that z_n is a Cauchy sequence in the Hilbert space ℓ_2. So it has a limit $\int_0^\infty T(\tau)bu(\tau)\,\mathrm{d}\tau$ in ℓ_2. ∎

The operator $T(t)$ has the form

$$T(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & & \\ 0 & e^{\lambda_2 t} & 0 & & \\ \vdots & 0 & e^{\lambda_3 t} & 0 & \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix} \tag{8}$$

From this, it follows that (7) can be written as

$$\begin{pmatrix} \int_0^\infty e^{\lambda_1 \tau} b_1 u(\tau)\,\mathrm{d}\tau \\ \int_0^\infty e^{\lambda_2 \tau} b_2 u(\tau)\,\mathrm{d}\tau \\ \vdots \end{pmatrix} \tag{9}$$

Note that each element in this vector can be seen as the Laplace transform of u at a certain point. So if we denote the Laplace transform of u by $\hat{u}$, then we can write this vector as

$$\begin{pmatrix} \hat{u}(-\lambda_1)b_1, & \hat{u}(-\lambda_2)b_2, & \cdots \end{pmatrix}^T \tag{10}$$

Hence the input operator is admissible if and only if the vector (10) lies in ℓ_2, i.e.

$$\sum_{n=1}^\infty |\hat{u}(-\lambda_n)b_n|^2 < \infty \tag{11}$$

for all $u \in \boldsymbol{L}_2(0,\infty)$.

Note the Laplace transform of an $L_2(0,\infty)$-function is an H_2-function, that is, analytic in the right half-plane and square integrable over every line parallel to the imaginairy axis (see Hoffman, 1962). The admissibility of the input vector b is then equivalent to summability properties of an H_2-function. The following theorem of Carleson (Garnett, 1981, Theorem 5.6, p. 33) is therefore relevant.

Theorem 1. *The following assertions are equivalent:*

1. *The measure σ is a Carleson measure, i.e. it is a positive measure on the right half-plane $\mathbb{C}_0^+ = \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$ satisfying*
$$\sigma\Big(R(h,w)\Big) \leq c_1 h \tag{12}$$
for all rectangles $R(h,w) := \{x + \jmath y \in \mathbb{C} \mid 0 < x < h, |y - w| < h\}$.
2. *For all $f \in H_2$, the following inequality holds*
$$\int_{\mathbb{C}_0^+} |f(s)|^2 \,\mathrm{d}\sigma(s) \leq c_2 \|f\|_{H_2}^2 \tag{13}$$

From this theorem we have the following corollary.

Corollary 1. *Consider system (1) with A and b given by (2) and (3), respectively. Then the input vector b is admissible if and only if the following holds:*
$$\sum_{-\lambda_n \in R(h,w)} |b_n|^2 \leq mh \tag{14}$$
for some m independent of h and w.

Proof. If we define the positive measure σ as
$$\sigma\Big(R(h,w)\Big) = \sum_{-\lambda_n \in R(h,w)} |b_n|^2$$
then it is easy to see that
$$\int_{\mathbb{C}_0^+} |f(s)|^2 \,\mathrm{d}\sigma(s) = \sum_{n=1}^{\infty} |f(-\lambda_n)|^2 |b_n|^2$$
The corollary follows from this and Theorem 1. ∎

Relation (14) is known as the *Carleson measure criterion.* We apply it to a simple example.

Example 1. Consider the heat equation with boundary control action:
$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t), \quad \frac{\partial z}{\partial x}(0,t) = 0, \quad \frac{\partial z}{\partial x}(1,t) = u(t), \quad z(x,0) = z_0(x)$$
From Example 3.3.5 of (Curtain and Zwart, 1995), we have that this partial differential equation can be formulated as (1), where $\lambda_n = -\pi^2(n-1)^2$, $b_1 = 0$, and $b_n = (-1)^{n+1}\sqrt{2}$. As can be seen from the eigenvalue at zero, this system is not stable.

However, it is easy to see that b is admissible for $T(t)$ if and only if it is admissible for $e^{-\varepsilon t}T(t)$. Hence we define now $\lambda_n^\varepsilon := \lambda_n - \varepsilon$ and check the inequality (13). Since all the eigenvalues lie on the real axis, we only have to check this inequality for $w = 0$. Hence we have to calculate

$$\sum_{n^2\pi^2+\varepsilon \in R(h,0)} |b_n|^2$$

There are approximately $\sqrt{h}/\pi$ n's in $R(h,0)$, and so this sum can be bounded by $c\sqrt{h}$, for some constant c. In other words, this input operator is admissible.

For other examples, we refer to (Ho and Russell, 1983).

3. Optimizability of the Zabczyk Example

The *Zabczyk example* (Zabczyk, 1975) has an infinitesimal generator which is block diagonal, where the blocks are Jordan ones with growing sizes. Let

$$A = \begin{pmatrix} A_1 & 0 & \cdots & & \\ 0 & A_2 & 0 & & \\ \vdots & 0 & A_3 & 0 & \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix} \tag{15}$$

where A_n is the $n \times n$ matrix

$$A_n = \begin{pmatrix} \lambda_n & 1 & 0 & \cdots & 0 \\ 0 & \lambda_n & 1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \\ \vdots & \ddots & \ddots & & \\ 0 & \cdots & & 0 & \lambda_n \end{pmatrix} \tag{16}$$

Furthermore, we assume that the λ_n are unequal and all have real parts equal to one. This operator generates a C_0-semigroup that does not satisfy the spectrum-determined growth assumption. As is shown in (Zabczyk, 1975) the growth of this semigroup is two, but the maximum over the real part of the eigenvalues is one. Since this system is unstable, it is logical to ask whether or not it is stabilizable by some feedback law. Here we shall investigate this question for one-dimensional feedback operators. Since we additionally want the input trajectories generated by the stabilizing feedback to be square integrable, our stabilizability question is equivalent to optimizability. We shall show that the answer to this question is negative.

We take the input vector b in the form (3), but now the b_n's are vectors in $\mathbb{C}^n$. We assume that all the elements in these vectors are uniformly bounded, i.e. the ℓ_∞-norm of the infinite vector b is finite.

The problem we discuss in this section is whether or not we can choose for every $z_0 \in \ell_2$ an input $u \in \boldsymbol{L}_2(0,\infty)$ such that the solution $z(t)$ of

$$\dot{z}(t) = Az(t) + bu(t), \qquad z(0) = z_0 \tag{17}$$

is square integrable, i.e.

$$\int_0^\infty \|z(t)\|^2 \, \mathrm{d}t < \infty \tag{18}$$

In other words, can the cost functional

$$J(z_0, u) = \int_0^\infty \|z(t)\|^2 + \|u(t)\|^2 \, \mathrm{d}t \tag{19}$$

be made finite for every initial condition? This is commonly known as *optimizability.* Since we are working with a general input operator, we have to be more precise what is meant by the solution of (17). We say that $(z(\cdot), u(\cdot))$ satisfies (17) if and only if

$$z_n(t) = e^{A_n t} z_{n,0} + \int_0^t e^{A_n(t-s)} b_n u(s) \, \mathrm{d}s$$

where $z_n(\cdot)$ and b_n denote the n-th component vector of $z(\cdot)$ and b, respectively. Hence we have decomposed the state according to the generator A, see (15).

In the next lemma, we show that the input that makes the cost finite can be chosen in a special way.

Lemma 2. *If system (17) is optimizable, then for each $z_0 \in \ell_2$ there exists an input u such that*

$$J(z_0, u) \leq M^2 \|z_0\|^2 \tag{20}$$

for some M independent of z_0 and u.

Proof. We prove this by using the Baire Category Theorem.

Step 1. First we remark that the square root of the cost functional can be seen as the norm on the Hilbert space $\mathcal{X} := \boldsymbol{L}_2((0,\infty); Z) \oplus \boldsymbol{L}_2(0,\infty)$. Let

$$V_N = \left\{ z_0 \in Z \mid \text{there exists an input } u \text{ such that } J(z_0, u) \leq N \right\}$$

Since the system is optimizable, we have that

$$\bigcup_{N=1}^{\infty} V_N = Z \tag{21}$$

In order to apply the Baire Category Theorem, we need to show that V_N is a closed subset for every N.

Step 2. Let z_0^n be a sequence in V_N which converges to z_0 in Z. Denote by $(z^n(\cdot), u^n(\cdot))$ the state and input trajectories, respectively, that satisfy $J(z_0^n, u^n) \leq N$. So in the Hilbert space $\mathcal{X}$, we have that $\|(z^n, u^n)\|_{\mathcal{X}}$ is a uniformly bounded sequence. Thus it contains a weakly converging subsequence. We denote the limit of this subsequence by (z, u). Since we can always delete the elements in the sequence (z^n, u^n) that are not part of this converging subsequence, we may without loss of generality assume that (z^n, u^n) is weakly convergent. For weakly convergent sequences, we have

$$\|(z,u)\|_{\mathcal{X}}^2 \leq \liminf_n \|(z^n, u^n)\|_{\mathcal{X}}^2 \leq N \tag{22}$$

Furthermore, it is not hard to see that (z, u) satisfies (17). Hence V_N is a closed subset.

Step 3. By the Baire Category Theorem, there exists an N_0 such that V_{N_0} contains an open ball $B(c, \rho)$. Let z_0 be an arbitrary element of Z, and let x be defined as

$$x = c + \frac{\rho}{2\|z_0\|} z_0$$

Then it is clear that $x \in B(c, \rho)$. Hence x satisfies $J(x, u^x) \leq N_0$ for some u^x. For c we can also construct an input u_c such that $J(c, u^c) \leq N_0$. Since the system is linear, it is easy to see that for the initial state z_0 and the input

$$u(\cdot) = \left[u^x(\cdot) - u^c(\cdot)\right] \frac{2\|z_0\|}{\rho}$$

the state is given by

$$z(\cdot) = \left[z^x(\cdot) - z^c(\cdot)\right] \frac{2\|z_0\|}{\rho}$$

Hence, since z and c are in $B(c, \rho) \subset V_{N_0}$,

$$\begin{aligned} \sqrt{J(z_0, u)} = \|(z,u)\|_{\mathcal{X}} &\leq \left[\|(z^x, u^x)\|_{\mathcal{X}} + \|(z^c, u^c)\|_{\mathcal{X}}\right] \frac{2\|z_0\|}{\rho} \\ &\leq \left[2\sqrt{N_0}\right] \frac{2\|z_0\|}{\rho} = M\|z_0\| \end{aligned}$$

But z_0 is arbitrary, so we have proved the assertion. ■

Taking the Laplace transform of (17), we obtain

$$\xi(s) = (sI - A)^{-1} z_0 + (sI - A)^{-1} b\omega(s) \tag{23}$$

where ω and ξ are the Laplace transforms of $u(\cdot)$ and $x(\cdot)$, respectively. Denote by $\boldsymbol{H}_2(\ell_2)$ the set of $\boldsymbol{H}_2$-functions with values in the Hilbert space ℓ_2. If the system is optimizable, then the complex-valued functions ξ and ω are in $\boldsymbol{H}_2(\ell_2)$ and $\boldsymbol{H}_2$, respectively.

Consider the initial condition $z_0 = \tilde{e}_n$, where

$$\tilde{e}_n = \begin{pmatrix} 0, & \cdots, & 0, & e_n^T, & 0, & \cdots \end{pmatrix}^T$$

with

$$e_n = \begin{pmatrix} \alpha^{-n+1}, \cdots, \alpha^{-1}, 1 \end{pmatrix}^T \tag{24}$$

and $\alpha > 1$.

Using the block structure of A and b we can write (23) as

$$\xi_n(s) = (sI_n - A_n)^{-1}\Big[e_n - b_n\omega^n(s)\Big] \tag{25}$$

where I_n denotes the identity on $\mathbb{C}^n$, ξ_n stands for the n-th block of ξ corresponding to $\tilde{e}_n$, and ω^n is the ω corresponding to $\tilde{e}_n$. By the special structure of A_n, we can write (25) as

$$\begin{pmatrix} \xi_{n,1}(s) \\ \vdots \\ \xi_{n,n}(s) \end{pmatrix} = \begin{pmatrix} \frac{1}{s-\lambda_n} & \cdots & \frac{1}{(s-\lambda_n)^n} \\ \vdots & \ddots & \\ 0 & & \frac{1}{s-\lambda_n} \end{pmatrix} \left[\begin{pmatrix} e_{n,1} \\ \vdots \\ e_{n,n} \end{pmatrix} + \begin{pmatrix} b_{n,1} \\ \vdots \\ b_{n,n} \end{pmatrix} \omega^n(s) \right] \tag{26}$$

Multiplying the top row by $(s-\lambda_n)^n$, we obtain

$$\Big[(s-\lambda_n)^{n-1}e_{n,1} + \cdots + e_{n,n}\Big] + \Big[(s-\lambda_n)^{n-1}b_{n,1} + \cdots + b_{n,n}\Big]\omega^n(s) = (s-\lambda_n)^n\xi_{n,1}(s) \tag{27}$$

We introduce the variable $\tau := s - \mathrm{Im}(\lambda_n)\jmath$. Since the real part of λ_n is 1, using (24) we obtain

$$\left[\left(\frac{\tau-1}{\alpha}\right)^{n-1} + \cdots + 1\right] + \Big[(\tau-1)^{n-1}b_{n,1} + \cdots + b_{n,n}\Big]\omega^n\Big(\tau + \mathrm{Im}(\lambda_n)\jmath\Big)$$
$$= (\tau-1)^n\xi_{n,1}\Big(\tau + \mathrm{Im}(\lambda_n)\jmath\Big) \tag{28}$$

Since the shift in the argument of $\xi_{n,1}$ and ω^n are purely imaginary, we have that $\xi_{n,1}(\cdot + \mathrm{Im}(\lambda_n)\jmath)$ and $\omega^n(\cdot + \mathrm{Im}(\lambda_n)\jmath)$ are still $\boldsymbol{H}_2$-functions. Furthermore, their norms do not change. Apart from that,

$$|\xi_{n,1}(\tau + \mathrm{Im}(\lambda_n)\jmath)| = \left|\int_0^\infty e^{-(\tau+\mathrm{Im}(\lambda_n)\jmath)t} x_{n,1}(t)\,\mathrm{d}t\right|$$

$$\leq \frac{1}{\sqrt{2\mathrm{Re}(\tau)}}\|x_{n,1}\|_{L_2(0,\infty)}, \quad \text{by the Cauchy-Schwarz inequality}$$

$$\leq \frac{M}{\sqrt{2\mathrm{Re}(\tau)}}\|\tilde{e}_n\|_{\ell_2}, \quad \text{by (20)}$$

$$\leq \frac{M}{\sqrt{2\mathrm{Re}(\tau)}}\frac{\alpha}{\sqrt{\alpha^2-1}} \tag{29}$$

Therefore we have that the sequence $\xi_{n,1}(\tau + \mathrm{Im}(\lambda_n)\jmath)$ is uniformly bounded on the disc $B(1, \frac{1}{2})$, i.e. the disc with center 1 and radius $1/2$.

Furthermore, we have

$$\|\omega^n(\,\cdot\, + \mathrm{Im}(\lambda_n)\jmath)\|_{H_2} = \|\omega^n(\cdot)\|_{H_2} = \|u^n\|_{L_2(0,\infty)} \leq M\|\tilde{e}_n\|_{\ell_2} \leq \frac{M\alpha}{\sqrt{\alpha^2-1}} \tag{30}$$

where we have used the fact that $\boldsymbol{H}_2$ and $\boldsymbol{L}_2(0,\infty)$ are isomorphic, and eqn. (20).

Now we have all the ingredients to prove the contradiction. Since the vector b has a bounded ℓ_∞-norm, we have that $\{(\tau-1)^{n-1}b_{n,1} + \cdots + b_{n,n}\}$ is a bounded sequence of functions in the disc $B(1, \frac{1}{2})$. Hence it has a convergent subsequence. We denote this limit function by $\mathcal{B}(\tau)$. For this subsequence, we see from (30) that the $\boldsymbol{H}_2$-sequence $\omega^n(\cdot + \mathrm{Im}(\lambda_n)\jmath)$ is uniformly bounded. Since $\boldsymbol{H}_2$ is a Hilbert space, we know that every bounded sequence has a weakly converging subsequence, hence $\{\omega^n(\cdot + \mathrm{Im}(\lambda_n)\jmath)\}$ has a weakly converging subsequence. We denote the index set of this final subsequence by n_k and the weak limit of $\{\omega^n(\cdot + \mathrm{Im}(\lambda_n)\jmath)\}$ by $\omega_\infty \in \boldsymbol{H}_2$.

Note that for $\boldsymbol{H}_2$-functions, weak convergence implies pointwise convergence. So letting $n \to \infty$ along the subsequence n_k, we see that for $\tau \in B(1, \frac{1}{2})$ eqn. (28) becomes

$$\frac{\alpha}{\alpha - \tau + 1} + \mathcal{B}(\tau)\omega_\infty(\tau) = 0 \tag{31}$$

Since ω_∞ is analytic in the right half-plane, and since the first function is meromorphic on the complex plane, we see from this equation that $\mathcal{B}$ has a meromorphic extension to the right half-plane. Furthermore, using the analyticity of ω_∞ once again, we see that $\mathcal{B}$ must have a pole at $\tau = \alpha + 1$. Now the construction of $\mathcal{B}$ is independent of the initial condition, hence independent of α, so it follows that $\mathcal{B}$ must have a pole at every real point larger than 1. This is impossible for a meromorphic function. Hence we have a contradiction, and so system (17) cannot be optimizable.

Note that lack of optimizability also excludes stabilizability for a large class of feedbacks. For more about the lack of stabilizability, we refer to the paper (Rebarber and Zwart, 1996).

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References

Curtain R.F. and Zwart H.J. (1995): *An Introduction to Infinite-Dimensional Systems Theory.* — TAM-21, New York: Springer Verlag.

Fattorini H.O. (1975): *Exact controllability of linear systems in infinite dimensional spaces*, In: Partial Differential Equations and Related Topics (J.A. Goldstein, Ed.). — (Program, Tulane, Univ., New Orleans, L.A., 1974), Lecture Notes in Math., v.446, Berlin: Springer Verlag, pp.166–183.

Garnett J.B. (1981): *Bounded Analytic Functions.* — New York: Academic Press.

Grabowski P. (1995): *Admissibility of observation functionals.* — Int. J. Control, v.62, No.5, pp.1161–1175.

Ho L.F. and Russell D.L. (1983): *Admissible input elements for systems in Hilbert space and a Carleson measure criterion.* — SIAM J. Control Optim., v.21, No.4, pp.614–640.

Parks P.C. (1973): *On how to shake a piece of string to a standstill.* In: Recent Mathematical Developments in Control (D.J. Bell, Ed.) — London: Academic Press, pp.267–287.

Hoffman K. (1962): *Banach Spaces of Analytic Functions.* — New York: Dover Publications, Inc.

Rebarber R. and Zwart H. (1996): *Open loop stabilizability of infinite-dimensional systems.* — submitted.

Russell D.L. (1967): *Nonharmonic Fourier series in control theory of distributed parameter systems.* — J. Math. Appl., v.18, No.3, pp.542–560.

Russell D.L. (1978): *Controllability and stabilizability theory for linear partial differential equations; recent progress and open questions.* — SIAM Review, v.20, No.4, pp.639–739.

Weiss G. (1988): *Admissible input elements for diagonal semigroups on ℓ^2.* — Systems and Control Letters, v.10, No.1, pp.79–82.

Weiss G. (1989): *Admissibility of unbounded control operators.* — SIAM J. Control Optim., v.27, No.3, pp.527–545.

Zabczyk J. (1975): *A note on C_0-semigroups.* — Bull. Acad. Polon. Sci. Sèr. Sci. Math. Astronom., v.23, No.6, pp.895–898.

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