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THICKNESS OPTIMIZATION OF A GEOMETRICALLY NON-LINEAR ARCH AT A LIMIT POINT

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An optimization method for geometrically non-linear mechanical structures based on a sensitivity gradient is proposed. This gradient is computed by using an adjoint state equation and the structure is analysed by means of a total Lagrangian formulation. This classical method is well-understood for regular cases, but standard equations (see e.g. Rousselet *et al.*, 1995) have to be modified for the limit-point case. The case of sensitivity of a bifurcation point is under development (see (Mróz and Haftka, 1994) for more details). An arc-length algorithm embedded in the optimization algorithm is built. These modifications introduce numerical problems which occur at limit points (Doedel *et al.*, 1991). All systems are very stiff and the quadratic convergence of the Newton-Raphson algorithm is lost, so higher-order derivatives with respect to state variables have to be computed (Wriggers and Simo, 1990). The thickness distribution of the arch is optimized for differentiable costs under linear and non-linear constraints. Numerical results of optimal design of arches undergoing small and large displacements are given and compared with analytic solutions. Related topics of shape optimization can be found in (Aubert and Rousselet, 1996), and theoretical results with details in (Aubert, 1996).

1. Sensitivity Analysis

Optimal design of beams and arches with respect to the thickness distribution, cross-section area or shape is a wide domain in structural mechanics and applied mathematics. This paper deals with the case where beams and arches are analysed with geometric non-linearities. The optimization process at regular states is first recalled: in order to compute the sensitivity gradient of the cost functional with respect to the design, one adjoint state equation is solved. This equation is modified to compute the sensitivity gradient at a critical (turning-point) state. A differentiable cost at the post-buckling state or the critical load are optimized. The second derivatives of the critical load with respect to the state variables and design variables are computed. With these two Hessians, the convergence in both the direct non-linear analysis and the optimization process is accelerated.

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1.1. Standard Equations

Our shape optimization method for structural mechanics is based on a shape gradient computed through an adjoint state equation. This construction is motivated by (Bernadou *et al.*, 1991; Habbal, 1992; Phelan *et al.*, 1991). Let $\mathcal{V}$ denote a functional space for the state variable and $\mathcal{X}$ denote another functional space for the design variables. In the arch problem where both ends are clamped, $\mathcal{V}$ is the Sobolev space $H_0^2(0,L)$ and $\mathcal{X}$ is $L^\infty(0,L)$. The state equation will be written in the form:
find $\phi \in \mathcal{V}$ such that

$$\forall v \in \mathcal{V}, \qquad G(h;\phi,\lambda)v = 0 \tag{1}$$

and the cost functional is defined by

$$j(h) = J(h;\phi) = \int_0^l \tilde{J}(h;\phi,\phi',\phi'',\lambda)\,\mathrm{d}s_0 \tag{2}$$

where $h \in \mathcal{X}$ stands for the *design* variable, $\phi \in \mathcal{V}$ denotes the *state* of the structure and λ is a scalar variable, e.g. a loading parameter. The quantity h may be the thickness or a parametrization of the geometry. The primes denote derivatives with respect to s_0 which is the curvilinear abscissa of the middle line (of the unloaded beam). The subset of admissible thickness distributions is denoted by $\mathcal{X}_{ad}$. Typically the constraints may be imposed that the total volume equals a constant with bounds on the thickness (see Subsection 3.1). We want to solve the following minimization problem:

$$\min_{h\in\mathcal{X}_{ad}} j(h) \tag{3}$$

Assuming that the following derivatives are meaningful, we can define the design sensitivity of the state equation (1) as follows:
find $\delta\phi \in \mathcal{V}$, $\delta\phi = \partial_h\phi\delta h$, such that

$$\forall v \in \mathcal{V}, \qquad \partial_\phi G(h;\phi,\lambda)[v,\delta\phi] = -\partial_h G(h;\phi,\lambda)[v,\delta h] \tag{4}$$

and the design sensitivity of the cost functional (2) by

$$\forall \delta h \in \mathcal{X}, \qquad \frac{\mathrm{d}j}{\mathrm{d}h}\delta h = \partial_h J(\phi,h)\delta h + \partial_\phi J(\phi,h)\delta\phi \tag{5}$$

The first term of the former equation is explicit in δh but the second term depends implicitly on δh through $\delta\phi$. To make it dependent explicitly on δh, one can invert (4) at the finite-element level (this is the so-called direct approach) or use the following adjoint state equation:
find $p \in \mathcal{V}$ such that

$$\forall v \in \mathcal{V}, \qquad \partial_\phi G(h;\phi,\lambda)^t[v,p] = -\partial_\phi J[v] \tag{6}$$

where the adjoint state p is the solution of this linear system. Then the design sensitivity of the cost functional becomes explicit:

$$\frac{\mathrm{d}j}{\mathrm{d}h}\delta h = \partial_h J(\phi,h)\delta h + \partial_h G(h;\phi,\lambda)[p,\delta h] \tag{7}$$

Definition 1. A solution of (1) is *regular* if $\partial_\phi G$ is invertible, see (Mróz, 1993).

Theorem 1. *At regular states the solution of (4) is unique and depends smoothly on h.*

The proof is based on the implicit-function theorem (see e.g. Rousselet *et al.*, 1995). In this case, (7) can be evaluated as soon as the state ϕ and the adjoint state p are computed.

1.2. Sensitivity Analysis at Limit Points

Let us consider the case where the cost functional is smooth but the linear mapping G_ϕ may not be invertible. For numerical analysis of the approximation of some bifurcation problems in partial differential equations, the reader is referred to (Crouzeix and Rappaz, 1989) and the references given there. The notion of the limit point is recalled following (Doedel *et al.*, 1991).

Definition 2. A solution to (4) is a limit or turning point at (ϕ, λ) if and only if

$$\dim\Big(\mathrm{Ker}(\partial_\phi G)\Big) = 1 \quad \text{and} \quad \partial_\lambda G \notin \mathrm{Im}(\partial_\phi G) \tag{8}$$

Let us consider a curve $\eta \to (h(\eta), \phi(\eta), \lambda(\eta))$, $\eta \in \mathbb{R}$ and let $\delta h, \delta\phi, \delta\lambda$ denote the differentials with respect to η. We differentiate (1) with respect to h, ϕ and λ:

$$\partial_h G \delta h + \partial_\phi G \delta\phi + \partial_\lambda G \delta\lambda = 0 \tag{9}$$

By Definition 2, $\partial_\phi G$ in (4) and (6) is not invertible.

Proposition 1. *The differential $\delta\phi$ is proportional to the right eigenvector.*

The proof of this classical result may be found e.g. in (Doedel *et al.*, 1991). If G_ϕ has zero as a simple eigenvalue, we denote by q (resp. p) the left (resp. right) associated eigenvector:

$$q^t \partial_\phi G = 0, \qquad \partial_\phi G p = 0 \tag{10}$$

At a fixed h, (9) is multiplied by q^t:

$$q^t \partial_\lambda G \, \delta\lambda = 0 \tag{11}$$

So either $\delta\lambda = 0$ and $q^t \partial_\lambda G \neq 0$, and we deal with a limit point, or $q^t \partial_\lambda G = 0$, and we deal with bifurcation. In what follows, we consider the first case of a limit point.

1.2.1. First-Order Derivative

We assume that ϕ and λ are differentiable with respect to h, so we can rewrite (9):

$$\partial_h G\ \delta h + \partial_\phi G \partial_h \phi \delta h + \partial_\lambda G \partial_h \lambda \delta h = 0 \tag{12}$$

We multiply (12) by q^t and we call λ^c the critical load, i.e. the load evaluated at limit points,

$$\partial_h \lambda^c = -\frac{q^t\ \partial_h G}{q^t\ \partial_\lambda G} \tag{13}$$

the denominator being non-zero by the definition of a turning point. The latter equation provides the derivative of the critical load with respect to the design variable. We use the arc-length method at turning points (see Doedel *et al.*, 1991). The incremental equation takes the form

$$\partial_\phi G \delta\phi + \partial_\lambda G \delta\lambda = 0 \tag{14}$$

with auxiliary relation

$$N(\delta\phi, \delta\lambda) = 0 \tag{15}$$

which can be e.g.

$$N(\delta\phi, \delta\lambda) = (\delta\phi)^2 + (\delta\lambda)^2 - 1 \tag{16}$$

To evaluate $\partial_h \phi$, we need to solve (12) with $\partial_\phi G$ singular at the turning point, however in view of (13) there is a solution to (9). The uniqueness of this solution is granted by differentiating the buckling condition (10):

$$\left(\partial_h q^t\right) \partial_\phi G + q^t \left(\partial^2_{\phi^2} G \partial_h \phi + \partial^2_{\phi\lambda} G \partial_h \lambda + \partial^2_{\phi h} G\right) = 0 \tag{17}$$

and multiplying (17) by p:

$$q^t \left(\partial^2_{\phi^2} G \partial_h \phi + \partial^2_{\phi\lambda} G \partial_h \lambda + \partial^2_{\phi h} G\right) p = 0 \tag{18}$$

Then the unknown $\delta q = \partial_h q$ vanishes and with the use of (12) a solution for $\partial_h \phi$ is found. Finally, from $||q||^2 = 1$ we obtain an orthogonality condition on q: $q\ \delta q = 0$, where the element δq is unique.

1.2.2. Second-Order Derivative

It is well-known that the quadratic convergence of the Newton-Raphson algorithm is lost at limit points. In order to recover this fast convergence, a second-order predictor and an arc-length algorithm which remains quadratic at the turning point are used. In both cases we need the second-order derivative of the critical load.

Second-order derivative of the critical load with respect to the state variable. In order to compute the second-order derivatives of the critical load with respect to the state and design variables, we need the second-order derivative of (1).

The quantities $\delta^2 h, \delta^2\phi, \delta^2\lambda$ denote the second-order derivatives with respect to the parameter η of the curve

$$\eta \to \Big(h(\eta), \phi(\eta), \lambda(\eta)\Big)$$

Thus,

$$\begin{aligned}&\partial^2_{h^2}G\,(\delta h)^2 + \partial^2_{\phi^2}G\,(\delta\phi)^2 + \partial^2_{\lambda^2}G\,(\delta\lambda)^2\\&\quad+2\partial^2_{h\phi}G\,(\delta h\delta\phi) + 2\partial^2_{h\lambda}G\,(\delta h\delta\lambda) + 2\partial^2_{\phi\lambda}G\,(\delta\phi\delta\lambda)\\&\quad+\partial_h G\,(\delta^2 h) + \partial_\phi G\,(\delta^2\phi) + \partial_\lambda G\,(\delta^2\lambda) = 0\end{aligned} \tag{19}$$

In the case of a fixed h (i.e. we set $\delta h = 0$), eqn. (19) takes the form

$$\begin{aligned}&\partial^2_{\phi^2}G\,(\delta\phi)^2 + \partial^2_{\lambda^2}G\,(\delta\lambda)^2 + 2\partial^2_{\phi\lambda}G\,(\delta\phi\delta\lambda)\\&\quad+\partial_\phi G\,(\delta^2\phi) + \partial_\lambda G\,(\delta^2\lambda) = 0\end{aligned} \tag{20}$$

Then we assume that λ is twice differentiable with respect to ϕ:

$$\delta\lambda = \partial_\phi\lambda\delta\phi, \qquad \delta^2\lambda = \partial^2_{\phi^2}\lambda(\delta\phi)^2 \tag{21}$$

so, in view of (21), (20) becomes

$$\begin{aligned}&\partial^2_{\phi^2}G\,(\delta\phi)^2 + \partial^2_{\lambda^2}G\,(\partial_\phi\lambda\delta\phi)^2 + 2\partial^2_{\phi\lambda}G\Big(\partial_\phi\lambda(\delta\phi)^2\Big)\\&\quad+\partial_\phi G\delta^2\phi + \partial_\lambda G\partial^2_{\phi^2}\lambda(\delta\phi)^2 = 0\end{aligned} \tag{22}$$

Then we multiply the above equation by q and we evaluate it at the turning point, where $\lambda = \lambda^c$, $\partial_\phi\lambda = 0$ and $\delta\phi$ is a multiple of p. We thus obtain the second-order derivative of the critical load with respect to the state variable:

$$\partial^2_{\phi^2}\lambda^c p^2 = -\frac{q^t\partial^2_{\phi^2}Gp^2}{p^t\partial_\lambda G} \tag{23}$$

Based on this equation, a second-order algorithm can be constructed to speed up the convergence at the turning point, see (Aubert, 1996).

Second-order derivative of the critical load with respect to the design variable. In the remainder of this section, we assume that ϕ and λ are twice differentiable with respect to h which is no longer fixed:

$$\begin{aligned}\delta\phi &= \partial_h\phi\delta h, \qquad & \delta^2\phi &= \partial^2_{h^2}\phi(\delta h)^2 + \partial_h\phi\delta^2 h\\ \delta\lambda &= \partial_h\lambda\delta h, \qquad & \delta^2\lambda &= \partial^2_{h^2}\lambda(\delta h)^2 + \partial_h\lambda\delta^2 h\end{aligned} \tag{24}$$

with

$$G = \partial_{x,y}\Pi(x,y)[\delta x, \delta y]$$
$$= \int_0^l EI(\kappa - \kappa_0)\delta\kappa \, \mathrm{d}s_0 + \int_0^l EA(\epsilon - \epsilon_0)\delta\epsilon \, \mathrm{d}s_0$$
$$+\lambda \int_0^l p(\phi y' - \psi' x) \, \mathrm{d}s_0 - \lambda \int_0^l (f\phi + g\psi) \, \mathrm{d}s_0 = 0 \tag{31}$$

and

$$\mathcal{V}_0 = \Big\{(x_0, y_0) \in (H^3(0,L))^2,\ x_0(0) = x_0(L) = 0,\ x_0'(0) = 0, x_0'(L) = 0\Big\}$$
$$\mathcal{V} = \Big\{(x,y) \in (H^2(0,L))^2,\ x(0) = x(L) = 0,\ y(0) = y(L) = 0\Big\}$$
$$\mathcal{X} = \Big\{h \in L^\infty(0,L)\Big\}$$

Here $\delta\kappa$ and $\delta\epsilon$ are respectively the derivatives of κ and ϵ with respect to the state variable ϕ. We notice that (30) is of the form (1) with $h = (x_0, y_0)$, $\phi = (x,y)$ and $\delta\phi = (\delta x, \delta y)$. Let us also remark that the boundary conditions included in the spaces $\mathcal{V}_0$ and $\mathcal{V}$ are taken just as an example; other types may be considered and are treated in of Section 3.

3. Numerical Results

In this section, numerical results are presented and compared for the linear case with well-known solutions, analytic or numerical, that we were able to find in the literature. For a large load, in some particular cases, it is easy to solve the equilibrium equation, but optimal shapes do not seem to be known analytically.

The first, very simple example is the cantilever beam subjected to a transversal load. We optimize the thickness distribution of the beam for various costs and we show the stability of the algorithm. Then we study a beam which is freely supported or clamped at both ends and we optimize the thickness distribution under compressive forces to find a minimal displacement. We recall the results from (Banichuk, 1983; 1990; Olhoff, 1986), where the analytical solutions are evaluated for these two examples, and we compare them with our solutions. The clamped arch under a concentrated load is the last and most complicated one. The problems are due to the fact that the shape of the beam at the buckling point is very far from the initial state and then the structure is very sensitive with respect to the initial conditions. We optimize the thickness distribution h.

We choose a non-differentiable local cost $j(s_0)$, but to approach the infimum or supremum of this functionals, we regularize the cost with L^p-norm (see Banichuk, 1981):

$$j_p(h) = \left[\int_0^l (j(h, s_0))^p \, \mathrm{d}s_0\right]^{1/p} \tag{32}$$

As p increases, we approach $\inf j$ by j_p. For the case when the cost in not differentiable, we refer the reader to (Habbal, 1992). We use a sequential quadratic-programming method (Laurence *et al.*, 1994) in order to solve the optimization problem. In all computations, ϵ_{opt} denotes the value used in the stopping criteria for the optimization process. In practice, ϵ_{opt} is a triplet composed of ϵ^{c}_{opt}, ϵ^{kt}_{opt}, ϵ^{cr}_{opt}. The algorithm CFSQP stops if the following conditions are met:

$$\left\{\begin{array}{lll} \frac{|J^{n+1}-J^n|}{J^n} & \leq \epsilon^{c}_{opt} & \text{cost} \\ \|\nabla_x L\| & \leq \epsilon^{kt}_{opt} & \text{Kuhn-Tucker vector} \\ \sum_{k=1}^{k=n_c} \|J_k(x_k)\| & \leq \epsilon^{cr}_{opt} & \text{constraints} \end{array}\right. \tag{33}$$

where J^n is the cost functional evaluated at the n-th iteration, L is the Lagrangian for the discrete problem and J_k are the equality constraints. Inequality constraints are under the same condition if they are active. By default, we choose $\epsilon^{c}_{opt} = 1. \times 10^{-4}$, $\epsilon^{kt}_{opt} = 1. \times 10^{-10}$ and $\epsilon^{cr}_{opt} = 1. \times 10^{-6}$.

All computations were performed on a SparcStation 5 with 64 MB of memory and on a Dec-Alpha 400 with 96 MB of memory. All times will be given for the SparcStation.

3.1. Optimal Design of a Freely-Supported Beam

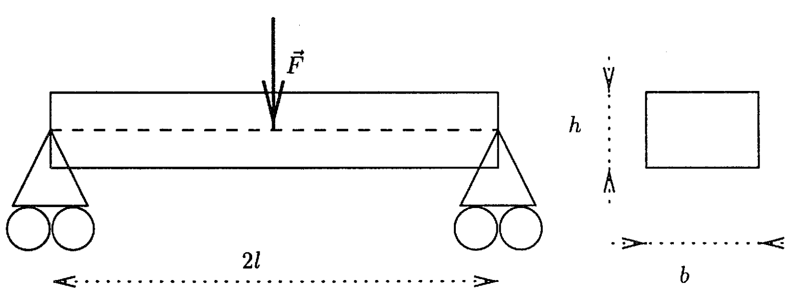

Fig. 2. Freely-supported beam.

3.1.1. On Results of Banichuk

Banichuk (1983, p.66) considers a freely-supported beam subjected to bending. The initial beam has a constant thickness and is shown in Fig. 2. The volume V and the length $2l$ are given. We have $S(x) = B_\alpha(x)h(x)$ for the cross-section area distribution and $D(x) = EI(x) = A_\alpha(x)h^\alpha(x)$, where α depends on the shape of the cross-section. If the cross-section is rectangular, then $\alpha = 3$. Here $h(x)$ is the thickness distribution and b is the constant width of the beam. The beam is subjected to a transverse point load applied at $x_0 = 0$, i.e. $F = \lambda\delta x(0)$. Banichuk optimizes the rectangular ($\alpha = 3$) cross-section area distribution $S(x)$ in order to minimize the displacement $w(0)$ of the middle point of the beam:

$$J(h) = |w_h(0)| \qquad \text{with} \qquad \int_{-l}^{+l} S(x)\,\mathrm{d}x = V \tag{34}$$

The equilibrium equation is $(Dw_{xx})_{xx} = q$ with boundary conditions

$$(w)_{x=+l} = (w)_{x=-l} = 0, \qquad (Dw_{xx})_{x=+l} = (Dw_{xx})_{x=-l} = 0 \tag{35}$$

The following results hold for $0 \leq x \leq 1$ and $\alpha = 3$:

$$h(x) = \frac{3V}{4bl}\sqrt{1 - \frac{x}{l}} \tag{36}$$

$$J_0 = w_0(0) = \frac{4Pl^6b^3}{3A_3V^3} \tag{37}$$

$$J_* = w_*(0) = \frac{64Pl^6b^3}{81A_3V^3} \tag{38}$$

$$w(x) = \frac{64Pl^6b^3}{81A_3V^3}(1 - \frac{x}{l})(3 - 2\sqrt{1 - \frac{x}{l}}) \tag{39}$$

where $w(x)$ is the optimal displacement and the solution is symmetric with respect to $x = 0$.

3.1.2. Proposed Model

We approximate the previous problem with the use of our beam model. We use the same physical parameters as in the previous example, but we have to restrict the thickness to be strictly positive: $h(x) \geq h_{\min} = h_0/3 > 0$. The lower bound on the thickness is to ensure reasonable conditioning of the tangent rigidity matrix. The functional $j = y(0) - y_0(0)$ is approximated by the smooth cost

$$j_p(x_0, y_0) = \left[\int_0^l \left(\sqrt{(y - y_0)^2}\right)^p \mathrm{d}s_0\right]^{1/p} \tag{40}$$

with $l = 1\,\mathrm{m}$, $3V/(4b) = 0.1$ and $64Pb^3l^6/(81A_3V^3) = 0.3$.

The same problem is solved with an increased load factor. The results are presented in Fig. 5 for a small load and Fig. 8 for a large load.

3.1.3. Comparison

We compare the analytic solution proposed by Banichuk with the results we obtain for a small load (see Fig. 5). The difference is due to the minimum thickness which cannot be zero on our model, because the system loses its stability for a very low thickness. When the load increases, the thickness distributions for the linear and non-linear example are not very different, because the deformations are essentially due to bending, and stretching is small.

cost	initial	optimal	improvement
analytical	J_0, eqn. (37)	J_*, eqn. (38)	$J_0/J_* = 0.47$
computed for $p = 02$	0.498	0.320	0.35
computed for $p = 04$	0.466	0.291	0.37
computed for $p = 10$	0.468	0.286	0.38
computed for $p = 20$	0.478	0.301	0.37
computed for $p = 30$	0.484	0.331	0.31
computed for $p = 40$	0.487	0.334	0.31
computed for $p = 50$	0.490	0.334	0.32

For $p > 20$ the optimal thickness evolves far from the optimal one described by Banichuk, and the algorithm finds another distribution, see Fig. 6. For this new distribution, we remark that the constraint on the minimum of the thickness is *not* active. The evolution of the cost functional for different values of p is shown in Fig. 7. We see that for large values of p the algorithm is slow. We guess that this is related to the fact that SQP does not work with a non-differentiable cost.

3.2. Optimal Design of a Compressed Rod with Pin-Jointed Supports

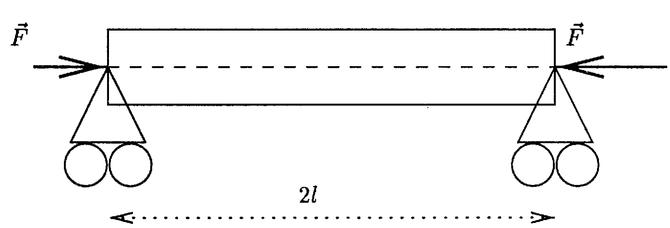

Fig. 3. Compressed rod with pin-jointed supports.

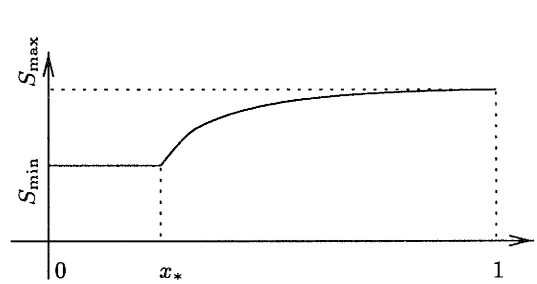

Fig. 4. Distribution of thickness with a lower-bound constraint.

3.2.1. Banichuk's Problem

For details, the reader is referred to (Banichuk, 1990, p.259). A pin-jointed beam compressed by two opposite forces at both ends is optimized; see Fig. 3 for a description of the beam and the boundary conditions. The cross-section is rectangular. This problem differs from the previous one because the beam exhibits a pitchfork bifurcation for a small value of the load parameter. The equilibrium equation and boundary conditions are respectively given by

$$EIu_{xx} + pu = 0 \tag{41}$$

$$u(0) = 0 = u(l) \tag{42}$$

where $\alpha = 3$, $EI = A_\alpha h^\alpha$. Then we minimize

$$j_p(x_0, y_0) = \left[\int_0^l \left(\sqrt{(x - x_0)^2 + (y - y_0)^2} \right)^p \mathrm{d}s_0 \right]^{1/p} \tag{43}$$

subject to

$$\int_0^l S(x)\,\mathrm{d}x = V \tag{44}$$

with $p = 2, 4, 6$, $l = 1\,\mathrm{m}$, $3V/4b = 0.1$ and $64l^6Pb^3/81A_3V^3 = 0.3$. The solution of this problem is implicitly given by the following equations:

$$\beta = \frac{S_{\min}}{S(l/2)}, \quad 0 \leq \beta \leq 1 \tag{45}$$

$$\psi(\beta) = \arctan\sqrt{\frac{\beta}{(\alpha+1)(1-\beta)}} \tag{46}$$

$$I_2(\alpha, \beta, s) = \frac{2}{15}(8 - 4\beta + 3\beta^2)\sqrt{1-\beta} \tag{47}$$

$$I_1(\alpha, \beta, s) = \int_\beta^s \frac{\xi^{(\alpha-1)/2}}{\sqrt{1-\xi}}\,\mathrm{d}\xi = \frac{2\sqrt{p_0}}{\sqrt{1-\xi}}(x - x_*) \tag{48}$$

$$S_{\min} = \frac{2\beta^{\alpha/2}\psi(\beta) + \sqrt{\alpha+1}I_1(\alpha,\beta,1)}{2\beta^{(\alpha+2)/2}\psi(\beta) + \sqrt{\alpha+1}I_2(\alpha,\beta)} \tag{49}$$

$$p_0 = (\alpha+1)\left(\frac{(I_2(\alpha,\beta) - \beta I_1(\alpha,\beta,1))S_{\min}}{(1 - S_{\min})\beta}\right)^2 \tag{50}$$

$$x_* = \frac{S_{\min}I_2(\alpha,\beta) - \beta I_1(\alpha,\beta,1)}{2S_{\min}(I_2(\alpha,\beta) - \beta I_1(\alpha,\beta,1))} \tag{51}$$

and the solution is symmetric with respect to $x = 0$.

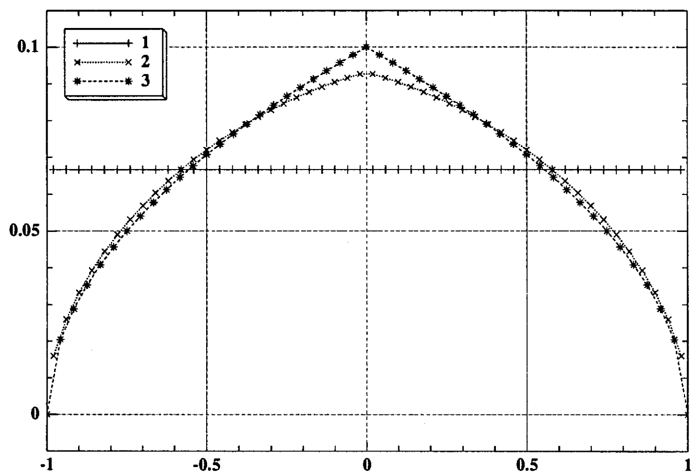

Fig. 5. Example 1: 1 initial thickness distribution, 2 optimal computed thickness, 3 optimal theoretical thickness.

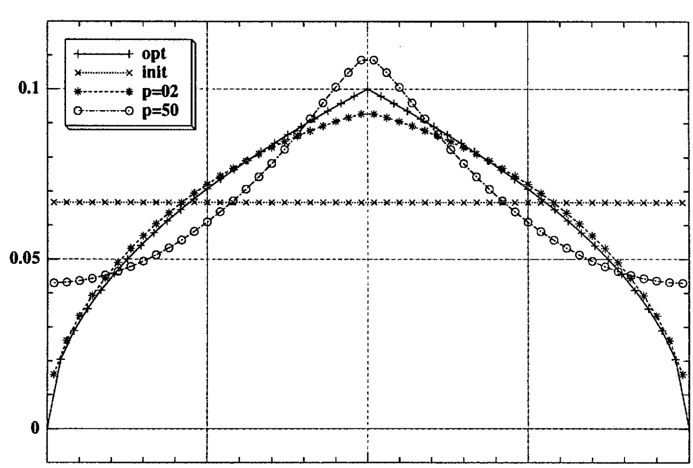

Fig. 6. Example 1: optimal thickness distribution, initial thickness distribution, $p = 02$, $p = 50$ optimal computed thickness.

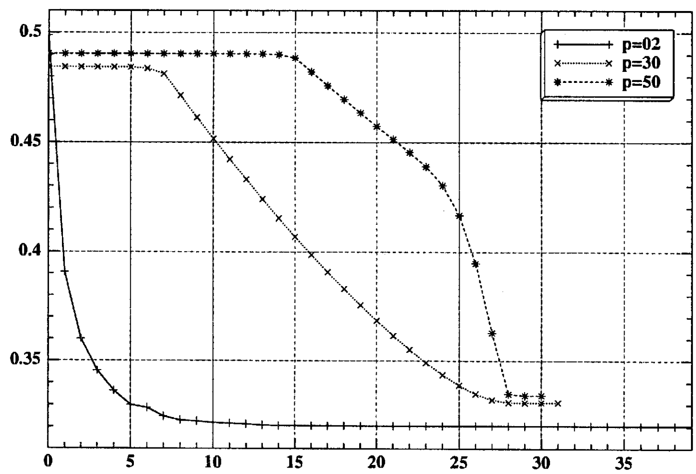

Fig. 7. Example 1: cost per iteration with a varying value of p.

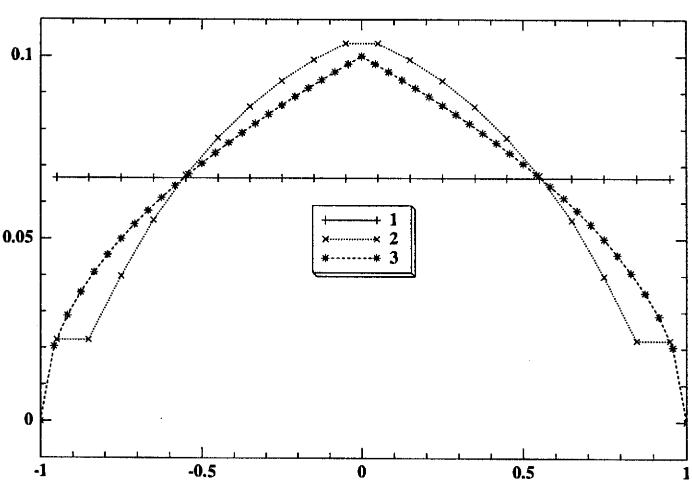

Fig. 8. Example 1: 1 initial thickness distribution, 2 optimal computed thickness with large load, 3 optimal theoretical thickness with small load.

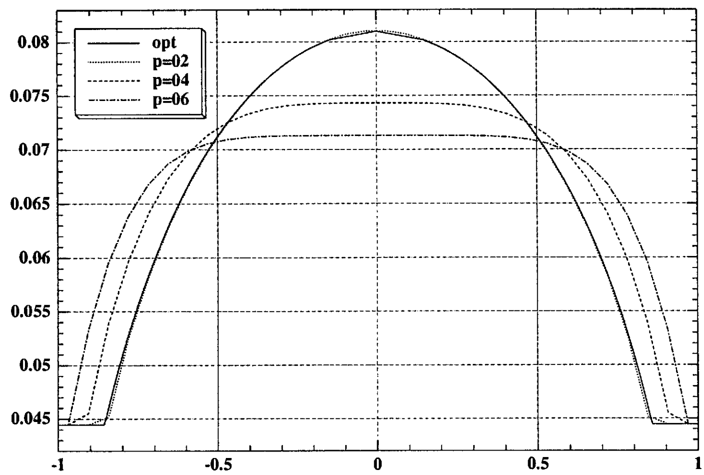

Fig. 9. Example 2: optimal theoretical thickness distribution, $p = 02$, $p = 04$ and $p = 06$ optimal computed thickness distribution with small load.

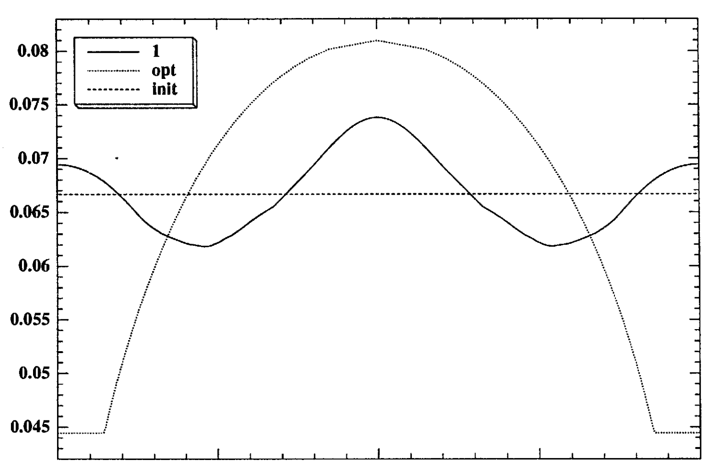

Fig. 10. Example 2: 1 initial thickness distribution, 2 optimal computed thickness distribution with large load, optimal theoretical thickness with small load.

3.2.2. Our Model

We approximate the problem 3.2.1. with our beam model. We use the same physical parameters, but we have to restrict the thickness to be strictly positive: $h(x) \geq h_{\min} = h_0/3 > 0$. The cost functional is given by (43). We consider the foregoing example, but with an increasing load. The results are presented in Fig. 9 for a small load and in Fig. 10 for a large load.

3.2.3. Comparison

We compare the analytic solution obtained by Banichuk and the results of our computations, see Fig. 9. Then we show new results when the load increases. In this case, the thickness distributions for the linear and non-linear example are very different, because the strain is due to bending and stretching, and hence the linear model equation cannot be considered to be accurate. With a large load, the constraint on the minimal thickness is not active.

cost	initial	optimal	improvement
analytical	J_0	J_*	$J_0/J_* = 0.06$
computed for $p = 2$	0.0147	0.0139	0.05
computed for $p = 4$	0.0143	0.0138	0.03
computed for $p = 6$	0.0146	0.0143	0.02

4. Conclusion

We validated the sensitivity analysis of a geometric non-linear arch. We proved that this method is efficient and that it remains stable at a turning point. The total cost needed to compute an optimal shape is 7 or 8 times higher than that of the finite-element analysis, but the successive shapes do not change a lot after the second iteration, and this shape may be sufficient in many cases.

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