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KINEMATICS OF FREE-FLOATING ROBOTS REVISITED

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In this paper, a decomposed expression for the kinematics of a free-floating robot with rotational joints is derived in detail. Some mathematical conditions on invertibility of a matrix used in a definition of the generalized Jacobian matrix are derived and proved for a general robot and for a planar robot (an n-pendulum). For the n-pendulum, closed-form kinematics are given. A comparison of the resulting kinematic equations for the simplest free-floating robot (a planar 2-pendulum), with the simplest mobile robot (a unicycle) is presented. Being an extended version of (Dulęba, 1996), this paper offers a family of models of nonholonomic systems.

1. Introduction

The kinematics of a free-floating robot is a transformation between the linear and angular velocities of the robot base (or the end-effector), and the velocities at the joints (directly controlled variables). Although there are many papers dealing with the kinematics of free-floating robots, the resulting equations sometimes differ significantly. The papers concerned with this subject can be divided into two main categories: those where derivation of the kinematics is given (Dubowsky and Papadopoulos, 1991; Mukherjee and Nakamura, 1992; Nagashima and Nakamura, 1992; Papadopoulos and Dubowsky, 1991; 1993; Vafa and Dubowsky, 1990), and those where the kinematics are used only to introduce a subject of interest (Dubowsky and Papadopoulos, 1993; Nakamura and Mukherjee, 1991; 1993; Umetani and Yoshida, 1988). The first group can be divided, in turn, according to the principles used to derive the kinematic equations. Nakamura and co-workers, (Nakamura and Mukherjee, 1991; 1993; Nagashima and Nakamura, 1992; Mukherjee and Nakamura, 1992) opt for a standard robotic approach with items (inertia matrices, positions) expressed in the coordinate frames attached to the joints. On the other hand, in the works (Dubovsky and Papadopoulos, 1991; 1993; Papadopoulos and Dubovsky, 1991; 1993) a barycentric approach is preferred with kinematic equations expressed in barycentric coordinates. Nevertheless, both approaches start from the angular and linear momentum conservation laws governing the motion of the free-floating robot. Therefore the resulting equations differ in form and some of them are even inaccurate (Nakamura and Mukherjee, 1991, eqn. (12), p. 502), whereas for some others a simpler form is allowed.

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The first aim of this paper is to obtain a decomposed form of the kinematic equations of free-floating robots. The decomposition separates quantities expressed in the manipulator's base coordinate frame from those situating the robot's base in the inertial frame. While comparing our equations with any others known in the robotic literature, one can notice their simplicity and ease of implementation. The kinematic equations have been derived based on a standard robotic approach—all quantities are expressed in coordinate frames attached to the manipulator's joints. The second aim is to formulate and to prove mathematical conditions on invertibility of a matrix used in the derivation of the so-called generalized Jacobian matrix (Umetani and Yoshida, 1988). The theorems formulated here impose very weak conditions on the invertibility, therefore any physical robot satisfies them easily. The third goal relies on finding a closed form of the kinematics for a family of planar pendulums (a source of models for free-floating robots). As a by-product of the derivation, we compare the simplest free-floating robot (a 2-pendulum) with its mobile counterpart, i.e. a unicycle. The results show, howewer, that although the form of the equations is the same, the equations for the free-floating robot are computationally much more complex than those for the mobile robot.

We end this section with a terminology remark. Some authors use the term 'free-flying robots' as an equivalent to free-floating robots (Mukherjee and Nakamura, 1992; Umetani and Yoshida, 1988). Recently, there has been a tendency to use the term 'free-floating robots' in the context of the robots satisfying conservation laws (nonholonomic), as opposed to the free-flying robots violating the laws (Dubowsky and Papadopoulos, 1991; 1993; Papadopoulos and Dubowsky, 1993). The free-flying robots (holonomic) are equipped with thruster jets or reaction wheels.

This paper is organized as follows. In Section 2 the kinematic equations of a free-floating robot are given based on a standard robotic approach. In Section 3 some theorems on invertibility of a matrix used in the definition of the generalized Jacobian matrix are formulated. In Section 4 compact kinematic equations for a family of n-pendulums are given and conditions for the invertibility of the aforementioned matrix are given. In Section 5 the kinematics of a 2-pendulum, being the simplest free-floating structure, are compared with the kinematics of the simplest mobile robot, i.e. a unicycle. Section 6 concludes the paper.

2. Kinematic Equations of a Free-Floating Robot

2.1. Notation

In this paper a free-floating rigid robot equipped only with rotational joints is considered. Several coordinate frames are introduced: an inertial frame (I), a frame attached to the base of the robot (V), a frame attached to the base of the manipulator (0), frames attached to consecutive joints of the robot (k), an end-effector frame (E), cf. Fig. 1. For convenience, it is assumed that the inertial frame has the origin at the mass centre of the free-floating robot. Throughout this paper the standard Denavit-Hartenberg notation will be adopted.

Following conventions used in (Mukherjee and Nakamura, 1992) a subscript alone denotes a variable expressed in a local coordinate frame, a transformation with both sub- and superscripts denotes a transformation from the frame labelled with the superscript to the frame denoted by the subscript. Below definitions and symbols extensively used are introduced (most of the items have their manipulator base frame (0) counterparts, cf. $z_I^k \leftrightarrow z_0^k$):

n the number of degrees of freedom of a manipulator.

R_i^j the (3×3) matrix of rotation $\in SO(3)$.

z_I^k the z-axis versor of the k-th coordinate frame expressed in the inertial frame, $z_I^k = R_I^k \cdot [0,0,1]^T$.

m_k the k-th link mass. The base of the robot has index 0 and the other indices run from the base of the manipulator till its end-tip.

s_k ($\in \mathbb{R}^3$) vector connecting the k-th frame origin with the mass centre of the k-th link (expressed in the k-th frame).

r_I^k ($\in \mathbb{R}^3$) vector connecting the origin of the inertial frame with the mass centre of the k-th link (expressed in the inertial frame).

d_I^k ($\in \mathbb{R}^3$) position of the k-th frame origin in the inertial frame.

ω_I^k the angular velocity of the k-th frame expressed in the inertial frame.

I_k^k the k-th link inertia matrix (3×3) expressed with respect to the k-th frame.

$\dot{} = \frac{\mathrm{d}}{\mathrm{d}t}$ the time derivative operator.

$\dot{\Theta}_1$ ($\in \mathbb{R}^6$) linear and angular velocity of the mass centre of the robot's base expressed in the inertial frame $\dot{\Theta}_1 = [\dot{d}_{Ix}^0, \dot{d}_{Iy}^0, \dot{d}_{Iz}^0, \omega_x, \omega_y, \omega_z]^T$.

q_k ($\in \mathbb{R}^1$), $k = 1, \dots, n$, the k-th joint variable of the manipulator.

Θ_2 ($\in \mathbb{R}^n$) a configuration of the manipulator $[q_1, \dots, q_n]^T$.

J_{Iv}^k ($\in \mathbb{R}^{3\times n}$) partial Jacobian matrix (until the k-th link inclusively) for linear velocities expressed in the inertial frame.

$\times$ as a superscript denotes an operator introducing the matrix vector product. For a vector $a = [a_x, a_y, a_z]^T$, $a^\times = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$; when used as an infix operator, $\times$ denotes a vector product.

E_s the identity matrix of rank s ($s \times s$).

2.2. Derivation

Derivation of the kinematics equation for a free-floating robot is based on two conservation laws. The conservation of linear momentum states that

$$\sum_{k=0}^{n} m_k \cdot \dot{r}_I^k = 0 \tag{1}$$

while the angular momentum conservation law yields

$$\sum_{k=0}^{n} (I_I^k \cdot \omega_I^k + m_k \cdot r_I^k \times \dot{r}_I^k) = 0 \tag{2}$$

For brevity, in eqns. (1) and (2) it has been assumed that the right-hand sides equal zero instead of a constant. Indeed, this assumption, made by most authors in the field of space robotics, not only simplifies the resulting equations but also prevents the free-floating robot from drift and spin. These features are important both from the control and the communication with the robot perspective. In fact, by a temporary use of specialized devices (reaction wheels, thruster jets) initial values of momenta can be set to arbitrary values.

Having introduced the notation and equations of conservation laws, we are ready to derive an equation connecting linear and angular velocities of the robot's base as a function of directly controlled velocities at the joints. The following definitions, taken from a primary course of robotics (Craig, 1981; Paul, 1981; Spong and Vidyasagar, 1989) are extensively used in the derivation:

- the definition of the angular velocity ω:

$$\dot{R} = \omega^{\times} R \tag{3}$$

- an angular velocity transformation between the k-th frame and the manipulator's base frame valid for a manipulator with rotational joints only:

$$\omega_I^k = \omega_I^0 + \sum_{i=1}^{k} z_I^{i-1} \dot{q}_i \tag{4}$$

- a chain rule of multiplying matrices of rotations ($\in SO(3)$):

$$R_i^k = R_i^j R_j^k \quad \text{where} \quad i \le j \le k, \quad R_i^i = E_3, \quad R_k^i = (R_i^k)^T \tag{5}$$

- a position of the k-th frame origin in the inertial frame:

$$d_I^k = d_I^0 + R_I^0 d_0^k = d_I^0 + \sum_{i=1}^{k} R_I^{i-1} d_{i-1}^i \tag{6}$$

- a position of the k-th link mass centre, $k = 0, 1, \ldots, n$, in the inertial frame:

$$r_I^k = R_I^k s_k + d_I^k = R_I^0 r_0^k + d_I^0 \tag{7}$$

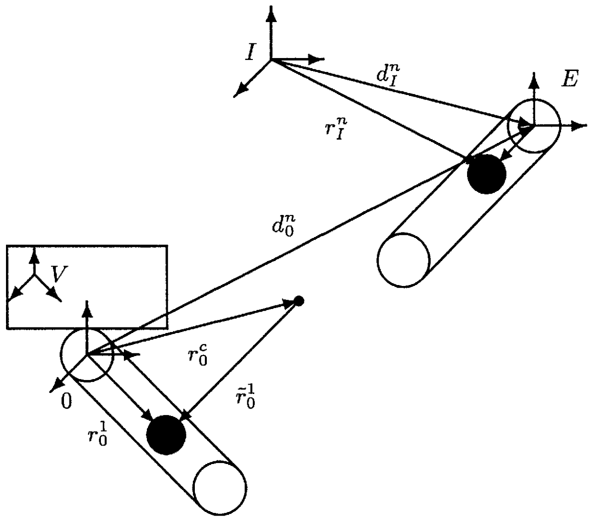

Fig. 1. A free-floating robot with attached coordinate frames.

Let us consider a free-floating robot presented in Fig. 1. The transformation between frames V and 0 is described by a constant matrix. The transformation between frames I and V can be viewed as a kinematic pair of zeroth order, i.e. having six degrees of freedom, whereas the transformation between the $(i-1)$-th and the i-th frame (for $i = 1, \ldots, n$) as a kinematic pair of fifth order (1 DOF).

To get an equation relating linear and angular velocities of the robot's base with velocities at the joints, a linear velocity of the k-th link mass centre should be calculated. By taking the time derivative of eqn. (7) and noting that s_k is time-independent, we obtain

$$\dot{r}_I^k = \omega_I^k \times (R_I^k s_k) + \dot{d}_I^k \tag{8}$$

Exploiting eqns. (4), (6), (7), the time derivative of eqn. (6) and the properties of the vector product leads to

$$\dot{r}_I^k = \dot{r}_I^0 + \omega_I^0 \times (r_I^k - r_I^0) + \sum_{i=1}^{k} \left(z_I^{i-1} \times (r_I^k - d_I^k) \right) \dot{q}_i + R_I^0 \dot{d}_0^k \tag{9}$$

It is easy to derive a formula for $\dot{d}_0^k$ as in the case of stationary robots (Spong and Vidyasagar, 1989):

$$\dot{d}_0^k = J_{0v}^k \, \dot{q} = \sum_{i=1}^{k} J_{0v}^{i,k} \, \dot{q}_i \tag{10}$$

where the columns of the linear velocity part $J_{0v}^{i,k}$ of the Jacobian matrix for rotational joints are defined by the well-known formula

$$J_{0v}^{i,k} = \begin{cases} z_0^{i-1} \times (d_0^k - d_0^{i-1}) & \text{for } \ i \leq k \\ 0 & \text{for } \ i > k \end{cases} \tag{11}$$

Substituting (10) into (9) and applying the identity $R(a \times b) = (Ra) \times (Rb)$ valid for any vectors $a, b \in \mathbb{R}^3$ and $R \in SO(3)$ we see that

$$\begin{aligned}\dot{r}_I^k &= \dot{r}_I^0 + \omega_I^0 \times (r_I^k - r_I^0) + \sum_{i=1}^{k} z_I^{i-1} \times (r_I^k - d_I^k)\dot{q}_i \\ &\quad + R_I^0 \sum_{i=1}^{k} \Big(z_0^{i-1} \times (d_0^k - d_0^{i-1}) \Big) \dot{q}_i \\ &= \dot{r}_I^0 + \omega_I^0 \times (r_I^k - r_I^0) + \sum_{i=1}^{k} z_I^{i-1} \times (r_I^k - d_I^{i-1})\dot{q}_i \end{aligned} \tag{12}$$

The last component of (12) is a partial Jacobian matrix (up to the k-th link inclusively) which will be denoted by J_{Iv}^k and defined as

$$J_{Iv}^k = \begin{bmatrix} J_{Iv}^{1,k} & \cdots & J_{Iv}^{k,k} & 0 & \cdots & 0 \end{bmatrix} \tag{13}$$

where $J_{Iv}^{i,k} = z_I^{i-1} \times (r_I^k - d_I^{i-1})$.

The linear momentum conservation law (eqn. (1)) expressed in matrix form is as follows:

$$\left[\Big(\sum_{k=0}^{n} m_k \Big) E_3 \qquad - \Big(\sum_{k=0}^{n} m_k r_I^{0k} \Big)^{\times} \right] \dot{\Theta}_1 + \left[\sum_{k=1}^{n} m_k J_{Iv}^k \right] \dot{\Theta}_2 = 0 \tag{14}$$

with $r_I^{0k} = r_I^k - r_I^0$.

Let us consider the angular momentum conservation law. By substituting (4) and (12) into (2) and applying $I_I^k = R_I^k I_k^k (R_I^k)^T$ the angular momentum conservation law leads to the matrix equation:

$$\begin{aligned} &\left[\Big(\sum_{k=0}^{n} m_k r_I^k \Big)^{\times} \qquad \sum_{k=0}^{n} \Big\{ R_I^k I_k^k (R_I^k)^T - m_k (r_I^k)^{\times} (r_I^{0k})^{\times} \Big\} \right] \dot{\Theta}_1 \\ &\qquad + \left[\sum_{k=0}^{n} m_k (r_I^k)^{\times} J_{Iv}^k + P_I \right] \dot{\Theta}_2 = 0 \end{aligned} \tag{15}$$

where $P_I = [P_I^1, P_I^2, \ldots, P_I^n]$ and $P_I^i = \sum_{k=i}^{n} (R_I^k I_k^k (R_I^k)^T) z_I^{i-1}$.

With the notation

$$m = \sum_{k=0}^{n} m_k, \qquad r_I^c = \frac{1}{m} \sum_{k=0}^{n} m_k \cdot r_I^k$$

$$\sum_{k=0}^{n} m_k \cdot r_I^{0k} = \sum_{k=0}^{n} m_k \cdot r_I^k - \sum_{k=0}^{n} m_k \cdot r_I^0 = m \cdot r_I^c - m \cdot r_I^0$$

where r_I^c is a vector connecting the mass centre of the robot with the origin of the inertial frame, eqns. (14) and (15) can be coupled into a single matrix equation:

$$\begin{bmatrix} mE_3 & -\left(m(r_I^c - r_I^0)\right)^\times \\ (mr_I^c)^\times & \sum_{k=0}^{n}\left\{R_I^k I_k^k (R_I^k)^T - m_k (r_I^k)^\times (r_I^{0k})^\times\right\} \end{bmatrix} \dot{\Theta}_1 + \begin{bmatrix} \sum_{k=0}^{n} m_k J_{Iv}^k \\ \sum_{k=0}^{n} m_k (r_I^k)^\times J_{Iv}^k + P_I \end{bmatrix} \dot{\Theta}_2 = 0 \tag{16}$$

If the inertial frame is chosen at the centre of mass of the robot, then $\sum_{k=0}^{n} m_k \cdot r_I^k = 0 \Rightarrow r_I^c = 0$. This assumption will be made henceforth. Under this assumption, eqn. (16) can be rewritten as

$$\begin{bmatrix} mE_3 & m(r_I^0)^\times \\ 0 & \sum_{k=0}^{n}\left\{R_I^k I_k^k (R_I^k)^T - m_k \left((r_I^k)^\times\right)^2\right\} \end{bmatrix} \dot{\Theta}_1 + \begin{bmatrix} \sum_{k=0}^{n} m_k J_{Iv}^k \\ \sum_{k=0}^{n} m_k (r_I^k)^\times J_{Iv}^k + P_I \end{bmatrix} \dot{\Theta}_2 = 0 \tag{17}$$

This form which can be found in (Mukherjee and Nakamura, 1992) is not the simplest possible. It is easy to derive the formulae:

$$J_{Iv}^k = R_I^0 J_{0v}^k, \quad P_I = R_I^0 P_0 \tag{18}$$

where P_0 is defined as its counterparts P_I with subscript 0 instead of I. Here r_I^k can be given a multiplicative form after a chain of transformations:

$$0 = \sum_{k=0}^{n} m_k r_I^k \overset{(7)}{=} \sum_{k=0}^{n} m_k (R_I^0 r_0^k + d_I^0) = m d_I^0 + R_I^0 \sum_{k=0}^{n} m_k r_0^k \overset{\text{def.}}{=} m d_I^0 + m R_I^0 r_0^c \tag{19}$$

where r_0^c is a vector connecting the origin of the manipulator's base frame with the mass centre of the robot. Substituting (19) into (7) we get

$$r_I^k = R_I^0 r_0^k - R_I^0 r_0^c = R_I^0 (r_0^k - r_0^c) \overset{\text{def.}}{=} R_I^0 \tilde{r}_0^k \tag{20}$$

where $\tilde{r}_0^k$ is the vector connecting the mass centre of the robot with the mass centre of the k-th link and expressed in the manipulator's base coordinate frame.

Based on (20) and using properties of the matrix vector product, the following formula can be derived:

$$\left((r_I^k)^\times\right)^2 = -R_I^0\left(|\tilde{r}_0^k|^2 E_3 - \tilde{r}_0^k(\tilde{r}_0^k)^T\right)(R_I^0)^T \tag{21}$$

After substitution of (5), (18) and (21), into (17), applying the identities: $(Rx)^\times = Rx^\times R^T$ valid for $x \in \mathbb{R}^3$ and $R \in SO(3)$, $R_I^0(R_I^0)^T = E_3$, $R_I^0 \cdot 0 \cdot (R_I^0)^T = 0$, we get a decomposed form of the kinematics:

$$\tilde{R}_I^0 \begin{bmatrix} mE_3 & m(\tilde{r}_0^0)^\times \\ 0 & \alpha \end{bmatrix} (\tilde{R}_I^0)^T \dot{\Theta}_1 + \tilde{R}_I^0 \begin{bmatrix} \sum_{k=0}^n m_k J_{0v}^k \\ \sum_{k=0}^n m_k (\tilde{r}_0^k)^\times J_{0v}^k + P_0 \end{bmatrix} \dot{\Theta}_2 = 0 \tag{22}$$

where $\alpha = \sum_{k=0}^n \{R_0^k I_k^k (R_0^k)^T + m_k(|\tilde{r}_0^k|^2 E_3 - \tilde{r}_0^k(\tilde{r}_0^k)^T)\}$ and $\tilde{R}_I^0 = \operatorname{diag}\{R_I^0, R_I^0\}$ (this notation will be used throughout the paper). In (22) similarity appears twice: $I_k^k \sim R_0^k I_k^k R_0^k$ and the mid-term of the matrix premultiplying $\dot{\Theta}_1$ is similar to the whole matrix.

In the sequel, we shall use a variant of the formula (22) where uncontrolled directly variables Θ_1 stand alone. Exploiting formulae for inverting block matrices, we get

$$\dot{\Theta}_1 = -\tilde{R}_I^0 \begin{bmatrix} \frac{1}{m}E_3 & -(\tilde{r}_0^0)^\times \\ 0 & E_3 \end{bmatrix} \begin{bmatrix} \sum_{k=0}^n m_k J_{0v}^k \\ \alpha^{-1}\left(\sum_{k=0}^n m_k (\tilde{r}_0^k)^\times J_{0v}^k + P_0\right) \end{bmatrix} \dot{\Theta}_2 \tag{23}$$

Here we have assumed the invertibility of α. In the next section we formulate conditions which secure its fulfilment.

3. Conditions for the Invertibility of the Matrix Premultiplying $\dot{\Theta}_1$

Nakamura and Mukherjee (1991) argued that the matrix premultiplying $\dot{\Theta}_1$ in (22) is invertible. Their arguments come from physical considerations and are as follows: for real robots and $\dot{\Theta}_2 = 0$ the momentums of the system are described by the first component of (22). For a non-zero vector $\dot{\Theta}_1$ it is physically impossible that the momentums be equal zero, therefore the matrix should be invertible. We try to validate the claim from a mathematical perspective. More precisely, some conditions for the invertibility of the matrix

$$\alpha = \sum_{k=0}^n \left\{R_0^k I_k^k (R_0^k)^T + m_k\left(|\tilde{r}_0^k|^2 E_3 - \tilde{r}_0^k(\tilde{r}_0^k)^T\right)\right\} \tag{24}$$

will be formulated.

Lemma 1. *Any matrix component in (24) is symmetric and non-negative definite.*

Proof. Obviously, the inertia matrix I_k^k is symmetric and non-negative definite. Premultiplying it by R and postmultiplying by R^T ($\in SO(3)$) do not influence this characteristics. Assume that $\tilde{r}_0^k = [a, b, c]^T$. Then

$$|\tilde{r}_0^k|^2 E_3 - \tilde{r}_0^k (\tilde{r}_0^k)^T = \begin{bmatrix} b^2 + c^2 & -ab & -ac \\ -ab & a^2 + c^2 & -bc \\ -ac & -bc & a^2 + b^2 \end{bmatrix} \tag{25}$$

Clearly, (25) is symmetric and its main minors are

$$b^2 + c^2, \qquad c^2(a^2 + b^2 + c^2), \qquad 0 \tag{26}$$

All of the minors are non-negative, so the matrix is non-negative definite. ■

Technical Lemma 2 taken from (Horn and Johnson, 1986) clarifies out conditions for the positive definiteness of a sum of symmetric non-negative definite matrices.

Lemma 2. *A sum of symmetric, non-negative definite matrices is symmetric and non-negative definite. Additionally, if in the sum there is at least one positive-definite matrix, then so is the sum.*

Theorem 1. *If for a robot any inertia matrix is positive-definite, then so is the matrix α (and hence it is invertible).*

Proof. It is an immediate consequence of Lemma 2. ■

Let us consider a more difficult case when $\forall k = 0, \dots, n \;\; I_k^k = 0_{3\times 3}$, i.e. the mass of each link is condensed at a single point. Before formulating a theorem concerning the invertibility of the matrix α in this case, let us state an auxiliary lemma.

Lemma 3. *For a point-mass system with at least three non-zero masses put at three non-collinear points, say A, B, C, there are three mass points A1, B1, C1 such that the centre of mass for the whole system is at none of the points A1, B1, C1, and the points are non-collinear.*

Proof. If the centre does not lie at A, B, or C, then $A1 = A$, $B1 = B$, and $C1 = C$. Otherwise, the centre of mass lies at one of these points, say A. In this case, the definition of the centre of mass guarantees that a non-zero mass at point D can be found, so that $A1 = D$, $B1 = B$, $C1 = C$, and the points $A1, B1, C1$ are non-collinear. ■

Theorem 2. *If for a robot $\forall k = 0, \dots, n$ $I_k^k = 0_{3\times 3}$, i.e. the total mass of each joint is concentrated at a single point, and, at any time moment, three non-zero masses are located at non-collinear points, then the matrix α is invertible.*

Proof. Let us consider a sum of two matrices from (25) derived from vectors $x = [a_1, b_1, c_1]^T$ and $y = [a_2, b_2, c_2]^T$. The main minors for the matrix are as follows:

$$\begin{cases} b_1^2 + c_1^2 + b_2^2 + c_2^2 \\ (c_1^2 + c_2^2)(a_1^2 + b_1^2 + c_1^2 + a_2^2 + b_2^2 + c_2^2) + (a_1 b_2 - a_2 b_1)^2 \\ (a_1^2 + b_1^2 + c_1^2 + a_2^2 + b_2^2 + c_2^2)\Big\{(a_2 b_1 - a_1 b_2)^2 + (a_2 c_1 - a_1 c_2)^2 + (b_1 c_2 - b_2 c_1)^2\Big\} \end{cases} \tag{27}$$

The second and third minors can be rewritten in a simpler form:

$$(c_1^2 + c_2^2)\big(\|x\|^2 + \|y\|^2\big) + (a_1 b_2 - a_2 b_1)^2, \quad \big(\|x\|^2 + \|y\|^2\big) \cdot \|x \times y\|^2 \tag{28}$$

Indeed, $(a_1 b_2 - a_2 b_1)^2 = \|x \times y\|^2$ when $c_1 = c_2 = 0$.

Theorem 2 and Lemma 3 guarantee the existence of non-collinear mass points $A1$, $B1$, $C1$, mentioned in the lemma. Three non-zero vectors are constructed by joining the mass centre with $A1$, $B1$, and $C1$. Among them at least two vectors are non-collinear and these vectors are denoted by x and y. Minor 1 (cf. (27)) is positive by the non-collinearity of x and y. Minor 3 is also positive by the non-collinearity of the non-zero vectors x, y, and $(\|x \times y\| = \|x\| \cdot \|y\| \cdot |\sin \angle(x, y)|)$. Minor 2 has either both c_1, c_2 equal to zero, and then its positiveness comes from the same argument as in the case of Minor 3, or at least one of c_i's is non-zero and the first term in Minor 2 is greater than zero. ■

3.1. The Generalized Jacobian Matrix for a Free-Floating Robot

In this subsection a functional dependence of the linear and angular velocities of the end-effector $(\dot{d}_I^n, \omega_I^n)^T$ on the directly controlled variables $\dot{\Theta}_2$ will be derived. Using (4) and computing the time derivative of (6), (10) for $k = n$, as well as the time derivative of (7) for $k = 0, \ldots,$ we get

$$\begin{bmatrix} \dot{d}_I^n \\ \omega_I^n \end{bmatrix} = \tilde{R}_I^0 \begin{bmatrix} E_3 & -(d_0^n - s_0)^\times \\ 0 & E_3 \end{bmatrix} (\tilde{R}_I^0)^T \dot{\Theta}_1 + \tilde{R}_I^0 \begin{bmatrix} J_{0v}^n \\ z_0^0, z_0^1, \ldots, z_0^{n-1} \end{bmatrix} \dot{\Theta}_2 \tag{29}$$

The indirectly controlled variables $\dot{\Theta}_1$ should be eliminated from (29). To this aim, (23) is applied. After further calculations, the final form of (29) is obtained:

$$\begin{bmatrix} \dot{d}_I^n \\ \omega_I^n \end{bmatrix} = \tilde{R}_I^0 \begin{bmatrix} -\dfrac{1}{m} h_1 + (d_0^n - r_0^c)^\times \alpha^{-1} h_2 + J_{0v}^n \\ -\alpha^{-1} h_2 + \left[z_0^0, z_0^1, \ldots, z_0^{n-1}\right] \end{bmatrix} \dot{\Theta}_2 \tag{30}$$

where α is defined by (24) and

$$h_1 = \sum_{k=0}^{n} m_k J_{0v}^k, \quad h_2 = \sum_{k=0}^{n} m_k (\tilde{r}_0^k)^\times J_{0v}^k + P_0 \tag{31}$$

or, in general form,

$$\begin{bmatrix} \dot{d}_I^n \\ \omega_I^n \end{bmatrix} = J_I^\star \, \dot{\Theta}_2 = \tilde{R}_I^0 \, J_0^\star \, \dot{\Theta}_2 \tag{32}$$

$J_0^\star$ depends on the directly controlled coordinates q and on kinematic (lengths, etc.) and dynamic (masses, etc.) parameters, but does not depend on any parameter relating the base of the manipulator with the inertial frame. Hence $J_0^\star$ can be called the *generalized manipulator's Jacobian matrix*, whereas $J_I^\star$ is said to be the *generalized Jacobian matrix* (Umetani and Yoshida, 1988).

Note that in a nonholonomic motion planning (23) and (30) should be supplemented by the equation

$$\dot{R}_I^0 = (\omega_I^0)^\times \cdot R_I^0 \tag{33}$$

which indicates that R_I^0 ($\tilde{R}_I^0$) varies as the free-floating robot moves itself.

4. The Free-Floating Planar n-Pendulum

Let us consider a family of pendulums and check how (30) manifests in practice. The following standard notation will be used:

$$\begin{aligned}
s_{ij} &= \sin\Big(\sum_{k=i}^{j} q_k\Big), & s_{\Theta ij} &= \sin\Big(\Theta + \sum_{k=i}^{j} q_k\Big) \\
c_{ij} &= \cos\Big(\sum_{k=i}^{j} q_k\Big), & c_{\Theta ij} &= \cos\Big(\Theta + \sum_{k=i}^{j} q_k\Big) \\
S_{ij} &= \sum_{k=i}^{j} a_k \, s_{1k}, & C_{ij} &= \sum_{k=i}^{j} a_k \, c_{1k} \\
S_{\Theta ij} &= \sum_{k=i}^{j} a_k \, s_{\Theta 1k}, & C_{\Theta ij} &= \sum_{k=i}^{j} a_k \, c_{\Theta 1k}
\end{aligned} \tag{34}$$

A pendulum under consideration is depicted in Fig. 2 with the task-space coordinates $[x, y, \Theta]^T$. The pendulum consists of point masses placed at the ends of appropriate links and it has only revolute joints with axes of rotation parallel to each other. Denavit-Hartenberg parameters of the n-pendulum are collected in Table 1.

Table 1. Denavit-Hartenberg parameters for a free-floating n-pendulum.

joint	Θ_i	α_i	a_i	d_i
0	0	0	a_0	0
1	q_1	0	a_1	0
⋮	⋮	⋮	⋮	⋮
n	q_n	0	a_n	0

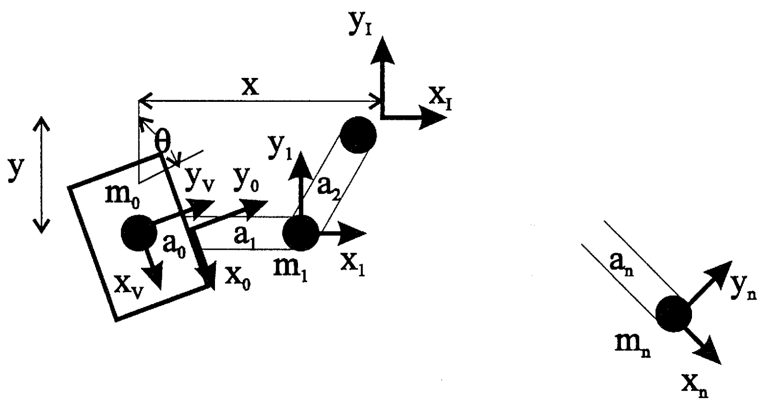

Fig. 2. Coordinate frames of the n-pendulum.

For the n-pendulum the following equalities are satisfied:

1. $r_0^k = d_0^k,\ \ I_k^k = 0_{(3\times 3)}\ |\ k = 0, 1, \ldots, n$

2. $P_0 = 0_{(3\times n)},\ R_I^0 = \begin{bmatrix} c_\Theta & -s_\Theta & 0 \\ s_\Theta & c_\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. $r_0^0 = (-a_0,\ 0,\ 0)^T$, partial kinematics

$$r_0^k = \begin{bmatrix} \sum_{i=1}^{k} a_i c_{1i} \\ \sum_{i=1}^{k} a_i s_{1i} \\ 0 \end{bmatrix} = \begin{bmatrix} r_{0x}^k \\ r_{0y}^k \\ 0 \end{bmatrix} = \begin{bmatrix} C_{1k} \\ S_{1k} \\ 0 \end{bmatrix}, \quad k = 1, \ldots, n \tag{35}$$

4. The indirectly controlled variables $(x, y, \Theta)^T$, (cf. Fig. 2), together with the controlled variables $(q_1, \ldots, q_n)^T$ form a state vector.

5. Due to the symmetry of masses along the z-axis and the assumption of parallel axes of rotations, $z_0^k = (0, 0, 1)^T,\ \ k = 0, \ldots, n-1,\ \ \dot{z} = 0$ and $(\omega_x, \omega_y, \omega_z)^T = (0, 0, \dot{\Theta})^T$.

According to the above relations the following equations are easy to derive: $\tilde{r}_0^k = (\star, \star, 0)^T$, ($\star$ denotes any value), so $\tilde{r}_0^k \tilde{r}_0^{k^T} = \begin{bmatrix} \star & \star & 0 \\ \star & \star & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $R_I^{0^T} \cdot \begin{bmatrix} 0 \\ 0 \\ \dot{\Theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\Theta} \end{bmatrix}$.

Substituting them into (22) we get

$$\begin{bmatrix} mE_3 & m\,R_I^0(\tilde{r}_0^0)^\times \\ 0 & \sum_{k=1}^{n} m_k|\tilde{r}_0^k|^2 E_3 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \\ 0 \\ 0 \\ \dot{\Theta} \end{bmatrix} + \begin{bmatrix} R_I^0 \sum_{k=0}^{n} m_k J_{0v}^k \\ \sum_{k=0}^{n} m_k (\tilde{r}_0^k)^\times J_{0v}^k \end{bmatrix} \dot{\Theta}_2 = 0 \tag{36}$$

The components of (36) are as follows:

$$J_{0v}^k = \begin{bmatrix} -S_{1k} & -S_{2k} & \cdots & -S_{kk} & 0 & \cdots & 0 \\ C_{1k}+a_0 & C_{2k} & \cdots & C_{kk} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{37}$$

$$\sum_{k=0}^{n} m_k(\tilde{r}_0^k)^\times J_{0v}^k = \left[\begin{matrix} 0 \\ 0 \\ \sum_{k=1}^{n} m_k(\tilde{r}_{0y}^k S_{1k} + \tilde{r}_{0x}^k C_{1k}) + a_0 \sum_{k=1}^{n} m_k \tilde{r}_{0x}^k \end{matrix}\right.$$

$$\left.\begin{matrix} \cdots & 0 & 0 \\ \cdots & 0 & 0 \\ \cdots & \sum_{k=n-1}^{n} m_k(\tilde{r}_{0y}^k S_{n-1,k} + \tilde{r}_{0x}^k C_{n-1,k}) & \sum_{k=n}^{n} m_k(\tilde{r}_{0y}^k S_{nk} + \tilde{r}_{0x}^k C_{nk}) \end{matrix}\right] \tag{38}$$

$$mR_I^0(\tilde{r}_0^0)^\times = m \begin{bmatrix} \star & \star & c_\Theta(d_{0y}^0 - r_{0y}^c) + s_\Theta(d_{0x}^0 - r_{0x}^c) \\ \star & \star & s_\Theta(d_{0y}^0 - r_{0y}^c) - c_\Theta(d_{0x}^0 - r_{0x}^c) \\ \star & \star & 0 \end{bmatrix} \tag{39}$$

where the coordinates of the centre of mass for the system expressed in the manipulator's base frame are

$$r_{0x}^c = \frac{1}{m}\Big(-m_0 a_0 + \sum_{i=1}^{n} m_i C_{1i}\Big), \quad r_{0y}^c = \frac{1}{m}\sum_{i=1}^{n} m_i S_{1i} \tag{40}$$

Substituting the above equations into (39), we get

$$
\begin{aligned}
M_{13} &= m \cdot \left\{ c_\Theta (d^0_{0y} - r^c_{0y}) + s_\Theta (d^0_{0x} - r^c_{0x}) \right\} \\
&= -\Big(\sum_{i=1}^{n} m_i S_{\Theta 1i} + s_\Theta a_0 \sum_{i=1}^{n} m_i \Big) \\
M_{23} &= m \cdot \left\{ s_\Theta (d^0_{0y} - r^c_{0y}) - c_\Theta (d^0_{0x} - r^c_{0x}) \right\} \\
&= \sum_{i=1}^{n} m_i C_{\Theta 1i} + c_\Theta a_0 \sum_{i=1}^{n} m_i
\end{aligned}
\tag{41}
$$

The new variables M_{13} and M_{23} are supplemented by M_{33} defined as

$$
M_{33} = \sum_{k=0}^{n} m_k |\tilde{r}^k_0|^2 = \sum_{k=0}^{n} m_k |r^k_I|^2 \tag{42}
$$

Collecting eqns. (37), (38), (41), (42) and skipping three identity equations in (36), we obtain the final formula for (36):

$$
\begin{bmatrix} m & 0 & M_{13} \\ 0 & m & M_{23} \\ 0 & 0 & M_{33} \end{bmatrix}
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\Theta} \end{bmatrix}
=
\begin{bmatrix}
\sum_{j=1}^{n} \sum_{i=j}^{n} m_i \, S_{\Theta ji} \, \dot{q}_j + \Big(\sum_{i=1}^{n} m_i \Big) a_0 \, s_\Theta \, \dot{q}_1 \\
-\sum_{j=1}^{n} \sum_{i=j}^{n} m_i \, C_{\Theta ji} \, \dot{q}_j - \Big(\sum_{i=1}^{n} m_i \Big) a_0 \, c_\Theta \, \dot{q}_1 \\
-\sum_{j=1}^{n} \sum_{i=j}^{n} m_i \, (\tilde{r}^i_{0y} S_{ji} + \tilde{r}^i_{0x} \, C_{ji}) \dot{q}_j - \Big(\sum_{i=1}^{n} m_i \tilde{r}^i_{0x} \Big) \, a_0 \, \dot{q}_1
\end{bmatrix}
\tag{43}
$$

Now, we formulate conditions for the invertibility of the matrix on the left-hand side. Clearly, the matrix will be invertible if M_{33} is always positive. M_{33} can be rewritten in a simpler form based on the Jacobi theorem. Let a set of pairs (mass, its position), i.e. (m_k, d^k_I), $k = 0, \ldots, n$ be given. The *inertia momentum* for the set w.r.t. any point S is given by the formula

$$
J_S = \sum_{k=0}^{n} m_k |d^k_I - S|^2 \tag{44}
$$

When S is chosen at the centre of mass, a simpler formula is obtained. We have

Jacobi Theorem. (Balk and Boltyanskii, 1987)

$$
J_Z = \frac{1}{m} \sum_{0 \le i < j \le n} m_i m_j |d^i_I - d^j_I|^2 \tag{45}
$$

For the case of the n-pendulum and S chosen at the mass centre of the mass system, we have

$$
M_{33} = \sum_{k=0}^{n} m_k |\tilde{r}^k_0|^2 = \frac{1}{m} \sum_{0 \le i < j \le n} m_i m_j |d^i_0 - d^j_0|^2 \tag{46}
$$

Now, we can state a condition for $M_{33} > 0$ in the case of a real free-floating n-pendulum. This condition will be formulated in a very weak form as follows:

Theorem 3.

$$\exists_{i\in[0,n-1]} m_i, m_{i+1}, a_{i+1} > 0 \implies \sum_{k=0}^{n} m_k |\tilde{r}_0^k|^2 > 0 \tag{47}$$

where a_i is the length of the i-th link.

Proof.

$$M_{33} = \frac{1}{m} \sum_{0\le i<j\le n} m_i m_j |d_0^i - d_0^j|^2 \ge \sum_{i=0}^{n-1} \frac{1}{m} m_i m_{i+1} a_{i+1}^2 > 0$$

For the case of a real n-pendulum it follows that $\forall_{i\in[0,n-1]} m_i, m_{i+1}, a_{i+1} > 0$. ■

5. Kinematics of the 2-Pendulum

In this section a closed-form kinematics of a 2-pendulum being the simplest free-floating robot will be derived. Additionally, it is assumed that $a_0 = 0$. The notation $m_{ij} = m_i + m_j$ will be used henceforth.

For the case of the 2-pendulum the components of eqns. (43) are as follows:

- $M_{13} = -m_{12} a_1 s_{\Theta 1} - m_2 a_2 s_{\Theta 12}$
- $M_{23} = m_{12} a_1 c_{\Theta 1} + m_2 a_2 c_{\Theta 12}$
- $M_{33} = \{m_0 m_{12} a_1^2 + m_2 m_{01} a_2^2 + 2 m_0 m_2 a_1 a_2 c_2\}/m$
- $r_{0x}^c = \{m_{12} a_1 c_1 + m_2 a_2 c_{12}\}/m$
- $r_{0y}^c = \{m_{12} a_1 s_1 + m_2 a_2 s_{12}\}/m$
- $\sum_{j=1}^2 \sum_{i=j}^2 m_i S_{\Theta ji} \dot{q}_j = \{m_{12} a_1 s_{\Theta 1} + m_2 a_2 s_{\Theta 12}\}\dot{q}_1 + m_2 a_2 s_{\Theta 12} \dot{q}_2$
- $\sum_{j=1}^2 \sum_{i=j}^2 m_i C_{\Theta ji} \dot{q}_j = \{m_{12} a_1 c_{\Theta 1} + m_2 a_2 c_{\Theta 12}\}\dot{q}_1 + m_2 a_2 c_{\Theta 12} \dot{q}_2$
- $\sum_{j=1}^2 \sum_{i=j}^2 m_i (\tilde{r}_{0y}^i S_{ji} + \tilde{r}_{0x}^i C_{ji}) = \{m_0 m_{12} a_1^2 + m_2 m_{01} a_2^2 + 2 m_0 m_2 a_1 a_2 c_2\}\dot{q}_1/m$
  $+\{m_2 m_{01} a_2^2 + m_0 m_2 a_1 a_2 c_2\}\dot{q}_2/m$

Substituting these equations into (43), we get

$$\begin{aligned}\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\Theta}\end{bmatrix} &= X_1\cdot\dot{q}_1 + X_2\cdot\dot{q}_2 = X_1\cdot u_1 + X_2\cdot u_2\\ &= \begin{bmatrix}0\\0\\-1\end{bmatrix}u_1 + \begin{bmatrix}X_{21}(\Theta,q_1,q_2)\\ X_{22}(\Theta,q_1,q_2)\\ X_{23}(q_2)\end{bmatrix}u_2 = \begin{bmatrix}0\\0\\-1\end{bmatrix}u_1\\ &+\begin{bmatrix}\frac{m_2a_1a_2}{m^2M_{33}}\{m_0m_{12}a_1c_{\Theta 1}s_2 + m_0m_2a_2s_{\Theta 12}c_2 - m_{01}m_{12}a_2s_{\Theta 1}\}\\ \frac{-m_2a_1a_2}{m^2M_{33}}\{-m_0m_{12}a_1s_{\Theta 1}s_2 + m_0m_2a_2c_{\Theta 12}c_2 - m_{01}m_{12}a_2c_{\Theta 1}\}\\ \frac{-m_2a_2}{mM_{33}}\{m_{01}a_2 + m_0a_1c_2\}\end{bmatrix}u_2\end{aligned} \tag{48}$$

The simplest mobile robot (a unicycle) depicted in Fig. 3 has the kinematics of the form (Laumond *et al.*, 1994):

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\Theta}\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}u_1 + \begin{bmatrix}\cos(\Theta)\\ \sin(\Theta)\\ 0\end{bmatrix}\cdot u_2 \tag{49}$$

with controls $u_1 = v\,\sin(\phi)$ and $u_2 = v\,\cos(\phi)$. Here v is the linear velocity of the vehicle and ϕ is the steering angle.

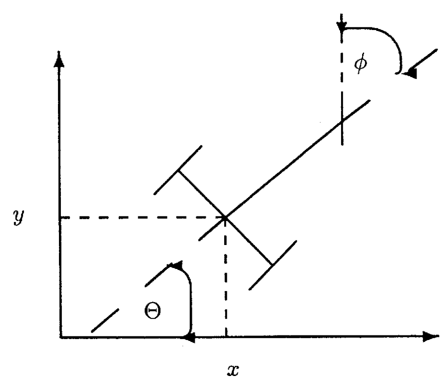

Fig. 3. External coordinates of a unicycle.

As can be seen from (48) and (49), the structure of both equations is the same, but the equation for the free-floating robot is much more computationally involved. A numerical complexity of models for n-pendulums will grow rapidly with the number of links n, cf. (43).

6. Conclusions

In this paper decomposed kinematic equations for a free floating robot with rotational joints have been presented. In our opinion the equations are the simplest possible, and therefore suitable for implementation. Conditions for the invertibility of a matrix used in the generalized Jacobian matrix have been formulated and proved for a general free-floating robot and its special, planar case, i.e. the n-pendulum. The conditions are very weak and any real robot easily satisfies them. Kinematics of a family of n-pendulums have been derived. They can be used as a real model of a nonholonomic system for testing purposes. As has been shown, the simplest free-floating robot has much more complicated kinematics than the simplest mobile robot, although both equations share the same structure.

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