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ROBUST STABILIZATION FOR UNCERTAIN TIME-VARYING DELAY CONSTRAINED SYSTEMS WITH DELAY-DEPENDENCE

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This paper is concerned with the problem of robust stabilization of linear time-varying delay systems containing saturating actuators in the presence of nonlinear parametric perturbations. Based on Razumikhin's approach to the stability of functional differential equations, we determine upper bounds on the time-varying delay such that the uncertain system under consideration is robustly globally or locally asymptotically stabilizable via memoryless state feedback control laws. The obtained bounds are given in terms of solutions to Lyapunov equations. Two numerical examples are included to illustrate the results.

Keywords: time-delay systems, stabilization, robustness, nonlinear parametric perturbations, saturating actuators.

1. Introduction

Time-delays are frequently encountered and their existence is often the source of instability and poor performance (Malek-Zaveri and Jamshidi, 1987). The problems of stability analysis and stabilization of dynamic systems with time-delay are, therefore, of theoretical and practical importance and have attracted considerable attention for several decades. Various techniques of stability and robust stability analysis have been proposed over the past few years, including delay-dependent stability criteria which include the information on the size of time-delay (Cheres *et al.*, 1989; Gu, 1997; 1999; Li and de Souza, 1997a; 1997b; Mori, 1985; Mori and Kokame, 1989; Shyu and Yan, 1993; Su and Huang, 1992; Thowsen, 1982; Xu, 1994), as well as those which are independent of time-delay (Chen and Latchman, 1995; Chen *et al.*, 1995; Han and Mehdi, 1998a; Hmamed, 1986; Kamen, 1982; Luo and Van den Bosch, 1997; Mori *et al.*, 1981; 1983; Trinh and Aldeen, 1994).

In many practical control problems, nonlinear actuators are also frequently met, and one of the common nonlinearities is saturation. If the saturating actuators are not considered, such a system will produce many difficulties, not only during starting-up and shifting-down, but also during sudden changes. Therefore, many researchers

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have investigated the stabilization of linear (time-delay) systems with saturating actuators. For example, using differential inequality techniques and the Bellman-Gronwall lemma, several sufficient conditions to guarantee the stability of the saturating time delay are derived in (Chen *et al.*, 1988). Improved results over those of (Chen *et al.*, 1988) are given in (Tissir and Hmamed, 1992). Some sufficient delay-independent conditions addressing the global (or local) asymptotic stabilization of a linear time-delay system with saturating controls are presented in (Klai *et al.*, 1994). Sufficient conditions on the time delay, which maintain the asymptotic stability of the closed-loop system via a state feedback control law, are obtained in (Su *et al.*, 1991). The case of a linear system with delayed input is treated in (Liu, 1995; Shen and Kung, 1989) where sufficient conditions are derived for stabilization via a state feedback controller, a dynamic controller and an observer-based controller.

Recently, a keen interest has been taken in robust stabilization for uncertain time-delay systems containing saturating actuators. For example, by using a matrix measure and comparison theory, a sufficient condition for the dynamic feedback compensator, independent of the delay, is derived in (Chou *et al.*, 1989). By employing Razumikhin's approach, an observer-based controller is designed to stabilize an uncertain time-delay system with a saturating actuator in (Han and Mehdi, 1998b). Some sufficient conditions are also proposed. These results are less restrictive than those derived in (Su *et al.*, 1989). By the Lyapunov-Krasovskii technique, the results in (Han and Mehdi, 1998b; Su *et al.*, 1989) are improved in (Han *et al.*, 1998). Using Razumikhin's approach, two sufficient delay-dependent criteria are given in (Niculescu *et al.*, 1996) for robust stabilization via memoryless state feedback control laws of uncertain time-delay systems with a saturating actuator. In (Niculescu *et al.*, 1996), the admissible uncertainties are assumed to be of linear time-varying forms, and upper bounds on the time delay are given in terms of solutions to appropriate finite-dimensional Riccati equations.

In this paper, we investigate the problem of robust stabilization of uncertain linear time-delay systems containing a saturating actuator. These uncertainties may be linear, non-linear and/or time-varying, but only norm bounds are known. Razumikhin's approach is employed to propose robust global or local asymptotic stability conditions.

This paper is organized as follows. Section 2 presents briefly the problem statement. Section 3 gives the robust global asymptotic stabilization result. A positive invariant and robust local asymptotic stability domain is determined in Section 4. In Section 5, two numerical examples illustrate the results. Finally, Section 6 gives conclusions.

Notation:

$\mathbb{R}$	the real number field
$\mathbb{R}_+$	the set of positive real numbers
$\mathbb{R}^n$	the n-dimensional real vector space
$\mathbb{R}^{n\times n}$	the space of real $n \times n$ matrices
x	a vector, $x = [x_1\ x_2\ \ldots\ x_n]^T,\ x_i \in \mathbb{R}$

$x^T(A^T)$	the transpose of a vector x (matrix A)
$\lambda_i[A]$	the i-th eigenvalue of a matrix A
$\sigma[A]$	the spectrum of a matrix A (the set of all eigenvalues of A)
$\lambda_{\max}[A]$	the maximum eigenvalue of a matrix $A = A^T$, i.e. $\lambda_{\max}[A] = \max_i\{\lambda_i[A]\}$
$\lambda_{\min}[A]$	the minimum eigenvalue of a matrix $A = A^T$, i.e. $\lambda_{\min}[A] = \min_i\{\lambda_i[A]\}$
$\|x\|$	the Euclidean norm of x, $\|x\| = \sqrt{x^T x}$
$\|A\|$	the norm of a matrix A defined as $\|A\| = \sqrt{\lambda_{\max}[A^T A]}$
$\mu(A)$	the matrix measure of a matrix A defined as $\mu(A) = \frac{1}{2}\lambda_{\max}[A + A^T]$
I	the identity matrix
$Q^{1/2}$	the square root of a symmetric positive definite matrix Q ($Q^{1/2} = V\Lambda^{1/2}V^T$, V being the eigenvector matrix of Q satisfying $VV^T = I$ and Λ the diagonal eigenvalues matrix of Q)
$x \leq y$	the partial ordering relation in $\mathbb{R}^n$ equivalent to $x_i \leq y_i$ for $i = 1, 2, \ldots, n$ and $x,\ y \in \mathbb{R}^n$

2. Problem Formulation

Consider the uncertain system described by the following differential-difference equation:

$$\begin{aligned}\dot{x}(t) &= A_0 x(t) + A_1 x\big(t - h(t)\big) + \big(B + \Delta B(t)\big)u(t) + f_0\big(x(t), t\big) \\ &\quad + f_1\left(x\big(t - h(t)\big), t\right), \quad \forall t > t_0 \geq 0 \end{aligned} \tag{1}$$

where $x(\cdot) \in \mathbb{R}^n$ is the state vector, $u(\cdot) \in \mathbb{R}^m$ is the control vector, A_0 and A_1 are matrices in $\mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. The time-varying delay $h(t)$ is a nonnegative, bounded and continuous function, i.e. $0 < h(t) < \bar{h}$, where $\bar{h}$ is a positive constant. Here, it is worth noticing that we do not require for the time derivative of the time-varying delay $h(t)$ to be less than one, i.e. $\dot{h}(t) < 1$. It is well-known that such an assumption is often needed in many works dealing with the stability problem for systems with time-varying delay (Ikeda and Ashida, 1979).

The initial condition for (1) is given by

$$x(\theta) = \varphi(\theta), \quad \theta \in [t_0 - \bar{h},\ t_0] \tag{2}$$

where $\varphi(\cdot)$ is a given continuous vector-valued function on the interval $[t_0 - \bar{h},\ t_0]$.

Here $\Delta B(t)$ is the time-varying perturbation matrix of input matrix B. The uncertainties $f_0(x(t), t)$ and $f_1(x(t - h(t)), t)$, which are smooth vector-valued functions satisfying $f_0(0, t) = 0$ and $f_1(0, t) = 0$, are unknown and represent the system's nonlinear parametric perturbations with respect to the current state $x(t)$ and the delayed state $x(t - h(t))$, respectively. In general, it is assumed that $\|\Delta B(t)\|$, $\|f_0(x(t), t)\|$

and $\|f_1(x(t-h(t)),t)\|$ are bounded, i.e.

$$\|\Delta B(t)\| \leq \delta \tag{3}$$

$$\|f_0(x(t),t)\| \leq \beta_0 \|x(t)\| \tag{4}$$

$$\left\|f_1\big(x\big(t-h(t)\big),t\big)\right\| \leq \beta_1 \left\|x\big(t-h(t)\big)\right\| \tag{5}$$

where $\delta > 0$, $\beta_0 > 0$ and $\beta_1 > 0$ are given.

The control vector $u(\cdot) \in \mathbb{R}^m$ is assumed to belong to a compact set $\Omega \subset \mathbb{R}^m$, $\forall t > t_0 \geq 0$, defined by

$$\Omega = \big\{u(\cdot) \in \mathbb{R}^m \;\big|\; -u_m \leq u(\cdot) \leq u_M;\; u_m,\, u_M \in \mathbb{R}_+^m\big\} \tag{6}$$

Assumption 1. The pair (A_0+A_1, B) is stabilizable and all the states of system (1) are available.

Note that Assumption 1, which is equivalent to the stabilizability of system (1) without time-delay and uncertainty, is necessary for the existence of a stabilizing memoryless state feedback control law for the system (1).

By implementing a saturated controller

$$u(\cdot) = \mathrm{sat}\big(Fx(\cdot)\big), \quad F \in \mathbb{R}^{m\times n} \tag{7}$$

the system (1) becomes

$$\begin{aligned}\dot{x}(t) &= A_0x(t) + A_1x\big(t-h(t)\big) + \big(B+\Delta B(t)\big)\mathrm{sat}\big(Fx(t)\big) + f_0\big(x(t),t\big)\\ &\quad + f_1\left(x\big(t-h(t)\big),t\right), \quad \forall t > t_0 \geq 0\end{aligned} \tag{8}$$

where the saturation term is given by

$$\mathrm{sat}\big(Fx(\cdot)\big) = \big[\mathrm{sat}\big([Fx(\cdot)]_1\big), \dots, \mathrm{sat}\big([Fx(\cdot)]_m\big)\big] \tag{9}$$

The operation range of the nonlinear saturation $\mathrm{sat}\left([Fx(\cdot)]_i\right)$, $i = 1, 2, \dots, m$ is considered inside the sector $[w, 1]$ which means that the graph of the nonlinearity lies between two straight lines passing through the origin with slopes w and 1, respectively, with $0 \leq w \leq 1$. The saturating actuator $\mathrm{sat}\left([Fx(\cdot)]_i\right)$ saturates at $-(u_m)_i$ or $(u_M)_i$, as shown in Fig. 1.

Note that the system (8) is of nonlinear nature. However, when the controls do not saturate, i.e. for any $x(t)$ such that $Fx(t) \in \Omega$, (8) admits the following model:

$$\begin{aligned}\dot{x}(t) &= \Big(A_0 + \big(B+\Delta B(t)\big)F\Big)x(t) + A_1x\big(t-h(t)\big) + f_0\big(x(t),t\big)\\ &\quad + f_1\left(x\big(t-h(t)\big),t\right), \quad \forall t > t_0 \geq 0\end{aligned} \tag{10}$$

which is valid only in the set of admissible states defined by

$$\mathcal{D}(F,\Omega) = \big\{x(\cdot) \in \mathbb{R}^m \;\big|\; -u_m \leq Fx(\cdot) \leq u_M;\; u_m,\, u_M \in \mathbb{R}_+^m\big\} \tag{11}$$

Before stating the problems to be treated in this paper, let us give the following definition.

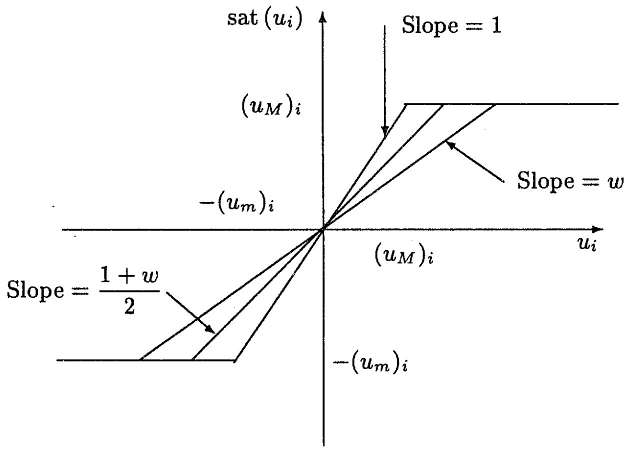

Fig. 1. The saturation function for the nonlinear sector $[w, 1]$, $0 \le w \le 1$.

Definition 1. A non-empty set $\mathcal{D} \subset \mathbb{R}^n$ is said to be *positively invariant* w.r.t. the motions of the system (8) if for every $\varphi(\theta) \in \mathcal{D}$ $(\forall \theta \in [t_0 - \bar{h},\ t_0])$, the motion $x(t, t_0, \varphi) \in \mathcal{D}$, $\forall t > t_0 \ge 0$.

Our purpose in this paper is:

1) to determine a static state feedback matrix F and to express some conditions for the parametric perturbations so that the closed-loop systems (8) is robustly globally asymptotically stable;
2) to determine a positive invariant and robust local asymptotic stability domain for (8) in which the behaviour is linear.

3. Robust Global Asymptotic Stabilization

From Assumption 1, there exists a matrix F such that $A_0 + A_1 + (1/2)(1+w)BF$ is Hurwitz with $0 \le w \le 1$. Then there exists a unique symmetric positive definite matrix P which is a solution to the Lyapunov equation

$$\left(A_0 + A_1 + \frac{1+w}{2}BF\right)^T P + P\left(A_0 + A_1 + \frac{1+w}{2}BF\right) = -Q \qquad (12)$$

for any given symmetric positive definite matrix Q.

By adding and subtracting $(A_1 + (1/2)(1+w)(B + \Delta B(t))F)\,x(t)$ to the right-hand side of (8), we get

$$\begin{aligned}\dot{x}(t) &= \left(A_0 + A_1 + \frac{1+w}{2}(B + \Delta B(t))F\right) x(t) + A_1\Big[x\big(t - h(t)\big) - x(t)\Big] \\ &\quad + f_0\big(x(t), t\big) + f_1\left(x\big(t - h(t)\big), t\right) \\ &\quad + \big(B + \Delta B(t)\big)\left(\text{sat}\big(Fx(t)\big) - \frac{1+w}{2}Fx(t)\right), \quad \forall t > t_0 \geq 0 \qquad (13)\end{aligned}$$

Without loss of generality, we supplement the definition of $x(t)$ on the interval $[t_0 - 2\bar{h}, t_0 - \bar{h}]$ by $x(\theta) = \varphi(\theta) = \varphi(t_0 - \bar{h})$, $\forall \theta \in [t_0 - 2\bar{h}, t_0 - \bar{h}]$. Since $x(t)$ is continuously differentiable $\forall t > t_0 \geq 0$, one can write (Hale, 1977)

$$x\big(t - h(t)\big) - x(t) = -\int_{-h(t)}^{0} \dot{x}(t + \theta)\,\mathrm{d}\theta \qquad (14)$$

Then (13) can also be written as

$$\begin{aligned}\dot{x}(t) &= \left(A_0 + A_1 + \frac{1+w}{2}(B + \Delta B(t))F\right)x(t) \\ &\quad - A_1 \int_{-h(t)}^{0} \dot{x}(t + \theta)\,\mathrm{d}\theta + f_0\big(x(t), t\big) + f_1\left(x\big(t - h(t)\big), t\right) \\ &\quad + \big(B + \Delta B(t)\big)\left(\text{sat}\big(Fx(t)\big) - \frac{1+w}{2}Fx(t)\right), \quad \forall t > t_0 \geq 0 \qquad (15)\\ x(\theta) &= \varphi(\theta), \quad \forall \theta \in \big[t_0 - 2\bar{h},\ t_0\big] \qquad (16)\end{aligned}$$

Remark 1. From the above discussion, any solution to the functional differential equation associated with (8) (with the initial condition (2)) is also a solution to the functional differential equations (15), (16). In view of this statement, a sufficient condition for the robust global asymptotic stability of (15), (16) is also a sufficient condition for the robust global asymptotic stability of (8) (with the initial condition (2)) and thus a sufficient condition for the robust global asymptotic stabilization of uncertain time-delay systems (1) and (2) via the memoryless state feedback control law given in Theorem 1 below.

Theorem 1. *Consider the system (1), (2) satisfying Assumption 1. If the state feedback matrix F is chosen in such a way that the inequality*

$$\begin{aligned}&0 \leq h(t) \leq \bar{h} < \bar{h}_G \\ &= \frac{\lambda_{\min}[G^T Q G] - 2\left(\frac{1-w}{2}\left\|G^T P B\right\| + \delta\left\|G^T P\right\|\right)\|FG\| - 2\left\|G^T P\right\|\|G\|(\beta_0 + \alpha\beta_1)}{2\alpha k_G}\end{aligned} \qquad (17)$$

is satisfied with

$$
\begin{aligned}
k_G = {} & \left\| G^T P A_1 \left(A_0 + \frac{1+w}{2} BF \right) G \right\| + \left\| G^T P A_1^2 G \right\| \\
& + \left(\frac{1-w}{2} \left\| G^T P A_1 B \right\| + \delta \left\| G^T P A_1 \right\| \right) \|FG\| \\
& + \left\| G^T P A_1 \right\| \|G\| (\beta_0 + \beta_1)
\end{aligned} \tag{18}
$$

then the system (1), (2) is robustly globally asymptotically stabilizable via the memoryless state feedback control law (7), where $G \in \mathbb{R}^{n\times n}$ *is any nonsingular matrix and* $\alpha = \sqrt{\lambda_{\max}[P]/\lambda_{\min}[P]}$ *and the symmetric positive definite matrices* P *and* Q *satisfy the Lyapunov equation (12).*

To prove Theorem 1, we have to make the following observation.

Observation 1. (Shyu and Yan, 1993) Consider the positive definite function

$$
\mathcal{V}\big(x(t)\big) = x^T(t) P x(t), \quad x(t) \in \mathbb{R}^n, \quad \forall t \geq t_0 - 2\bar{h} \tag{19}
$$

where P is the unique symmetric positive definite solution to (12). Assume that there exists a constant $q > 1$ such that

$$
\mathcal{V}\left(x\big(t - h(t)\big)\right) < q^2 \mathcal{V}\big(x(t)\big) \tag{20}
$$

Then we have

$$
\left\| x\big(t - h(t)\big) \right\| < q\alpha \|x(t)\| \tag{21}
$$

where α is given in Theorem 1.

Proof of Theorem 1. Let us take the positive definite function (19) as a Lyapunov function candidate for (15). Then the time derivative of $\mathcal{V}(x(t))$ along the trajectories of system (15) is as follows:

$$
\begin{aligned}
\dot{\mathcal{V}}\big(x(t)\big) = {} & -x^T(t) Q x(t) + (1+w) x^T(t) P \Delta B(t) F x(t) \\
& - 2x^T(t) P A_1 \int_{-h(t)}^{0} \dot{x}(t+\theta)\, \mathrm{d}\theta \\
& + 2x^T(t) P \Big(f_0\big(x(t), t\big) + f_1\left(x\big(t - h(t)\big), t\right) \Big) \\
& + 2x^T(t) P \big(B + \Delta B(t)\big) \left(\operatorname{sat}\big(Fx(t)\big) - \frac{1+w}{2} F x(t) \right)
\end{aligned} \tag{22}
$$

By introducing a nonsingular matrix G, we get

$$\begin{aligned}\dot{\mathcal{V}}(x(t)) = & -\big(G^{-1}x(t)\big)^T G^T Q G\big(G^{-1}x(t)\big) + (1+w)\big(G^{-1}x(t)\big)^T G^T P \Delta B(t) F G \\ & \times \big(G^{-1}x(t)\big) - 2\big(G^{-1}x(t)\big)^T G^T P A_1 \int_{-h(t)}^{0} G G^{-1}\dot{x}(t+\theta)\,\mathrm{d}\theta \\ & + 2\big(G^{-1}x(t)\big)^T G^T P\Big(f_0\big(GG^{-1}x(t),t\big) + f_1\big(GG^{-1}x(t-h(t)),t\big)\Big) \\ & + 2\big(G^{-1}x(t)\big)^T G^T P\big(B + \Delta B(t)\big) \\ & \times \left(\mathrm{sat}\,\big(FG(G^{-1}x(t))\big) - \frac{1+w}{2} FG\big(G^{-1}x(t)\big)\right) \end{aligned} \tag{23}$$

In view of Fig. 1, and based on the definition of the norm function, it is easy to show that (Chen *et al.*, 1988)

$$\left\| \mathrm{sat}\,\big(FG(G^{-1}x(t))\big) - \frac{1+w}{2} FG\big(G^{-1}x(t)\big) \right\| \le \frac{1-w}{2} \|FG\| \, \big\|G^{-1}x(t)\big\| \tag{24}$$

From (3), (4) and (24), we have

$$\begin{aligned}(1+w)\big(G^{-1}x(t)\big)^T G^T P \Delta B(t) F G\big(G^{-1}x(t)\big) \\ \le \delta(1+w)\, \big\|G^T P\big\| \, \|FG\| \, \big\|G^{-1}x(t)\big\|^2 \end{aligned} \tag{25}$$

and

$$2\big(G^{-1}x(t)\big)^T G^T P f_0\big(GG^{-1}x(t),t\big) \le 2\beta_0 \big\|G^T P\big\| \, \|G\| \, \big\|G^{-1}x(t)\big\|^2 \tag{26}$$

with

$$\begin{aligned}2\big(G^{-1}x(t)\big)^T G^T P\big(B + \Delta B(t)\big) \left(\mathrm{sat}\,\big(FG(G^{-1}x(t))\big) - \frac{1+w}{2} FG\big(G^{-1}x(t)\big)\right) \\ \le (1-w)\Big(\big\|G^T P B\big\| + \delta \big\|G^T P\big\| \Big) \|FG\| \, \big\|G^{-1}x(t)\big\|^2 \end{aligned} \tag{27}$$

From (3)–(5), (24) and Observation 1, we get

$$
\begin{aligned}
&-2\big(G^{-1}x(t)\big)^T G^T P A_1 \int_{-h(t)}^{0} G G^{-1}\dot{x}(t+\theta)\,\mathrm{d}\theta \\
&\le \left\| -2\big(G^{-1}x(t)\big)^T G^T P A_1 \int_{-h(t)}^{0} G G^{-1}\dot{x}(t+\theta)\,\mathrm{d}\theta \right\| \le 2\left\| \big(G^{-1}x(t)\big)^T G^T P A_1 \right. \\
&\quad \times \left. \int_{-h(t)}^{0} \left[\left(A_0 + \frac{1+w}{2}(B+\Delta B(t+\theta))F\right) G G^{-1}x(t+\theta)\right] \mathrm{d}\theta \right\| \\
&\quad + 2\left\| \big(G^{-1}x(t)\big)^T G^T P A_1 \int_{-h(t)}^{0} A_1 G G^{-1}x\big(t+\theta-h(t+\theta)\big)\,\mathrm{d}\theta \right\| \\
&\quad + 2\left\| \big(G^{-1}x(t)\big)^T G^T P A_1 \int_{-h(t)}^{0} f_0\big(G G^{-1}x(t+\theta), t+\theta\big)\,\mathrm{d}\theta \right\| \\
&\quad + 2\left\| \big(G^{-1}x(t)\big)^T G^T P A_1 \int_{-h(t)}^{0} f_1(G G^{-1}x\big(t+\theta-h(t+\theta)\big), t+\theta)\,\mathrm{d}\theta \right\| \\
&\quad + 2\left\| \big(G^{-1}x(t)\big)^T G^T P A_1 \int_{-h(t)}^{0} (B+\Delta B(t+\theta)) \right. \\
&\quad \times \left. \left(\mathrm{sat}\left(FG\big(G^{-1}x(t+\theta)\big)\right) - \frac{1+w}{2} FG\big(G^{-1}x(t+\theta)\big)\right) \mathrm{d}\theta \right\| \\
&\le 2q\alpha\bar{h}\left[\left\| G^T P A_1 \left(A_0 + \frac{1+w}{2}BF\right) G \right\| + \delta\frac{1+w}{2}\left\|G^T P A_1\right\| \|FG\| \right. \\
&\quad + \left\|G^T P A_1^2 G\right\| + \beta_0\left\|G^T P A_1\right\| \|G\| + \beta_1\left\|G^T P A_1\right\| \|G\| \\
&\quad \left. + \frac{1-w}{2}\left\|G^T P A_1 B\right\| \|FG\| + \delta\frac{1-w}{2}\left\|G^T P A_1\right\| \|FG\| \right] \left\|G^{-1}x(t)\right\|^2 \\
&= 2q\alpha\bar{h}\left[\left\| G^T P A_1 \left(A_0 + \frac{1+w}{2}BF\right) G \right\| \right. \\
&\quad + \|FG\| \left(\frac{1-w}{2}\left\|G^T P A_1 B\right\| + \delta\left\|G^T P A_1\right\|\right) \\
&\quad \left. + \left\|G^T P A_1^2 G\right\| + \left\|G^T P A_1\right\| \|G\| (\beta_0+\beta_1) \right] \left\|G^{-1}x(t)\right\|^2 \\
&= 2q\alpha\bar{h}k_G \left\|G^{-1}x(t)\right\|^2
\end{aligned}
\tag{28}
$$

and

$$2\left(G^{-1}x(t)\right)^T G^T P f_1\left(GG^{-1}x(t-h(t)),t\right) \le 2q\alpha\beta_1 \left\|G^T P\right\| \|G\| \left\|G^{-1}x(t)\right\|^2 \tag{29}$$

From (25)–(29), we have

$$\begin{aligned}\dot{\mathcal{V}}(x(t)) \;\le\; -\Big\{\lambda_{\min}\left[G^T Q G\right] - 2\|FG\| \left(\frac{1-w}{2}\left\|G^T PB\right\| + \delta\left\|G^T P\right\|\right) \\ -2\beta_0 \left\|G^T P\right\| \|G\| - 2q\alpha\bar{h}k_G - 2q\alpha\beta_1\left\|G^T P\right\| \|G\|\Big\} \left\|G^{-1}x(t)\right\|^2\end{aligned} \tag{30}$$

If (17) holds, then there exists a constant

$$1 < q < \frac{\lambda_{\min}\left[G^T Q G\right] - 2\|FG\|\left(\frac{1-w}{2}\left\|G^T PB\right\| + \delta\left\|G^T P\right\|\right) - 2\beta_0\left\|G^T P\right\| \|G\|}{2\alpha\bar{h}k_G + 2\alpha\beta_1\|G^T P\| \|G\|} \tag{31}$$

such that

$$\begin{aligned}\lambda_{\min}\left[G^T Q G\right] - 2\|FG\|\left(\frac{1-w}{2}\left\|G^T PB\right\| + \delta\left\|G^T P\right\|\right) \\ -2\beta_0\left\|G^T P\right\| \|G\| - 2q\alpha\left(\bar{h}k_G + \beta_1\left\|G^T P\right\| \|G\|\right) = \epsilon \;>\; 0\end{aligned} \tag{32}$$

which implies, for any $x(t) \neq 0$,

$$\dot{\mathcal{V}}(x(t)) \le -\epsilon\left\|G^{-1}x(t)\right\|^2 \tag{33}$$

Thus, according to the Razumikhin theorem (Hale, 1977), the system (15) and hence (8) is robustly globally asymptotically stable, therefore (1), (2) is robustly globally asymptotically stabilizable. ∎

Remark 2. When $G = Q^{-\frac{1}{2}}$, then (17) becomes

$$\begin{aligned}&0 \le h(t) \le \bar{h} < \bar{h}_Q \\ &= \frac{1 - 2\left(\frac{1-w}{2}\|Q^{-\frac{1}{2}}PB\| + \delta\|Q^{-\frac{1}{2}}P\|\right)\|FQ^{-\frac{1}{2}}\| - 2\|Q^{-\frac{1}{2}}P\| \, \|Q^{-\frac{1}{2}}\|\,(\beta_0 + \alpha\beta_1)}{2\alpha k_Q}\end{aligned} \tag{34}$$

with

$$\begin{aligned}k_Q \;=\; &\left\|Q^{-\frac{1}{2}}PA_1\left(A_0 + \frac{1+w}{2}BF\right)Q^{-\frac{1}{2}}\right\| + \left\|Q^{-\frac{1}{2}}PA_1^2Q^{-\frac{1}{2}}\right\| \\ &+ \left(\frac{1-w}{2}\left\|Q^{-\frac{1}{2}}PA_1B\right\| + \delta\left\|Q^{-\frac{1}{2}}PA_1\right\|\right)\left\|FQ^{-\frac{1}{2}}\right\| \\ &+ \left\|Q^{-\frac{1}{2}}PA_1\right\| \left\|Q^{-\frac{1}{2}}\right\| (\beta_0 + \beta_1)\end{aligned} \tag{35}$$

When $G = I$, from (17) we have

$$0 \le h(t) \le \bar{h} < \bar{h}_I$$
$$= \frac{\lambda_{\min}[Q] - 2\left(\frac{1-w}{2}\|PB\| + \delta\|P\|\right)\|F\| - 2\|P\|(\beta_0 + \alpha\beta_1)}{2\alpha k_I} \tag{36}$$

with

$$k_I = \left\|PA_1\left(A_0 + \frac{1+w}{2}BF\right)\right\| + \|PA_1^2\| + \left(\frac{1-w}{2}\|PA_1B\| + \delta\|PA_1\|\right)\|F\| + \|PA_1\|(\beta_0 + \beta_1) \tag{37}$$

It is easy to verify that

$$\bar{h}_I \le \bar{h}_Q \tag{38}$$

In fact, since

$$\bar{h}_Q = \frac{1 - 2\left(\frac{1-w}{2}\|Q^{-\frac{1}{2}}PB\| + \delta\|Q^{-\frac{1}{2}}P\|\right)\|FQ^{-\frac{1}{2}}\| - 2\|Q^{-\frac{1}{2}}P\|\,\|Q^{-\frac{1}{2}}\|(\beta_0 + \alpha\beta_1)}{2\alpha k_Q}$$
$$\ge \frac{1 - 2\left(\frac{1-w}{2}\|Q^{-\frac{1}{2}}\|\,\|PB\| + \delta\|Q^{-\frac{1}{2}}\|\,\|P\|\right)\|F\|\,\|Q^{-\frac{1}{2}}\|}{2\alpha\|Q^{-\frac{1}{2}}\|k_I\|Q^{-\frac{1}{2}}\|} - \frac{2\|Q^{-\frac{1}{2}}\|\,\|P\|\,\|Q^{-\frac{1}{2}}\|(\beta_0 + \alpha\beta_1)}{2\alpha\|Q^{-\frac{1}{2}}\|k_I\|Q^{-\frac{1}{2}}\|}$$
$$= \frac{\lambda_{\min}[Q] - 2\left(\frac{1-w}{2}\|PB\| + \delta\|P\|\right)\|F\| - 2\|P\|(\beta_0 + \alpha\beta_1)}{2\alpha k_I} = \bar{h}_I \tag{39}$$

Remark 3. When the uncertainties $f_0(x(t), t)$ and $f_1(x(t - h(t)), t)$ are linear and time-varying parametric perturbations, i.e. they are of the form

$$f_0(x(t), t) = \Delta A_0(t)x(t), \quad f_1(x(t - h(t)), t) = \Delta A_1(t)x(t - h(t)) \tag{40}$$

with known upper norm bounds

$$\|\Delta A_0(t)\| \le \beta_0, \quad \|\Delta A_1(t)\| \le \beta_1 \tag{41}$$

where $\beta_0 > 0$ and $\beta_1 > 0$ are given, (1) becomes

$$\dot{x}(t) = (A_0 + \Delta A_0(t))x(t) + (A_1 + \Delta A_1(t))x(t - h(t)) + (B + \Delta B(t))u(t), \quad \forall t \ge t_0 > 0 \tag{42}$$

It is easy to check that the results of Theorem 1 also apply to (42).

Remark 4. When $f_0(x(t),t) = 0$ and $f_1(x(t-h(t)),t) = 0$, the condition (36), with (37) in mind, becomes

$$0 \leq h(t) \leq \bar{h} < \bar{h}_{I0} = \frac{\lambda_{\min}[Q] - (1-w)\,\|PB\|\,\|F\|}{2\alpha\left(\|PA_1^2\| + \frac{1-w}{2}\|PA_1B\|\,\|F\| + \left\|PA_1\left(A_0 + \frac{1+w}{2}BF\right)\right\|\right)} \tag{43}$$

In case the delayed state $x(t-h(t))$ is not measurable and only the current state $x(t)$ is available, Chen *et al.* (1988) gave the following condition:

$$0 \leq h(t) \leq \bar{h} < \bar{h}_{\text{Chen}} = \frac{-\mu\left(A_0 + A_1 + \frac{1+w}{2}BF\right) - \frac{1-w}{2}\|B\|\,\|F\|}{\|A_1\|\left(\|A_1\| + \frac{1-w}{2}\|B\|\,\|F\| + \left\|A_0 + \frac{1+w}{2}BF\right\|\right)} \tag{44}$$

where $\mu\left(A_0 + A_1 + (1/2)(1+w)BF\right) < 0$. In this case, from the Lyapunov equation (12), we have $P = I$ (therefore $\alpha = 1$) and $\lambda_{\min}[Q] = -2\mu\left(A_0 + A_1 + (1/2)(1+w)BF\right)$ and, noting that $\|MN\| \leq \|M\|\,\|N\|$ for any matrices M and N, it is clear that $\bar{h}_{\text{Chen}} \leq \bar{h}_{I0}$. When $\mu\left(A_0 + A_1 + (1/2)(1+w)BF\right) \geq 0$, by condition (44) no conclusion can be made. However, we can use (43) in this paper to design a controller for the considered system. This means that our result is less restrictive than that given in (Chen *et al.*, 1988).

From the above analysis, the design procedure is proposed:

Algorithm 1:

Step 1. Check the upper norm bounds of the system uncertainties and adequately choose the eigenvalues λ_i, $i = 1, 2, \ldots, n$ in the open left complex half-plane.

Step 2. Find the corresponding F by any eigenvalue assignment technique and check if the robust stability conditions are satisfied. If so, then go to Step 4.

Step 3. Shift the eigenvalues to the left in the open left complex half-plane according to $\lambda_i := \lambda_i - \Delta\lambda_i$, $i = 1, 2, \ldots, n$, where $\Delta\lambda_i$, $i = 1, 2, \ldots, n$ are non-negative real numbers, and go to Step 2.

Step 4. Obtain the memoryless state feedback controller from (7).

4. Positive Invariant and Robust Local Asymptotic Stability Domain

Since the pair $(A_0 + A_1, B)$ is stabilizable (Assumption 1), there exists a matrix F such that $(A_0 + A_1 + BF)$ is Hurwitz, and then for any given symmetric positive definite matrix Q_1, there exists a unique symmetric positive definite matrix P_1 being a solution to the following Lyapunov equation:

$$(A_0 + A_1 + BF)^T P_1 + P_1 (A_0 + A_1 + BF) = -Q_1 \tag{45}$$

In much the same way as for (15), the system (10) can be written as

$$\begin{aligned} \dot{x}(t) &= \Big(A_0 + A_1 + \big(B + \Delta B(t)\big)F\Big)x(t) \\ &\quad - A_1 \int_{-h(t)}^{0} \bigg\{ \Big(A_0 + \big(B + \Delta B(t)\big)F\Big)x(t+\theta)A_1 x\big(t+\theta-h(t+\theta)\big) \\ &\quad + f_0\big(x(t+\theta), t+\theta\big) + f_1\big(x\big(t+\theta-h(t+\theta)\big), t+\theta\big) \bigg\}\,\mathrm{d}\theta \\ &\quad + f_0\big(x(t), t\big) + f_1\big(x\big(t-h(t)\big), t\big), \quad \forall t > t_0 \geq 0 \end{aligned} \tag{46}$$

with the initial condition (16). In this case, a positive invariant and robust local asymptotic stability domain of linear behaviour for (8) can be determined. The result is given below.

Theorem 2. *Consider the system (1), (2) satisfying Assumption 1. Let $\mathcal{V}(x(t)) = x^T(t)P_1x(t)$, with P_1 being the unique symmetric positive definite solution to the Lyapunov equation (45). If the state feedback matrix $F \in \mathbb{R}^{m\times n}$ is chosen to satisfy the inequality*

$$\begin{aligned} 0 &\leq h(t) \leq \bar{h} < \bar{h}_{GL} \\ &= \frac{\lambda_{\min}\left[G^TQ_1G\right] - 2\delta\left\|G^TP_1\right\|\|FG\| - 2\left\|G^TP_1\right\|\|G\|(\beta_0 + \alpha_1\beta_1)}{2\alpha_1 k_{GL}} \end{aligned} \tag{47}$$

with

$$\begin{aligned} k_{GL} &= \left\|G^TP_1A_1(A_0 + BF)G\right\| + \left\|G^TP_1A_1^2G\right\| \\ &\quad + \delta\left\|G^TP_1A_1\right\|\|FG\| + \left\|G^TP_1A_1\right\|\|G\|(\beta_0 + \beta_1) \end{aligned} \tag{48}$$

where $G \in \mathbb{R}^{n\times n}$ is any nonsingular matrix and $\alpha_1 = \sqrt{\lambda_{\max}[P_1]/\lambda_{\min}[P_1]}$, then there exists a positive scalar η for which the domain

$$\mathcal{D}(\mathcal{V}, \eta) = \left\{x(\cdot) \in \mathbb{R}^n \;\middle|\; \mathcal{V}\big(x(\cdot)\big) \leq \eta,\ \eta > 0\right\} \tag{49}$$

satisfying

$$\mathcal{D}(\mathcal{V}, \eta) \subseteq \mathcal{D}(F, \Omega) \tag{50}$$

is a positive invariant and robust local asymptotic stability domain of linear behaviour for (8).

Proof. If the state feedback gain F is chosen in such a way that the asymptotic stability of $A_0 + A_1 + BF$ is assured, the quadratic function $\mathcal{V}(x(t)) = x^T(t)P_1x(t)$ is positive definite. Computing the time derivative of $\mathcal{V}(x(t))$ along the trajectories of (46), we obtain

$$\begin{aligned}\dot{\mathcal{V}}(x(t)) &= -\left(G^{-1}x(t)\right)^T G^T Q_1 G\left(G^{-1}x(t)\right)\\ &\quad + 2\left(G^{-1}x(t)\right)^T G^T P_1 \Delta B(t) F G\left(G^{-1}x(t)\right)\\ &\quad - 2\left(G^{-1}x(t)\right)^T G^T P_1 A_1 \int_{-h(t)}^{0} GG^{-1}\dot{x}(t+\theta)\,\mathrm{d}\theta + 2\left(G^{-1}x(t)\right)^T G^T P_1\\ &\quad \times \Big(f_0\big(GG^{-1}x(t),t\big) + f_1\big(GG^{-1}x(t-h(t)),t\big)\Big) \end{aligned} \tag{51}$$

Following similar arguments as in the proof of Theorem 1, we get

$$\begin{aligned}\dot{\mathcal{V}}(x(t)) &\le -\Big\{\lambda_{\min}\left[G^TQ_1G\right] - 2\delta\,\|FG\|\,\left\|G^TP_1\right\| - 2\beta_0\left\|G^TP_1\right\|\,\|G\|\\ &\qquad - 2q\alpha_1\left(\bar{h}k_{GL} + \beta_1\left\|G^TP_1\right\|\,\|G\|\right)\Big\}\left\|G^{-1}x(t)\right\|^2\end{aligned} \tag{52}$$

If (47) holds, then there exists a constant

$$1 < q < \frac{\lambda_{\min}\left[G^TQ_1G\right] - 2\delta\,\|FG\|\,\left\|G^TP_1\right\| - 2\beta_0\left\|G^TP_1\right\|\,\|G\|}{2\alpha_1\bar{h}k_{GL} + 2\alpha_1\beta_1\left\|G^TP_1\right\|\,\|G\|} \tag{53}$$

such that

$$\begin{aligned}&\lambda_{\min}\left[G^TQ_1G\right] - 2\delta\,\|FG\|\,\left\|G^TP_1\right\|\\ &\qquad -2\beta_0\left\|G^TP_1\right\|\,\|G\| - 2q\alpha_1\left(\bar{h}k_{GL} + \beta_1\left\|G^TP_1\right\|\,\|G\|\right) = \epsilon_1 > 0\end{aligned} \tag{54}$$

which implies that for any $x(\cdot) \neq 0$,

$$\dot{\mathcal{V}}(x(t)) \le -\epsilon_1\left\|G^{-1}x(t)\right\|^2 \tag{55}$$

Thus $\mathcal{V}(x(t)) = x^T(t)P_1x(t)$ is a Lyapunov function of (46), and so is (10), generating an elliptic domain $\mathcal{D}(\mathcal{V},\eta)$. Hence, since the scalar η is chosen to satisfy (50), the set $\mathcal{D}(\mathcal{V},\eta)$ defined in (49) is a positive invariant and robust local asymptotic stability domain in which the behaviour is linear. ■

Remark 5. When $G = Q_1^{-\frac{1}{2}}$, the condition (47) becomes

$$\begin{aligned}&0 \le h(t) \le \bar{h} < \bar{h}_{QL}\\ &\quad = \frac{1 - 2\delta\left\|Q_1^{-\frac{1}{2}}P_1\right\|\left\|FQ_1^{-\frac{1}{2}}\right\| - 2\left\|Q_1^{-\frac{1}{2}}P_1\right\|\left\|Q_1^{-\frac{1}{2}}\right\|(\beta_0 + \alpha_1\beta_1)}{2\alpha_1 k_{QL}}\end{aligned} \tag{56}$$

with

$$k_{QL} = \left\|Q_1^{-\frac{1}{2}} P_1 A_1 (A_0 + BF) Q_1^{-\frac{1}{2}}\right\| + \left\|Q_1^{-\frac{1}{2}} P_1 A_1^2 Q_1^{-\frac{1}{2}}\right\| + \delta \left\|Q_1^{-\frac{1}{2}} P_1 A_1\right\| \left\|F Q_1^{-\frac{1}{2}}\right\| + \left\|Q_1^{-\frac{1}{2}} P_1 A_1\right\| \left\|Q_1^{-\frac{1}{2}}\right\| (\beta_0 + \beta_1) \quad (57)$$

Also, if we set $G = I$, the condition (47) becomes

$$0 \le h(t) \le \bar{h} < \bar{h}_{IL} = \frac{\lambda_{\min}[Q_1] - 2\delta \|P_1\| \|F\| - 2\|P_1\| (\beta_0 + \alpha_1 \beta_1)}{2\alpha_1 k_{IL}} \quad (58)$$

with

$$k_{IL} = \|P_1 A_1 (A_0 + BF)\| + \left\|P_1 A_1^2\right\| + \delta \|P_1 A_1\| \|F\| + \|P_1 A_1\| (\beta_0 + \beta_1) \quad (59)$$

Similarly to Remarks 2 and 3, it is easy to prove that $\bar{h}_{IL} \le \bar{h}_{QL}$ and the results of Theorem 2 are also true for the system (42).

Remark 6. Note that Theorems 1 and 2 can be easily extended to the case of multiple time-varying delays, using similar arguments to those developed in the present paper.

Now we present a design procedure to find a positive invariant and robust local asymptotic stability domain of linear behaviour for (8).

Algorithm 2:

Step 1. Use a similar method to that in Algorithm 1 to find the state feedback gain matrix F.

Step 2. Find a scalar $\eta > 0$ such that $\mathcal{D}(\mathcal{V}, \eta) \subseteq \mathcal{D}(F, \Omega)$. It suffices to choose η_i, $i = 1, 2, \ldots, m$ for which the elliptic domain is tangent at x_i for different hyperplanes $f_i x_i = u_m^i$, $f_i x_i = u_M^i$, $i = 1, 2, \ldots, m$. Then the contact points x_i are obtained as the solution to

$$\begin{bmatrix} P_1 & f_i^T \\ f_i & 0 \end{bmatrix} \begin{bmatrix} x_i \\ \omega_i \end{bmatrix} = \begin{bmatrix} 0 \\ \gamma_i \end{bmatrix} \quad (60)$$

for $\gamma_i = -u_m^i$ or $\gamma_i = u_M^i$, $i = 1, 2, \ldots, m$, where the ω_i's are the Lagrange multipliers. For each of these points, compute $x_i^T P_1 x_i = \eta_i$ and then the suitable η satisfying $\mathcal{D}(\mathcal{V}, \eta) \subseteq \mathcal{D}(F, \Omega)$ is obtained as $\eta = \min_i \eta_i$.

5. Numerical Examples

Example 1. Let us consider the uncertain time-delay system

$$\begin{aligned}\dot{x}(t) &= A_0 x(t) + A_1 x\big(t-h(t)\big) + \big(B+\Delta B(t)\big)u(t) + f_0\big(x(t),t\big)\\ &\quad + f_1\left(x\big(t-h(t)\big),t\right), \quad \forall t>0\end{aligned} \tag{61}$$

with

$$A_0 = \begin{bmatrix} 0.5 & 0 \\ 0.1 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad h(t) = 0.2 + 0.1\sin(t)$$

$$\Delta B(t) = \begin{bmatrix} [-0.1,\ 0] \\ 0 \end{bmatrix}, \quad f_0(x(t),t) = \begin{bmatrix} 0.2\sin\big(x_1(t)\big) \\ 0.2\sin\big(x_2(t)\big) \end{bmatrix}$$

$$f_1\left(x\big(t-h(t)\big),t\right) = \begin{bmatrix} 0.05\sin\big(x_1\big(t-h(t)\big)\big) \\ 0.05\sin\big(x_2\big(t-h(t)\big)\big) \end{bmatrix}$$

Then

$$\|\Delta B(t)\| \le 0.1 \quad (\delta = 0.1), \qquad \|f_0(x(t),t)\| \le 0.2\,\|x(t)\| \quad (\beta_0 = 0.2)$$

$$\left\|f_1\left(x\big(t-h(t)\big),t\right)\right\| \le 0.05\left\|x\big(t-h(t)\big)\right\| \quad (\beta_1 = 0.05)$$

Assume that all the states of the system (61) are available and the control variable is constrained as follows:

$$-1.5 \le u(t) \le 2$$

It is easy to check that the considered pair $\big(A_0 + A_1, B\big)$ is controllable, so it is stabilizable, i.e. Assumption 1 is satisfied. Let $w = 0.5$. According to Algorithm 1, we find $F = [-1\ 0]$ such that $A_0 + A_1 + (1/2)(1+w)BF = \left[\begin{smallmatrix} -1.5 & 0 \\ 0.1 & -1.5 \end{smallmatrix}\right]$ is Hurwitz. In the present case, the choice of the symmetric positive definite matrix $Q = \left[\begin{smallmatrix} 3 & -0.1 \\ -0.1 & 3 \end{smallmatrix}\right]$ yields $P = I$ as the unique symmetric positive definite solution to (12). From (34), we have

$$0 \le h(t) \le \bar{h} = 0.3 < \bar{h}_Q = 0.3781$$

With Theorem 1 in mind, we conclude that the memoryless state feedback control law $u(t) = \operatorname{sat}(Fx(t)) = -\operatorname{sat}(x_1(t))$ is robustly globally asymptotically stabilizing for the system (61). ♦

Example 2. Let us consider the constrained uncertain time-delay system

$$\begin{aligned}\dot{x}(t) &= A_0 x(t) + A_1 x\big(t-h(t)\big) + \big(B+\Delta B(t)\big)u(t) + f_0\big(x(t),t\big)\\ &\quad + f_1\left(x\big(t-h(t)\big),t\right), \quad \forall t>0\end{aligned} \tag{62}$$

with

$$A_0 = \begin{bmatrix} 1 & 1.5 \\ 0.3 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ 1 \end{bmatrix}, \quad h(t) = 0.2 + 0.2\sin(t)$$

$$\Delta B(t) = \begin{bmatrix} 0.2\cos(t) \\ 0 \end{bmatrix}, \quad f_0(x(t),t) = \begin{bmatrix} 0.3\sin\left(x_1(t)\right) \\ 0.3\sin\left(x_2(t)\right) \end{bmatrix}$$

$$f_1\left(x\left(t-h(t)\right),t\right) = \begin{bmatrix} 0 \\ 0.1\sin\left(x_2\left(t-h(t)\right)\right) \end{bmatrix}$$

Then

$$||\Delta B(t)|| \le 0.2 \quad (\delta = 0.2), \qquad ||f_0(x(t),t)|| \le 0.3\,||x(t)|| \quad (\beta_0 = 0.3)$$

$$\left\|f_1\left(x\left(t-h(t)\right),t\right)\right\| \le 0.1\left\|x\left(t-h(t)\right)\right\| \quad (\beta_1 = 0.1)$$

Assume that all the states of (62) are available and the control variable is constrained as follows:

$$-2 \le u(t) \le 0.5$$

It is easy to check that the considered pair $(A_0 + A_1, B)$ is controllable, so it is stabilizable, i.e. Assumption 1 is met. Using a pole assignment technique, e.g. $\sigma[A_0 + A_1 + BF] = \{-2, -2.15\}$, we have $F = [-0.3 \ \ -0.15]$. Choosing $Q_1 = \left[\begin{smallmatrix} 4 & 1 \\ 1 & 4.3 \end{smallmatrix}\right]$ yields $P = I$ as the unique symmetric positive definite solution to (45). From (56) we have

$$0 \le h(t) \le \bar{h} = 0.4 < \bar{h}_{QL} = 0.454$$

According to Theorem 2, we can determine the set $\mathcal{D}(\mathcal{V}, \eta)$, with $\mathcal{V}(x(t)) = x^T(t)P_1x(t) = x^T(t)x(t)$, in which (62) has a linear behaviour and the control does not saturate. Applying Algorithm 2, we obtain $\eta = 20/9$. ♦

6. Conclusion

The problem of robust stabilization for uncertain time-delay systems containing a saturating actuator has been addressed. Delay-dependent criteria for robust global or local asymptotic stabilization have been obtained by using the Razumikhin theorem. Based on the symmetric positive definite solutions to Lyapunov equations, the proposed criteria have given upper bounds on the time-varying delay. Two numerical examples have illustrated the obtained results.

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References

Chen B.-S., Wang S.-S. and Lu H.-C. (1988): *Stabilization of time-delay systems containing saturating actuators.* — Int. J. Contr., Vol.47, pp.867–881.

Chen J. and Latchman H.A. (1995): *Frequency sweeping tests for stability independent of delay.* — IEEE Trans. Automat. Contr., Vol.AC-40, pp.1640–1645.

Chen J., Xue D.M. and Shafai B. (1995): *On sufficient conditions for stability independent of delay.* — IEEE Trans. Automat. Contr., Vol.AC-40, pp.1675–1680.

Cheres E., Gutman E. and Palmor Z.J. (1989): *Stabilization of uncertain dynamic systems including state delay.* — IEEE Trans. Automat. Contr., Vol.AC-34, pp.1199–1203.

Chou J.-H., Horng I.-R. and Chen B.-S. (1989): *Dynamical feedback compensator for uncertain time-delay systems containing saturating actuator.* — Int. J. Contr., Vol.49, pp.961–968.

Gu K. (1997): *Discritized LMI set in the stability problem of linear uncertain time delay systems.* — Int. J. Contr., Vol.68, pp.923–934.

Gu K. (1999): *A genaralized discretization scheme of Lyapunov functional in the stability problem of uncertain time delay systems* — Int. J. Robust Nonlin. Contr., Vol.9, pp.1–14.

Hale J. (1977): *Theory of Functional Differential Equations.* — New York: Spring-Verlag.

Han Q.-L. and Mehdi D. (1998a): *Comments on 'Robust control for parameter uncertain delay systems in state and control input'.* — Automatica, Vol.34, pp.1665–1666.

Han Q.-L. and Mehdi D. (1998b): *Robust observer-based stabilisation of uncertain linear constrained time-delay systems.* — Proc. 5th Int. Symp. *Methods and Models in Automation and Robotics*, Międzyzdroje, Poland, Vol.2, pp.427–432.

Han Q.-L. Mehdi D. and Tang Z.-Z. (1998): *Stability robustness analysis for observer-based uncertain continuous-time time-delay systems containing saturating actuators.* — Proc. 37th IEEE Conf. *Decision Control*, Tampa, Florida, USA, pp.811–812.

Hmamed A. (1986): *On the stability of time-delay systems: new results.* — Int. J. Contr., Vol.43, pp.321–324.

Ikeda M. and Ashida T. (1979): *Stabilization of linear systems with time-varying delay.* — IEEE Trans. Automat. Contr., Vol.AC-24, pp.369–370.

Kamen E.W. (1982): *Linear systems with commensurate time delays: stability and stabilization independent of delay.* — IEEE Trans. Automat. Contr., Vol.AC-27, pp.367–375.

Klai M., Tarbouriech S. and Burgat C. (1994): *Some independent time-delay stabilization of linear systems with saturating controls.* — Proc. 1994 IEE Conf. *Control*, Vol.2, pp.1358–1363.

Li X. and de Souza C.E. (1997): *Criteria for robust stability and stabilization of uncertain linear systems with state delay.* — Automatica, Vol.33, pp.1657–1662.

Li X. and de Souza C.E. (1997): *Delay-dependent robust stability and stabilization of uncertain linear delay systems: A linear matrix inequality approach.* — IEEE Trans. Automat. Contr., Vol.AC-42, pp.1144–1148.

Liu P.-L. (1995): *Stabilization of input delay constrained systems with delay dependence.* — Int. J. Syst. Sci., Vol.26, pp.245–255.

Luo J.S. and Van den Bosch P.P.J. (1997): *Independent of delay stability criteria for uncertain linear state space models.* — Automatica, Vol.33, pp.171–179.

Malek-Zavarei M. and Jamshidi M. (1987): *Time Delay Systems: Analysis, Optimization and Applications.* — North-Holland.

Mori T. (1985): *Criteria for asymptotic stability of linear time-delay systems.* — IEEE Trans. Automat. Contr., Vol.AC-30, pp.158–161.

Mori T., Fukuma N. and Kuwahara M. (1981): *Simple stability criteria for single and composite linear systems with time delay.* — Int. J. Contr., Vol.34, pp.1175–1184.

Mori T. and Kokame H. (1989): *Stability of* $\dot{x}(t) = Ax(t) + Bx(t-\tau)$. — IEEE Trans. Automat. Contr., Vol.AC-34, pp.460–462.

Mori T., Noldus E. and Kuwahara M. (1983): *A way to stabilize linear systems with delayed state.* — Automatica, Vol.19, pp.571–573.

Niculescu S.-I., Dion J.-M. and Dugard L. (1996): *Robust stabilization for uncertain time-delay systems containing saturating actuators.* — IEEE Trans. Automat. Contr., Vol.AC-41, pp.742–747.

Shen J.-C. and Kung F.-C. (1989): *Stabilization of input delay systems with saturating actuator.* — Int. J. Contr., Vol.50, pp.1667–1680.

Shyu K.-K. and Yan J.-J. (1993): *Robust stability of uncertain time-delay systems and its stabilization by variable structure control.* — Int. J. Contr., Vol.57, pp.237–246.

Su T.J. and Huang C.G. (1992): *Robust stability of delay dependence for linear uncertain systems.* — IEEE Trans. Automat. Contr., Vol.AC-37, pp.1656–1659.

Su T.-J., Liu P.-L. and Tsay J.-T. (1991): *Stabilization of delay-dependence for saturating actuator systems.* — Proc. 30th Conf. Decision Contr., Brighton, U.K., pp.2891–2892.

Su T.-J., Sun Y.-Y. and Kuo T.-S. (1989): *Robust stabilization of observer-based linear constrained uncertain time-delay systems.* — Contr. Th. Adv. Tech., Vol.5, pp.205–213.

Thowsen A. (1982): *A transformation for stability analysis of linear delay systems.* — Int. J. Syst. Sci., Vol.13, pp.1371–1378.

Tissir E. and Hmamed A. (1992): *Further results on the stabilization of time-delay systems containing saturating actuators.* — Int. J. Syst. Sci., Vol.23, pp.615–622.

Trinh H. and Aldeen M. (1994): *On the stability of linear systems with delayed perturbations.* — IEEE Trans. Automat. Contr., Vol.AC-39, pp.1948–1951.

Xu B. (1994): *Comments on 'Robust stability of delay dependence for linear uncertain systems.'* — IEEE Trans. Automat. Contr., Vol.AC-39, p.2365.

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