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STABILITY OF MULTI-DIMENSIONAL DISCRETE SYSTEMS WITH VARYING STRUCTURE

MICHAEL DYMKOV*, IVAN GAISHUN*

A new class of multi-dimensional discrete systems with varying structure is introduced. The notions of total solvability, p_t-boundedness, p_t-stability and asymptotic stability are defined. For studying properties of the solutions for the considered systems a curvilinear composition of mappings along discrete curves is used. Total solvability conditions similar to the Frobenius ones are obtained. Sufficient conditions for p_t-stability and asymptotic stability based on the Lyapunov functional method are also established. A two-dimensional system of Volterra equations is presented as an example of equations with varying structure.

Keywords: multi-dimensional systems, total solvability, curvilinear composition, stability

1. Introduction

Dynamical systems with after-effects appear in various theoretical and applied problems and have been in the focus of research over the last twenty years. Such systems are used to simulate a variety of real processes, such as the unsteady motion of bodies in continuous medium (the phenomenon of aero-auto-elasticity) (Belozerkovskii *et al.*, 1980), oscillations in long transmission lines (Brayton, 1967), interaction of populations (Gopalsamy, 1992) and many others. In contrast to classic dynamic processes, a characteristic feature of such processes is the dependence on the entire prehistory $\tau < t$ at each moment t. In particular, for the systems described by equations such behaviour can be interpreted as varying the domain of definition and the range of values for these equations in the course of time. For example, a discrete equation with a varying structure can be treated as a one-step equation in a sequence $\{E_t\}$ of generally distinct spaces rather than in some fixed space; in this case, at time t the right-hand sides of the one-step equation are functions from the space E_t into another space E_{t+1}. For continuous cases (i.e. for the systems described by some differential equations) the varying structure can be determined by using the so-called scales of convenient spaces (see Treves and Duchateau, 1971). It should be noted here that in the continuous case an important place is occupied by the linear control

* Institute of Mathematics of the National Academy of Sciences of Belarus, 11, Surganov Str., 220072, Minsk, Belarus, e-mail: {dymkov,gaishun}@im.bas-net.by

systems described by differential equations with unbounded operators acting in Banach spaces. The theory of differential equations in scales of Banach spaces, worked out by Ovsyannikov (1965) and others, allows many results in the theory of ordinary differential equations to be generalized to partial differential equations.

On the other hand, the theory of multi-dimensional differential equations (in other words, Pfaff differential equations) and their discrete versions has attracted much attention in the last decades (see references in Izobov, 1998; Kaczorek, 1994; Myshkis, 1998; Perov, 1968). These objects were introduced long ago and their first applications were connected with differential geometry (Sternberg, 1970). Pfaff equations were then used in elasticity theory, mathematical physics, magnetohydrodynamics, control theory and other engineering problems (Lyrie, 1975). The main characteristic of such systems is their overdetermination in the sense that the number of equations is greater than that of the unknown functions. Consequently, the classes of totally integrable systems are the most interesting since the boundary Cauchy problem has a unique solution (Gaishun, 1983).

In this paper, we introduce a new class of mathematical objects which unify discrete equations with varying structure and multi-dimensional discrete equations. Here the goal is to provide mathematical tools which could contribute to a better understanding of (physical) phenomena in nonlinear dynamics (Thompson and Stewart, 1986). To investigate some properties of the model, we make use of an analogy to dynamic systems of simple structure. Specifically, we generalize the concepts of p_t-embedding introduced by Gaishun (1997) and define the notions of p_t-boundedness, p_t-stability and asymptotic stability. In order to study these notions, we use a curve-linear composition of mappings and the Lyapunov functional method.

2. Basic Notation and Definitions

Let $\mathbb{Z}_+$ be the set of non-negative integers, $E_t = E_{(t_1,t_2)}$, $t = (t_1, t_2) \in \mathbb{Z}_+^2$ be a sequence of non-empty sets, and let $f^{(1)}_{(t_1,t_2)} : E_{(t_1,t_2)} \to E_{(t_1+1,t_2)}$, $f^{(2)}_{(t_1,t_2)} : E_{(t_1,t_2)} \to E_{(t_1,t_2+1)}$ be some mappings. The system of equations

$$\begin{cases} x(t_1+1, t_2) = f^{(1)}_{(t_1,t_2)}\big(x(t_1,t_2)\big), \\ x(t_1, t_2+1) = f^{(2)}_{(t_1,t_2)}\big(x(t_1,t_2)\big), \quad (t_1,t_2) \in \mathbb{Z}_+^2 \end{cases} \tag{1}$$

will be called a discrete multi-dimensional (two-dimensional) system with varying structure.

Example 1. As an example of multi-dimensional discrete systems with varying structure we consider below the system described by multi-dimensional discrete Volterra equations that has been applied to a number of problems e.g. in systems theory (Schetzen, 1980), physics and ecology (Tsonis, 1992). Write $\mathbb{Z}^+_{[t_1,t_2]} = \{(i_1, i_2) \in \mathbb{Z}_+^2,\ i_1 \le t_1,\ i_2 \le t_2\}$. Let $\mathcal{F}(\mathbb{Z}^+_{[t_1,t_2]}, E)$ be the set of all mappings $f\colon\ \mathbb{Z}^+_{[t_1,t_2]} \to E$, where E

is a non-empty set equipped with an algebraic structure. Consider in E the following two-dimensional linear discrete Volterra system:

$$\begin{cases} \xi_{(t_1+1,t_2)} = \sum_{(i_1,i_2)\in\mathbb{Z}^+_{[t_1,t_2]}} a^{(1)}_{(i_1,i_2)}\xi_{(t_1-i_1,t_2-i_2)}, \\ \xi_{(t_1,t_2+1)} = \sum_{(i_1,i_2)\in\mathbb{Z}^+_{[t_1,t_2]}} a^{(2)}_{(i_1,i_2)}\xi_{(t_1-i_1,t_2-i_2)}, \end{cases} \quad (t_1,t_2)\in\mathbb{Z}^2_+ \tag{2}$$

with respect to an unknown function $\xi:\ \mathbb{Z}^2_+ \to E$. Here $a^{(s)}_{(i_1,i_2)}$, $s=1,2$, $(i_1,i_2)\in\mathbb{Z}^+_2$ are given elements from E. To transform (2) to (1) we define the mapping $F^{(1)}_{(t_1,t_2)}:\ \mathcal{F}(\mathbb{Z}^+_{[t_1,t_2]},E)\to\mathcal{F}(\mathbb{Z}^+_{[t_1+1,t_2]},E)$ as

$$\left(F^{(1)}_{(t_1,t_2)}x\right)(s_1+1,s_2) = \sum_{(i_1,i_2)\in\mathbb{Z}^+_{[s_1,s_2]}} a^{(1)}_{(i_1,i_2)}x(s_1-i_1,s_2-i_2),$$

$$\left(F^{(1)}_{(t_1,t_2)}x\right)(0,s_2) = x(0,s_2), \quad (s_1,s_2)\in\mathbb{Z}^+_{[t_1,t_2]} \tag{3}$$

and the mapping $F^{(2)}_{[t_1,t_2]}:\ \mathcal{F}(\mathbb{Z}^+_{[t_1,t_2]},E)\to\mathcal{F}(\mathbb{Z}^+_{[t_1,t_2+1]},E)$ on the analogy of $F^{(1)}_{[t_1,t_2]}$. Finally, the desired two-dimensional discrete system with varying structure

$$\begin{cases} x(t_1+1,t_2) = F^{(1)}_{(t_1,t_2)}\big(x(t_1,t_2)\big), \\ x(t_1,t_2+1) = F^{(2)}_{(t_1,t_2)}\big(x(t_1,t_2)\big), \end{cases} \quad (t_1,t_2)\in\mathbb{Z}^2_+, \tag{4}$$

where $x(t_1,t_2)\in\mathcal{F}(\mathbb{Z}^2_{[t_1,t_2]},E)$ for all $(t_1,t_2)\in\mathbb{Z}^2_+$, is obtained. The resulting system is equivalent to the original one in the sense that there exists a bijection between the solution sets of these systems. ♦

As mentioned above, the number of equations for the systems under consideration is in general greater than that of the unknown functions. Therefore, these systems are of interest for which there exist unique solutions. In order to characterize these cases, we give below the definition of totally solvable systems. But first, we assume that the family of sets $E_{(t_1,t_2)}$ satisfies the following conditions:

$$E_{(t_1+1,t_2)}\subset E_{(t_1,t_2)},\quad E_{(t_1,t_2+1)}\subset E_{(t_1,t_2)},\quad (t_1,t_2)\in\mathbb{Z}^2_+.$$

We say that a mapping $\varphi(t_1,t_2):\ \mathbb{Z}^2_+\to E_{(0,0)}$, $\varphi(t_1,t_2)\in E_{(t_1,t_2)}$, $(t_1,t_2)\in\mathbb{Z}^2_+$ is a solution to (1) if it satisfies (1) for all $(t_1,t_2)\in\mathbb{Z}^2_+$.

Definition 1. System (1) is said to be *totally solvable* if for any $(t^0,x^0)\in\mathbb{Z}^2_+\times E_{t^0}$ there exists a unique solution $x(t_1,t_2):\ \mathbb{Z}^+_{t^0}\to E_{t^0}$, where $x(t_1,t_2)\in E_{(t_1,t_2)}$ for $(t_1,t_2)\in\mathbb{Z}^+_{t^0}=\{(t_1,t_2)\in\mathbb{Z}^2_+,\ t_1\geq t^0_1,\ t_2\geq t^0_2\}$, that satisfies the initial condition (Cauchy condition) $x(t^0_1,t^0_2)=x^0$.

For brevity, the solution to (1) that satisfies the initial data (t^0,x^0) is denoted by $x(t,t^0,x^0)$. To represent the solutions to (1) in a convenient form we introduce the notions of a discrete curve (or path) and a curvilinear composition of mapping along the discrete curve.

Denote by $\mathcal{L}(t^1,\dots,t^N)$ the set $\{t^1,t^2,\dots,t^N\}$ of points from $\mathbb{Z}_+^2$ such that: (a) for any $i=1,2,\dots,N-1$ the inequalities $t_s^{i+1}\ge t_s^i$, $s=1,2$ are satisfied, and (b) $(t_1^{i+1}-t_1^i)+(t_2^{i+1}-t_2^i)=1$ for all $i=1,2,\dots,N-1$. The set $\mathcal{L}=\mathcal{L}(t^1,\dots,t^N)$ is called a discrete curve (or path) connecting t^1 with t^N.

Now, introduce the following compositions of maps:

$$\left(f_t^{(i)^p}\right)(x)=\underbrace{f_t^{(i)}\left(f_t^{(i)}\dots f_t^{(i)}(x)\dots\right)}_{p \text{ times}},\quad \left(f_t^{(i)^0}\right)(x)=x,\quad i=1,2, \tag{5}$$

$$\left(f_t^{(1)}\circ f_{\bar t}^{(2)}\right)(x)=f_t^{(1)}\left(f_{\bar t}^{(2)}(x)\right),$$

$$\left(\overleftarrow{\prod_{i=1}^{m}}\circ f_{t^i}^{(1)^{p_i}}\right)(x)=\left(f_{t^m}^{(1)^{p_m}}\circ\dots\circ f_{t^1}^{(1)^{p_1}}\right)(x),$$

where p and p_i are some positive integers. The mapping π defined by the formula

$$\pi(x)=\left(\overleftarrow{\prod_{\mathcal{L}(t^1,\dots,t^N)}}\circ f_t^{(1)\Delta p_1}f_t^{(2)\Delta p_2}\right)(x)$$

$$=\left(\overleftarrow{\prod_{i=1}^{N-1}}\circ f_{t^i}^{(1)t_1^{i+1}-t_1^i}f_{t^i}^{(2)t_2^{i+1}-t_2^i}\right)(x) \tag{6}$$

is called a curvilinear composition of functions $f^{(1)}$ and $f^{(2)}$ along the curve $\mathcal{L}(t^1,\dots,t^N)$.

In general, the value of $\pi(x)$ depends on all points $(t^1,\dots,t^N)$ of the path $\mathcal{L}(t^1,\dots,t^N)$ that connects t^1 and t^N. Hence, some additional conditions are needed to guarantee the independence of $\pi(x)$ from the choice of the curve $\mathcal{L}(t^1,\dots,t^N)$. It is shown below that under conditions guaranteeing total solvability of (1) the value of $\pi(x)$ depends only on the initial and terminal points t^1 and t^N, respectively. For that purpose, the notion of an elementary transformation for a discrete curve $\mathcal{L}=\mathcal{L}(t^1,\dots,t^N)$ should be defined. Without loss of generality, we assume that there are points t^r, t^{r+1} and t^{r+2} from $\mathcal{L}$ such that $t_1^{r+2}-t_1^r=1$, $t_2^{r+2}-t_2^r=1$. We say that the discrete curve $\mathcal{L}=\mathcal{L}(t^1,\dots,t^N)$ is transformed with an elementary transformation at t^r, if $\mathcal{L}(t^1,\dots,t^N)$ is replaced by the curve $\mathcal{L}'=\mathcal{L}'(t^1,\dots,t^r,\bar t^{r+1},t^{r+2},\dots,t^N)$, where $\bar t^{r+1}=(t_1^r+t_2^{r+1}-t_2^r,t_2^r+t_1^{r+1}-t_1^r)$. This allows us to define equivalence of discrete curves.

Definition 2. We say that any two discrete curves $\mathcal{L}$ and $\mathcal{L}'$ are *equivalent* if $\mathcal{L}$ and $\mathcal{L}'$ can be converted by elementary transformations to the same discrete path.

It can be proved that all the discrete curves connecting t^1 with t^N are equivalent. Conversely, all the discrete curves which are equivalent to $\mathcal{L}$ connect just the same points t^1 and t^N.

3. Solvability Problem

In this section, we investigate the existence and uniqueness of the solution to (1). As mentioned above, the number of equations in (1) is greater than the number of unknown functions. This results in the necessity of having some additional conditions for solvability of (1). As is well-known (Hartman, 1964), for the ordinary Pfaff differential equations such conditions are Frobenius ones. Some details for more general cases can be found e.g. in (Gaishun, 1983). Below, similar conditions are obtained for multi-dimensional discrete systems with varying structure.

Theorem 1. *Equation (1) is totally solvable if and only if the following equality for the map composition:*

$$f^{(1)}_{(t_1,t_2+1)} \circ f^{(2)}_{(t_1,t_2)} = f^{(2)}_{(t_1+1,t_2)} \circ f^{(1)}_{(t_1,t_2)} \tag{7}$$

is fulfilled for all $(t_1, t_2) \in \mathbb{Z}^2_+$.

Proof. (Necessity) Let $x(t_1,t_2)$, $(t_1,t_2) \in \mathbb{Z}^2_+$ be a solution to (1). For any $(t_1,t_2) \in \mathbb{Z}^2_+$ we obtain from (1) for the left and right-hand sides of (7)

$$\begin{aligned}\Big(f^{(1)}_{(t_1,t_2+1)} \circ f^{(2)}_{(t_1,t_2)}\Big)\big(x(t_1,t_2)\big) &= f^{(1)}_{(t_1,t_2+1)}\Big(f^{(2)}_{(t_1,t_2)}\big(x(t_1,t_2)\big)\Big)\\ &= f^{(1)}_{(t_1,t_2+1)}\big(x(t_1,t_2+1)\big) = x\big(t_1+1,t_2+1\big),\\ \Big(f^{(2)}_{(t_1+1,t_2)} \circ f^{(1)}_{(t_1,t_2)}\Big)\big(x(t_1,t_2)\big) &= f^{(2)}_{(t_1+1,t_2)}\Big(f^{(1)}_{(t_1,t_2)}\big(x(t_1,t_2)\big)\Big)\\ &= f^{(2)}_{(t_1+1,t_2)}\big(x(t_1+1,t_2)\big) = \big(x(t_1+1,t_2+1\big),\end{aligned}$$

respectively. Thus the necessity is proved.

(Sufficiency) We will directly construct the desired solution. Let t be an arbitrary point of $\mathbb{Z}^2_+$. Now we calculate the curvilinear composition $\pi(x)$ for $f^{(1)}$, $f^{(2)}$ along two sole discrete paths connecting the points $t=(t_1,t_2)$ and $\bar{t}=(t_1+1,t_2+1)$ and establish then the conditions to guarantee the independence of $\pi(x)$ from the choice of these curves. It is obvious that the obtained result does not depend on the choice of the paths iff (7) is satisfied. Keeping in mind the definition of the equivalence for discrete curves, we have that the curvilinear composition $\pi(x)$ does not depend on the choice of the paths connecting t^1 and t^N iff (7) is true. Hence

$$\pi(x) = \left(\overset{\leftarrow}{\prod_{(t^1,t^N)}} \circ f^{(1)\Delta t_1}_t f^{(2)\Delta t_2}_t\right)(x). \tag{8}$$

Finally, define the function $x = x(t,t^0,x^0)$ by the formula

$$x\big(t,t^0,x^0\big) = \left(\overset{\leftarrow}{\prod_{(t^0,t)}} \circ f^{(1)\Delta t_1}_t f^{(2)\Delta t_2}_t\right)(x^0), \tag{9}$$

where (t^0, x^0) is an arbitrary element from $\mathbb{Z}_+^2 \times E_{t^0}$. It is clear that if we establish that (9) is a solution to (1), then the theorem will be proved. Indeed, the initial condition $x(t^0, t^0, x^0) = x^0$ is fulfilled by the definition of the curvilinear composition. Also, since any curvilinear composition satisfies

$$\left(\overleftarrow{\prod_{\mathcal{L}(t^1,\dots,t^m,\dots,t^N)}} \circ f_t^{(1)\Delta t_1} f_t^{(2)\Delta t_2} \right) = \left(\overleftarrow{\prod_{\mathcal{L}(t^m,t^N)}} f_t^{(1)\Delta t_1} f_t^{(2)\Delta t_2} \right) \circ \left(\overleftarrow{\prod_{\mathcal{L}(t^1,t^m)}} f_t^{(1)\Delta t_1} f_t^{(2)\Delta t_2} \right),$$

from (1) we have

$$\begin{aligned} x(t_1+1, t_2, t^0, x^0) &= \left(\overleftarrow{\prod_{((t_1^0,t_2^0),(t_1+1,t_2))}} f_t^{(1)\Delta t_1} f_t^{(2)\Delta t_2} \right)(x^0) \\ &= \left(f^{(1)}_{(t_1+1,t_2)} \circ \overleftarrow{\prod_{((t_1^0,t_2^0),(t_1,t_2))}} f_t^{(1)\Delta t_1} f_t^{(2)\Delta t_2} \right)(x^0) \\ &= f^{(1)}_{(t_1+1,t_2)} \Big(x\big(t, t^0, x^0\big) \Big), \end{aligned}$$

which means that the function (9) satisfies the first equation of (1). Similary, it can be shown that (9) satisfies the second equation of (1). It is clear that the function given by (9) is unique. ■

Remark 1. The discrete system (1) can be interpreted as an approximation of the multi-dimensional system described by integrable differential equations. Accordingly, the following problem is of interest: What relations are between the solvability properties of the discrete and continuous systems? It should be pointed here that the determination of a multi-dimensional differential system with varying structure is not simple. In particular, a definition of such an object can be given by using the so-called scales of convenient spaces (Ovsyannikov, 1965; Treves and Duchateau, 1971). It can be shown that solvability of the discrete system does not imply that of the continuous one even if we consider systems with ordinary (non-varying) structure.

Example 2. Consider the following two-dimensional differential system:

$$\frac{\partial y}{\partial t_1} = (1 + y^2), \quad \frac{\partial y}{\partial t_2} = 2(1 + y^2) \tag{10}$$

with respect to an unknown function $y = y(t_1, t_2)$. It is easy to show that the Frobenius conditions (Hartman, 1964) are fulfilled, i.e.

$$\frac{\partial(1+y^2)}{\partial t_2} + \frac{\partial(1+y^2)}{\partial y} 2(1+y^2) = \frac{\partial(2(1+y^2))}{\partial t_1} + \frac{\partial(2(1+y^2))}{\partial y}(1+y^2).$$

Hence $4y(1+y^2) = 4y(1+y^2)$. Thus, the system under consideration is integrable. We construct a disctere approximation to (10) by substituting the corresponding Euler differences

$$\frac{1}{h}\Big[y\big(k_1 h, (k_2+1)h\big) - y\big(k_1 h, k_2 h\big)\Big], \quad k = (k_1, k_2) \in \mathbb{Z}_+^2$$

in lieu of the partial derivatives $\partial y(t_1,t_2)/\partial t_1$, $\partial y(t_1,t_2)/\partial t_2$ in (10). The resulting discrete two-dimensional system has the following form:

$$\begin{cases} x(k_1+1, k_2) = x(k_1,k_2) + h\big(1 + x^2(k_1,k_2)\big) \equiv f_1\big(k, x(k)\big), \\ x(k_1, k_2+1) = x(k_1,k_2) + 2h\big(1 + x^2(k_1,k_2)\big) \equiv f_2\big(k, x(k)\big). \end{cases}$$

In this case the solvability conditions for (7) are

$$f_2\big(k_1+1, k_2, f_1(k,x)\big) = f_1\big(k_1, k_2+1, f_2(k,x)\big), \quad (k_1,k_2) \in \mathbb{Z}_+^2,$$

that implies that the equality $4h^2 = 2h^2$ must be fulfilled for $h \neq 0$. This contradiction means that the discrete system is not solvable for any $h > 0$. Thus, the given example demonstrates that the solvability property of multi-dimensional systems can be disturbed under the passage from a continuous system to its discrete approximation. ♦

4. Stability and Boundedness Problems

In this section, we use the notion of comparability for dynamical systems to study some properties of (1). We generalize the concepts of p_t-embedding introduced by Gaishun (1997) and give some sufficient conditions for p_t-boundedness, p_t-stability and asymptotic stability.

First, assume that $E_{(t_1,t_2)}$, $(t_1,t_2) \in \mathbb{Z}_+^2$ is a collection of normed spaces over the field $\mathbb{R}$ (or $\mathbb{C}$) of real (complex) numbers. Let V be another normed space over $\mathbb{R}$ (or $\mathbb{C}$), and $p_{(t_1,t_2)} : E_{(t_1,t_2)} \to V$, $(t_1,t_2) \in \mathbb{Z}_+^2$, be a given sequence of mappings. By $|\cdot|_V$, $|\cdot|_t$ we denote the norms in the spaces V and E_t, $t \in \mathbb{Z}_+^2$, respectively. Without loss of generality we assume that $t^0 = 0$.

Definition 3. We say that the solution $x = x(t_1,t_2,x^0)$, $(t_1,t_2) \in \mathbb{Z}_+^2$ to system (1) is *p_t-bounded* if there exists a positive number $c(x^0)$ such that $|p_{(t_1,t_2)}(x(t_1,t_2,x^0))|_V \le c(x^0)$ for all $(t_1,t_2) \in \mathbb{Z}_+^2$. We say that system (1) has the property of p_t-boundedness if each solution to system (1) is p_t-bounded.

Let $f_t^{(1)}(0) = f_t^{(2)}(0) = 0$ in (1) for all $t = (t_1,t_2) \in \mathbb{Z}_+^2$. In this case, it is obvious that $x(t_1,t_2) = x(t_1,t_2,0) \equiv 0$ is a solution to (1). Here and subsequently, the zero elements of various spaces are denoted by the same symbol 0 if it is not confusing.

Definition 4. The zero solution $x(t_1,t_2) = 0$, $(t_1,t_2) \in \mathbb{Z}_+^2$ to system (1) is called *p_t-stable* if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x^0|_0 < \delta$ implies $|p_{(t_1,t_2)}(x(t_1,t_2,x^0))|_V < \varepsilon$ for all $(t_1,t_2) \in \mathbb{Z}_+^2$.

Definition 5. The zero solution $x(t_1,t_2)=0$, $(t_1,t_2)\in\mathbb{Z}^2_+$ to system (1) is said to be *p_t-asymptotically stable* if it is p_t-stable and there exists a $\mu>0$ such that $|p_{(t_1,t_2)}(x(t_1,t_2,x^0))|_V\to 0$ as $t_1+t_2\to\infty$ for $|x^0|_0<\mu$.

It should be noted that any path along which $t_1+t_2\to\infty$ is not fixed in the above definitions.

In the sequel, the Lyapunov-functional method is used to study the boundedness and stability problems for system (1). Let $v_{(t_1,t_2)}:\ E_{(t_1,t_2)}\to\mathbb{R}_+$ be a sequence of functionals (here $\mathbb{R}_+$ is the set of non-negative real numbers). We say that a functional sequence $\{v_{(t_1,t_2)}\}$ is p_t-positive if there exists a Hahn function $w:\ \mathbb{R}_+\to\mathbb{R}_+$ (i.e. a continuous increasing function vanishing at the point $s=0$) such that

$$v_{(t_1,t_2)}(x)\geq w\big(|p_{(t_1,t_2)}(x)|_V\big)\quad\text{for all}\quad (t_1,t_2)\in\mathbb{Z}^2_+,\quad x\in E_{(t_1,t_2)}. \tag{11}$$

If, in addition, $w(s)\to\infty$ as $s\to\infty$, then the sequence $\{v_{(t_1,t_2)}\}$ is said to be p_t-infinite. Now the sequences $\{\Delta_i v_{(t_1,t_2)}\}$, $i=1,2$ of differences of functionals $v_{(t_1,t_2)}$ on the solutions to system (1) are respectively constructed as

$$\begin{aligned}\Delta_1 v_{(t_1,t_2)}\big(x(t_1,t_2)\big)&=v_{(t_1+1,t_2)}\big(x(t_1+1,t_2)\big)-v_{(t_1,t_2)}\big(x(t_1,t_2)\big)\\&=v_{(t_1+1,t_2)}\Big(f^{(1)}_{(t_1,t_2)}\big(x(t_1,t_2)\big)\Big)-v_{(t_1,t_2)}\big(x(t_1,t_2)\big),\\\Delta_2 v_{(t_1,t_2)}\big(x(t_1,t_2)\big)&=v_{(t_1,t_2+1)}\big(x(t_1,t_2+1)\big)-v_{(t_1,t_2)}\big(x(t_1,t_2)\big)\\&=v_{(t_1,t_2+1)}\Big(f^{(2)}_{(t_1,t_2)}\big(x(t_1,t_2)\big)\Big)-v_{(t_1,t_2)}\big(x(t_1,t_2)\big).\end{aligned}$$

Note here that the functionals used in studying the stability and boundedness properties of solutions to (1) are traditionally called the Lyapunov functionals.

Theorem 2. *If there exists a functional sequence $\{v_{(t_1,t_2)}\}$ such that for any $c\geq 0$ the inequalities $v_{(t_1,t_2)}(x)\leq c$, $(t_1,t_2)\in\mathbb{Z}^2_+, x\in E_{(t_1,t_2)}$ imply the uniform boundedness of $|p_{(t_1,t_2)}(x)|, x\in E_{(t_1,t_2)}$ with respect to (t_1,t_2), and for any $x(t_1,t_2)\in E_{(t_1,t_2)}$ the inequalities*

$$\Delta_i v_{(t_1,t_2)}\big(x(t_1,t_2)\big)\leq 0,\quad i=1,2 \tag{12}$$

are fulfilled for all $(t_1,t_2)\in\mathbb{Z}^2_+$, then system (1) has the property of p_t-boundedness.

Proof. Applying (12) step by step we have $v_{(t_1,t_2)}(x(t_1,t_2))\leq v_{(0,0)}(x(0,0))$ for any $x(t_1,t_2)\in E_{(t_1,t_2)}$. Therefore, according to the assumption of the theorem, the value $|p_{(t_1,t_2)}(x)|$ is uniformly bounded on any solution $x(t_1,t_2)=x(t_1,t_2,x^0)$ of system (1).

Theorem 3. *If for system (1) there exists a p_t-positive sequence $\{v_{(t_1,t_2)}\}$ of Lyapunov functionals such that*

$$v_{(0,0)}\left(x^0\right)\leq w_1\left(|x^0|_0\right) \tag{13}$$

(w_1 is a Hahn function) and the inequalities

$$\Delta_i v_{(t_1,t_2)}(x(t_1,t_2)) \le 0, \quad i = 1,2 \tag{14}$$

are valid for all $(t_1,t_2) \in \mathbb{Z}^2_+$, *then the zero solution of (1) is* p_t*-stable.*

Proof. By analogy to Theorem 2, applying (14) yields that $v_{(t_1,t_2)}(x(t_1,t_2)) \le v_{00}(x(0,0))$ for all $x(t_1,t_2) \in E_{(t_1,t_2)}$, $(t_1,t_2) \in \mathbb{Z}^2_+$. Hence we have $w(|p_t(x)|_V) \le w_1(|x(0,0)|_0)$ and therefore $|p_t(x)|_V \le w^{-1}(w_1(|x(0,0)|_0))$. Due to the continuty of the functions w^{-1} and w_1 and the relation $w^{-1}(w_1(0)) = 0$ there exists a number $\delta > 0$ such that $w^{-1}(w_1(|x(0,0)|_0) < \varepsilon$ for $|x(0,0)|_0) < \delta$, i.e. $|p_{(t_1,t_2)}(x)|_V < \varepsilon$, $(t_1,t_2) \in \mathbb{Z}^2_+$. ■

Theorem 4. *If there exists a sequence* $\{v_{(t_1,t_2)}\}$ *of functionals satisfying inequalities (11) and (13) and the inequality*

$$\Delta_i v_{(t_1,t_2)}(x(t_1,t_2)) \le -w_2\left(|p_{(t_1,t_2)}(x(t_1,t_2))|_V\right) \tag{15}$$

is valid for all $(t_1,t_2) \in \mathbb{Z}^2_+$, *where* w_2 *is a Hahn function, then the zero solution to system (1) is* p_t*-asymptotically stable.*

Proof. According to Theorem 3 the zero solution is p_t-stable. Suppose now that this solution is not p_t-asymptotically stable, i.e. in an arbitrary small neighborhood of the zero of the space $E_{(0,0)}$ there exists an element x^0 such that $|p_{(\tau_1,\tau_2)}(x(\tau_1,\tau_2,x^0))|_V \ge \varepsilon_0$ for some positive number ε_0 and some sequence T of $\{(\tau_1,\tau_2)\}$, $(\tau_1,\tau_2) \in \mathbb{Z}^2_+$, $\tau_1 + \tau_2 \to \infty$. According to (15) we have

$$v_{(t_1,t_2)}(x(t_1,t_2)) \le v_{(0,0)}\left(x^0\right) - \sum_{(s_1,s_2)\in\mathbb{Z}^{(t_1,t_2)}_{(0,0)}} w_2\left(|p_{(s_1,s_2)}(x(s_1,s_2))|_V\right), \tag{16}$$

where $\mathbb{Z}^{(t_1,t_2)}_{(0,0)} = \{(s_1,s_2) \in \mathbb{Z}^2_+,\ 0 \le s_1 \le t_1,\ 0 \le s_2 \le t_2\}$. Furthermore, since $w_2(|p_{(\tau_1,\tau_2)}(x(\tau_1,\tau_2,x^0))|_V \ge w_2(\varepsilon_0)$ for $\tau_1 + \tau_2 \in T$, we get

$$\sum_{(s_1,s_2)\in\mathbb{Z}^{(t_1,t_2)}_{(0,0)}} w_2\left(|p_{(s_1,s_2)}(x(s_1,s_2))|_V\right) \to \infty \quad \text{as} \quad t_1 + t_2 \to \infty. \tag{17}$$

In this case, (16) contradicts (11). Therefore, the zero solution to (1) is p_t-asymptotically stable, which completes the proof. ■

Remark 2. The theorems given in this section yield some sufficient conditions for p_t-stability and p_t-asymptotic stability for the considered systems. It should be pointed out that their use is substantially restricted by the absence of methods for constructing convenient Lyapunov functionals. Consequently, it is of interest to continue the investigation to determine some new classes of the systems whose solutions properties are identical in some sense to the properties of some ordinary (non-varying) discrete systems (the so-called embedding problem). Since stability theory for the latter has been quite extensively developed, it follows that we can study the stability of embeddable systems.

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References

Belozerkovskii S.M., Kochetov Yu.A., Krasovskii A.A. and Novitskii V.V. (1980): *Introduction to Aero-Autoelasticity.* — Moscow: Nauka, (in Russian).

Brayton R.K. (1967): *Nonlinear oscillations in distributed network.* — Quart. Appl. Math., Vol.24, No.4, pp.289–301.

Dymkov M. and Gaishun I. (1997): *Discrete equations with varying structure and quadratic optimization problem for discrete linear Volterra equation.* — Proc. 4th Int. Symp. Methods and Models in Automation and Robotics MMAR'97, Międzyzdroje, Poland, (Domek S. *et al.*, Eds.), Warsaw, Vol.1, pp.123—128.

Gaishun I. (1983): *Multidimensional Integrable Differential Equations.* — Minsk: Nauka i Tekhnika, (in Russian).

Gaishun I. (1997): *Discrete equations of varying structure and stability of solutions .* — Diff. Eqns., Vol.33, No.12, pp.1611–1617, (in Russian).

Gopalsamy K. (1992): *Equations of Mathematical Ecology.* — Dordrecht: Kluwer.

Hartman P. (1964): *Ordinary Differential Equations.* — New York: Wiley.

Izobov N. (1998): *Existence of linear Pfaff systems with denumerable set of distinct characteristic sets of solutions.* — Diff. Eqns., Vol.34, No.6, pp.735–734, (in Russian).

Kaczorek T. (1994): *Singular continuous 2-D linear systems.* — Bull. Pol. Acad. Techn. Sci., Vol.42, No.4, pp.501–513.

Lyrie K. (1975): *Optimal Control for Problems of Mathematical Physics.* — Moscow: Nauka, (in Russian).

Myshkis A. (1998): *Differential equations with multi-dimensional time without conditions of total integrability.* — Proc. Inst. Math., Nat. Acad. Sci. Belarus, Vol.2, pp.128–134, (in Russian).

Ovsyannikov L.V. (1965): *Singular operator in scale of Banach spaces.* — Dokl. Akad. Nauk SSSR, Vol.163, No.4, pp.819–822, (in Russian).

Perov A. (1968): *On the theory of stationary differential equtions with multi-dimensional time.* — Diff. Eqns., Vol.4, No.7, pp.1289–1298, (in Russian).

Schetzen M. (1986): *The Volterra and Wiener Theories of Nonlinear Systems.* — New York: Wiley.

Sternberg S. (1970): *Lectures on Differential Geometry.* — Moscow: Nauka, (in Russian).

Thompson J.M.T. and Stewart H.B. (1986): *Nonlinear Dynamics and Chaos.* — New York: Wiley.

Treves J. and Duchateau E. (1971): *An abstract Cauchy-Kovalevski theorem in scales of Gervey classes*, In: Simposia Math., Academic Press, N.Y. — Vol.7, pp.135–163.

Tsonis A.A. (1992): *Chaos. From Theory to Applications.* — New York: Plenum Press.

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