ANALYTIC INTERPOLATION AND THE DEGREE CONSTRAINT[†]

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Analytic interpolation problems arise quite naturally in a variety of engineering applications. This is due to the fact that analyticity of a (transfer) function relates to the stability of a corresponding dynamical system, while positive realness and contractiveness relate to passivity. On the other hand, the degree of an interpolant relates to the dimension of the pertinent system, and this motivates our interest in constraining the degree of interpolants. The purpose of the present paper is to make an overview of recent developments on the subject as well as to highlight an application of the theory.

Keywords: analytic interpolation, uniformly optimal control, spectral analysis of time-series

1. Introduction

The present paper concerns a basic problem in analytic function theory with a history going back to the beginning of the 20th century. This problem is to *interpolate prescribed values and successive derivatives on a given set of points in the unit disc by means of an analytic function with magnitude bounded by one,* or alternatively, *by means of an analytic function having positive real part.* Variants of the problem were studied by Carathéodory, Féjer, Pick, and Nevanlinna more than 80 years ago, and connections can be traced to earlier mathematics on the moment problem by Chebyshev and Markov (Akhiezer, 1965; Grenander and Szegö, 1958; Walsh, 1956). The history of the problem is long, and in the second half of the 20th century the subject evolved into a beautiful chapter of modern operator theory in the works of Sz-Nagy, Koráni, Foias, Sarason, Ball, Helton, and many others, see, e.g., (Ball and Helton, 1983; Sarason, 1967).

Our purpose is not to review the many elegant theories which emerged from this journey, but rather to focus on a twist to this basic problem motivated by an engineering study over 30 years ago. Youla and Saito (1967) studied the problem of lossless coupling between a given generator and load. The problem of designing such a lossless coupling turns out to be equivalent to an analytic interpolation problem. It was in this context that the question of determining the minimal degree interpolant

† This research was supported in part by the NSF and AFOSR.

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was first raised. The same question was raised again a few years later by Kalman in the context of times-series modeling (Kalman, 1982) where the degree of the interpolant relates to the dimension of a modeling filter. At the present time the question of minimal degree has not received a definitive solution yet. However, there is a body of research spawned by the degree question which has led to an elegant characterization of interpolants, with degree bounded by the number of interpolating constraints. This research, which utilized non-traditional tools in the study of the analytic interpolation problem, is the subject of this paper.

2. The Analytic Interpolation Problem

For simplicity, we begin by discussing the case where the interpolation points are distinct. In this case the problem is named after Nevanlinna and Pick (see Walsh, 1956).

Given a set of n distinct points $\{z_1, z_2, \ldots z_n\}$ in the unit disc $\mathcal{D}$ and another set of values $\{w_1, w_2, \ldots, w_n\}$ also in $\mathcal{D}$, Pick answered the question of whether a function f exists which interpolates the given values w_k at the points z_k, and is analytic in the unit disc with modulus bounded by one, i.e., f belongs to the unit ball in $\mathcal{H}_\infty$ herein denoted by $\mathcal{S}$. The necessary and sufficient condition for the existence of such a function is that the so-called Pick matrix

$$P = \left[\frac{1 - w_k \bar{w}_\ell}{1 - z_k \bar{z}_\ell}\right]_{k,\ell=1}^{n}$$

is positive semi-definite. On the other hand, Nevanlinna successively reduced the number of constraints and simplified the problem so as to characterize all solutions. The tool used was the Schur algorithm which relies on the fact that $f(z_1) = w_1$ and has the required properties if and only if it is of the form

$$f(z) = \frac{w_1 - \frac{z - z_1}{1 - \bar{z}_1 z} f_1(z)}{1 - \bar{w}_1 \frac{z - z_1}{1 - \bar{z}_1 z} f_1(z)} \tag{1}$$

with $f_1(z)$ also in $\mathcal{S}$. In order for f to satisfy the remaining interpolation constraints, f_1 must satisfy $n-1$ constraints inherited through (1). Assuming for simplicity that P is positive definite, after n steps, the general solution can be expressed in the form of a linear fractional transformation

$$f(z) = \frac{a(z) + b(z) f_n(z)}{c(z) + d(z) f_n(z)} \tag{2}$$

on a free parameter $f_n(z) \in \mathcal{S}$. The coefficient matrix $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is J-unitary, i.e., it satisfies $M^* J M = J$ with $J := \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. This fact can be easily checked and guarantees that $1-|f|^2$ and $1-|f_n|^2$ have the same sign on the unit circle. Alternative methods exist where M is constructed directly by J-inner/outer factorization of a suitably constructed matrix whose range contains the graphs of possible solutions (Ball and Helton, 1983; Francis, 1987).

Analogous results hold for the case where interpolation requires the use of functions having positive real part in $\mathcal{D}$—this is the so-called *Carathéodory* class of functions and will be denoted by $\mathcal{C}$. The Pick matrix which corresponds to this problem is

$$P = \left[\frac{w_k - \bar{w}_\ell}{1 - z_k \bar{z}_\ell}\right]_{k,\ell=1}^{n},$$

and the positive semi-definiteness of P is again necessary and sufficient for the existence of positive-real interpolants which can again be parametrized similarly via a linear fractional transformation.

3. The Set of Interpolants with Degree $\leq n-1$

The entries of M are obviously rational functions and the 'central' solution, which corresponds to $f_n = 0$, is of degree $n-1$. In general, however, the parametrization given by (2) gives no insight into the class of interpolants of small degree. In particular, it may happen that even when $P > 0$ and the interpolation problems has many solutions, no solution of degree strictly less than $n-1$ exists. To determine whether this is the case one needs to test the solvability of complicated semi-algebraic conditions. Determining the minimal degree solution can again be approached via semi-algebraic conditions. However, in general, such an approach is intractable for all but the simplest cases where $n = 2, 3$ or 4 (cf. the Appendix of (Georgiou, 1983) for an analogous discussion on the Carathéodory interpolation problem). This road-block motivated interest in studying solutions of degree $n-1$. The set of such interpolants with degree $\leq n-1$ turns out to have an interesting characterization which is explained below. The statement of the theorem was originally derived in the context of the equivalent problem of interpolation by functions having positive real part instead of being contractive (Georgiou, 1983; 1987a; 1987b; Byrnes *et al.*, 1995; 1997; Georgiou, 1999).

Theorem 1. *Consider interpolation data (z_k, w_k) for $k = 1, \ldots, n$, as above and assume that $P > 0$. If $\sigma(z)$ is any polynomial of degree $\leq n-1$, having all roots in $|z| \geq 1$, then there exists a unique pair of polynomials $(\alpha(z), \beta(z))$ (modulo a common constant factor of magnitude 1), each of degree $\leq n-1$ with all their roots in $|z| \geq 1$ such that*

(i) $f = \beta/\alpha$ *is in* $\mathcal{S}$,

(ii) $f(z_k) = w_k$ *for* $k = 1, \ldots, n$, *and*

(iii) $\overline{\alpha(z)}\alpha(z) - \overline{\beta(z)}\beta(z) = \overline{\sigma(z)}\sigma(z)$

for $|z| = 1$.

Remark 1. The result holds true for the more general Carathéodory-Féjer interpolation where f is specified along with a number of its derivatives at various points in the disk (Georgiou, 1999), i.e., when

$$f^{(\ell_k)}(z_k) = w_{k,\ell_k} \text{ for } k = 1, \ldots, n_0, \text{ and } \ell_k = 0, \ldots, n_k. \tag{3}$$

The Pick matrix corresponding to such a situation is a bit more complicated (cf. the last section and (Georgiou, 1999)).

The above theorem gives a complete parametrization of candidate graph symbols (α, β) for interpolants of degree bounded by n. It should be noted that (α, β) may have common factors which will then be shared with σ. In fact, common factors do appear when an f exists with degree strictly $< n-1$. In this case the map $\sigma \to f$ is not injective whereas the map $\sigma \to (\alpha, \beta)$ is.

Remark 2. In the case where the interpolants are required to belong to $\mathcal{C}$, the analogous result holds with condition (iii) replaced by

(iii′) $$\overline{\alpha(z)}\beta(z) - \overline{\beta(z)}\alpha(z) = \overline{\sigma(z)}\sigma(z) \text{ for } |z| = 1.$$

Other than that, the result holds verbatim (Georgiou, 1999).

Analogous results hold in the case of interpolation with matrix-valued functions. In particular, the case of interpolation with positive real matrix-valued functions with its partial sequence of Laurent coefficients specified (i.e., interpolation with multiplicity at the origin) has been dealt with in (Georgiou, 1983).

The existence statement of Theorem 1 was given in (Georgiou, 1983; 1987a; 1987b) and the proof uses degree theory. It is based on a homotopy of maps $\sigma \to (\alpha, \beta)$ which is continuous in the data $(w_1, \dots, w_n)$. When the interpolating values are trivial, i.e., $w_1 = w_1 = \cdots = 0$, then the map sends $\sigma \to (\sigma, 0)$. An argument involving invariance of the topological degree is used to establish that, as long as the Pick matrix remains positive, a solution always exists. The uniqueness statement of the theorem was conjectured in (Georgiou, 1983; 1987b). It was proven in (Byrnes *et al.*, 1997; Byrnes *et al.*, 1995) for the case where σ has no root on the boundary of the circle using two different lines of argument. The proof was completed in (Georgiou, 1999) to encompass the case of σ having roots on the boundary. Since then, an alternative approach to both existence and uniqueness was introduced in (Byrnes *et al.*, 1999) for the Carathéodory problem (where all the z_k's coincide) and generalized in (Byrnes *et al.*, 2000a; 2000b; 2000c) to the Nevanlinna-Pick problem. It is based on a functional which is suitably selected on the basis of σ, and which has a unique minimum at the corresponding interpolant f. The theorem applies to choices of σ devoid of roots on $|z| = 1$.

For simplicity in the statement of the following result, the σ, α, β can be taken as elements of the co-invariant subspace

$$\mathcal{K} := \mathcal{H}_2 \ominus B(z)\mathcal{H}_2,$$

where $B(z) = \prod_{k=1}^{n}(z - z_1)/(1 - \bar{z}_1 z)$, since such elements are rational with common denominator $\prod_{k=1}^{n}(1 - \bar{z}_1 z)$ and numerator of degree $\leq n-1$. This notation simplifies the formalism and algebra in the general context of Sarason/Carathéodory-Féjer interpolation (Byrnes *et al.*, 2000c; Georgiou, 1987b).

Theorem 2. *Given $\sigma \in \mathcal{K}$ having its roots in $|z| > 1$, define*

$$\mathbb{I}_\sigma(f) = \int_{-\pi}^{\pi} \log\Big(1 - \big|f(e^{i\theta})\big|^2\Big)\big|\sigma(e^{i\theta})\big|^2 \,\mathrm{d}\theta. \tag{4}$$

The value

$$\sup_{f\in\mathcal{S}}\left\{\mathbb{I}_\sigma(f) : f(z_k) = w_k \ \ \textit{for} \ \ k = 1,\dots,n\right\} \tag{5}$$

is attained for a unique f *in the interior of* $\mathcal{S}$*. Furthermore, this* f *is of the form*

$$f = \beta/\alpha, \tag{6}$$

α *and* β *being outer functions in* $\mathcal{K}$ *satisfying*

$$\overline{\alpha(z)}\alpha(z) - \overline{\beta(z)}\beta(z) = \overline{\sigma(z)}\sigma(z) \ \textit{for} \ |z| = 1. \tag{7}$$

The pair (α, β) *is uniquely defined (modulo sign) in terms of* f *and* σ *via (6) and (7). Conversely, if* f *satisfies (6), (7) and the interpolation conditions, then it is the unique solution to the optimization problem (5).*

Remark 3. For the case of $\mathcal{C}$-interpolation, the analogous result requires $\mathbb{I}_\sigma(f)$ in (4) to be replaced by

$$\mathbb{I}_\sigma(f) = \int_{-\pi}^{\pi} \log\left(\Re f(e^{i\theta})\right) |\sigma(e^{i\theta})|^2 \,\mathrm{d}\theta. \tag{8}$$

and (7) by (iii$'$) as in Remark 2.

The advantage of the Theorem 2 is that it presents f as a solution to an optimization problem. It should be noted that the primal optimization problem stated in the theorem requires optimization in infinitely many variables, namely the function f itself, with finitely many constraints. However, the dual problem, which is convex, requires optimization in finitely many variables. This has been explored in (Byrnes *et al.*, 1999; 2000a; 2000b) and forms the basis of numerical algorithms given in these references.

4. Relevance in Spectral Analysis

In this section we overview one of the examples which to a large degree motivated the development of the theory. The context is spectral analysis of time-series using second-order statistics.

Assume that $\{u_\ell : \ell = \dots, -1, 0, 1, \dots\}$ is a zero-mean, stationary stochastic process, with unknown spectral power distribution. The problem at hand is to characterize all admissible power spectra which are consistent with a collection of second-order statistics (estimated, e.g., from an observation record for the process u_k). Recall that the power distribution of u_k is in general a nonnegative measure $\mathrm{d}\sigma(\theta)$ with $\theta \in [-\pi, \pi]$, and it is completely specified by the infinite sequence of its Fourier coefficients

$$R_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jm\theta}\,\mathrm{d}\sigma(\theta),$$

where $j = \sqrt{-1}$ and $m = 0, \pm 1, \ldots$. (Note that $R_{-m} = \bar{R}_m$.) In this case, the function

$$f(z) = \frac{1}{2} R_0 + R_1 z + R_2 z^2 + \cdots \tag{9}$$

is in $\mathcal{C}$ and $\mathrm{d}\sigma(\theta)$ is obtained via the radial limits of the real part of $f(z)$. It turns out that second-order statistics impose analytic interpolation constraints on f.

The standard example is when these second-order statistics consist of finitely many samples of the autocorrelation function, i.e., $\{R_0, R_1, \ldots, R_{n-1}\}$. A significant part of the literature on modern nonlinear spectral analysis techniques (Stoica and Moses, 1997) is based on this paradigm. In this case the Pick matrix corresponding to the relevant interpolation problem is simply a Toeplitz matrix formed out of the partial autocorrelation sequence. The theory which allows parametrization of all admissible $\mathcal{C}$-functions and, as a consequence, all admissible power spectra is classical and has connections to the theory of the Szegö orthogonal polynomials (Geronimus, 1961). We will discuss a framework which reveals the connection with the general Nevanlinna-Pick problem.

Consider a number of points $\{z_k : k = 1, \ldots, n\}$ in $\mathcal{D}$, and the family of first-order filters

$$x_\ell^{(k)} = z_k x_{\ell-1}^{(k)} + u_\ell \text{ for } \ldots, -1, 0, 1, \ldots \tag{10}$$

It is easy to show that the output covariances

$$p_k := \mathcal{E}\{|x_\ell^{(k)}|^2\}$$

of these filters impose the following interpolation constraints on the function f in (9) (see Byrnes *et al.*, 2000b):

$$f(z_k) = \frac{1 - |z_k|^2}{2} p_k.$$

A more general formulation involves a stable dynamical system

$$x_\ell = A x_\ell + B u_\ell \tag{11}$$

with $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{n \times 1}$ driven by u_ℓ. The state covariance $P = \mathcal{E}\{x_\ell x_\ell^*\}$ is shown in (Georgiou, 2001) to decompose into a sum $WE + EW^*$ where E is the solution of the Lyapunov equation $E = BB^* + AEA^*$ while W is a matrix which commutes with A. The interpolation conditions imposed by the state-covariance statistics on f are now of the form

$$f(A) = W. \tag{12}$$

This condition includes the case where there are repeated poles and hence interpolation constraints on the derivatives of f as well. More specifically, because A is a cyclic matrix and W commutes with A, W needs to be a polynomial function of A (see Gantmacher, 1959, p.222). So let $v(z) = v_0 + v_1 z + \cdots + v_{n-1} z^{n-1}$ be such a polynomial and $v(A) = W$. Then (12) is equivalent to (3), where the values

$w_k^{\ell_k}$ represent the values of $v(z)$ on the spectrum of A. In this case, the Pick matrix of the relevant interpolation problem is precisely the state-covariance P (Georgiou, 2001).

The filter (11) may represent the dynamics of a measuring aparatus with different sensors (such as an antenna array (Georgiou, 2000b)), or the dynamics and filtering of suitable post-processing of recorded time-series data for u_k. The choice of such a 'processing' state-filter (11) (or in the special case of a bank of filters (10) the values for z_k's) affects the dependence of the covariance statistics on the power spectrum of u_k. In particular, spectra which are consistent with the state statistics deviate less from the actual spectrum within the passband. Thus, such filters can be tuned prior to computing covariance statistics. This allows improved resolution of any subsequent models derived from such statistics, within the bandwidth of (11) or of (10). Benefits acquired thereof as compared to other methods have been documented in (Byrnes *et al.*, 2000a; 2000b; Georgiou, 2000a; 2001).

Compelling arguments for parsimonious representations and for low dimension models have been advanced over the years by Kalman, Rissanen, Akaike and others. In the context of time series analysis, typically, an auto-regressive/moving average (ARMA) modeling filter for u_k is sought in the form

$$\hat{u}_k + \alpha_1 \hat{u}_{k-1} + \alpha_m \hat{u}_{k-m} = \sigma_0 \nu_k + \cdots + \sigma_m \nu_{k-m},$$

where ν_k represents zero-mean and unit-variance white noise. Then, it is the dimension m of the filter which provides a measure of 'complexity'. Theorem 1 allows a complete parametrization of all modeling filters of dimension $\leq n-1$. This is summarized in our final proposition. The proof is a direct consequence of Theorem 1 (cf. Remarks 1 and 2, (Georgiou, 1987a; 1999)).

Proposition 1. *Let u_k, A, B and P be as above. Given any polynomial $\sigma(z) = \sigma_0 + \cdots + \sigma_{n-1} z^{n-1}$ with degree $\leq n-1$ and roots outside $\mathcal{D}$, there exists a unique polynomial $\alpha(z) = 1 + \alpha_1 z + \cdots + \alpha_{n-1} z^{n-1}$ so that*

$$T(z) = \frac{\sigma_0 + \cdots + \sigma_m z^m}{1 + \alpha_1 z + \cdots + \alpha_m z^m}$$

is a transfer function of an ARMA model for u_k which is consistent with the state-covariance statistics P.

Thus, the proposition states that all ARMA models of dimension $\leq n-1$ which are consistent with the given second-order statistics P are parametrized by a choice of a stable numerator for $T(z)$, i.e., by an arbitrary selection of the coefficients of the moving-average part (subject to the stability requirement).

5. Concluding Remark

Analytic interpolation is encountered in a variety of engineering applications (Delsarte *et al.*, 1981; Helton, 1982; Kalman, 1982; Tannenbaum, 1982; Youla and Saito, 1967;

Zames, 1981). Invariably, the interpolating function relates to a dynamical system, while the degree of the function relates to the dimension of the system (e.g., see (Byrnes *et al.*, 2000a; 2000b) for application in uniformly optimal control and in the modeling of time series). Thus, the set of solutions with degree bounded by $n - 1$ represents a set of parsimonious solutions. Further, this set has a rather elegant parametrization which has been exploited in (Byrnes *et al.*, 2000a; 2000b; Georgiou, 2000a; 2001) for high resolution spectral analysis of time series.

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