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REMARKS ABOUT ENERGY TRANSFER IN AN RC LADDER NETWORK

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The problem of energy transfer in an RC-ladder network is considered. Using the maximum principle, an algorithm for constructing optimal control is proposed, where the cost function is the energy delivered to the network. In the case considered, optimal control exists. Numerical simulations were performed using Matlab.

Keywords: optimal control, energy transfer, approximation of an RC-long line

1. Introduction

The problem of determining optimal controls is one of fundamental problems in control theory and its applications (Athans and Falb, 1969; Bryson and Ho, 1972). Analytic solutions exist only in particular examples. Below we will investigate such an example.

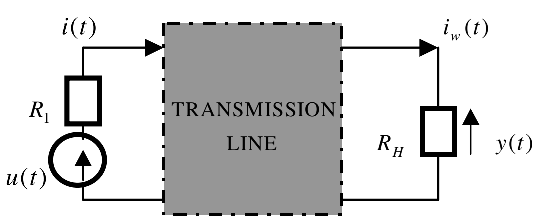

Fig. 1. Scheme of energy transfer.

We consider an electric network shown in Fig. 1. The resistance of the voltage source R_1 and the output resistance R_H are given. Let

$$J(u) = \int_0^T u(t)i(t)\,\mathrm{d}t, \tag{1}$$

where $J(u)$ is the energy delivered to the network and T is the time horizon. Let the output current be $i_w(0) = 0$ and $t \in [0,T]$. Assume that T and E are fixed and consider the following problem: Find u_o such that

$$J(u) \geq J(u_o), \quad \forall\, u \tag{2a}$$

and

$$\int_0^T y(t)i_w(t)\,\mathrm{d}t = E, \tag{2b}$$

where E is the energy producing heat on the resistance R_H.

Remark 1. Consider the electric network shown in Fig. 2. We have

$$J(u) = \frac{1}{R_1 + R + R_H}\int_0^T u(t)^2\,\mathrm{d}t$$

$$E = \frac{R_H}{(R_1 + R + R_H)^2}\int_0^T u(t)^2\,\mathrm{d}t. \tag{3}$$

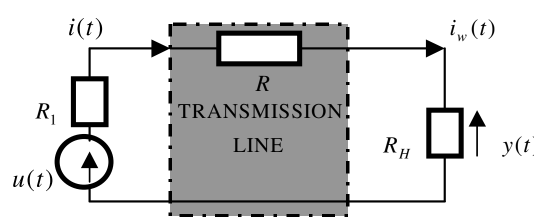

Fig. 2. Electric resistance network.

The energy E producing heat on the resistance R_H is given. From (3) we obtain many controls u satisfying

$$\int_0^T u(t)^2\,\mathrm{d}t = E(R_1 + R + R_H)^2/R_H,$$

$$J(u) = E\left(1 + \frac{R_1 + R}{R_H}\right). \tag{4}$$

If $u(t) = \bar{u} = \mathrm{const}$, then

$$\bar{u} = (R_1 + R + R_H)\sqrt{\frac{E}{R_H T}}. \qquad \blacklozenge$$

Remark 2. Consider a homogeneous long electric RC transmission line, i.e. one where the parameters per the unit length (resistance r and capacity c) are constant and independent of the co-ordinate z. An infinitesimal part of the long line is described by the equation

$$rc\frac{\partial x(t,z)}{\partial t} = \frac{\partial^2 x(t,z)}{\partial z^2}, \quad 0 \le t, \quad 0 \le z \le l. \tag{5}$$

♦

Remark 3. Let $z = ih$, $h = l/n$, $i = 0, 1, \dots, n$ and $x(t,(2k-1)h/2) = x_k(t)$, $k = 1, 2, \dots, n$. We have

$$\frac{\partial^2 x(t,z)}{\partial z^2} \approx \frac{1}{h}\left(\frac{x(t,z+h)-x(t,z)}{h} - \frac{x(t,z)-x(t,z-h)}{h}\right)$$

for $z = (2k-1)h/2$ and $k = 1, 2, \dots, n$. Then the RC transmission line can be approximated by the RC ladder network shown in Fig. 3, where $R = rl$ and $C = cl$ (Butkovskii, 1965, p. 314).

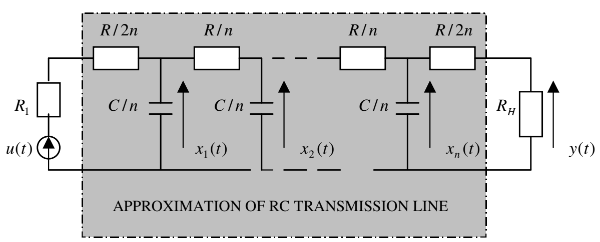

Fig. 3. RC ladder network.

♦

2. Electric RC Ladder Network

Consider the electric RC ladder network shown in Fig. 3. Its parameters R, R_1, R_H and C are known. The system shown in Fig. 3 can be described by the equation (Mitkowski, 1994; 1997; 2000):

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ x(t) &= \left[x_1(t) \ \ x_2(t) \ \ \dots \ \ x_n(t)\right]^T, \\ y(t) &= Wx(t), \end{aligned} \tag{6}$$

where A is the $n \times n$ real tridiagonal Jacobi matrix,

$$\begin{aligned} A &= [a_{ij}], \quad a_{ij} = 0 \ \text{ for } \ |i-j| > 1, \\ a_{ii} &= \frac{n^2}{RC}, \quad i = 2, 3, \dots, n-1, \\ a_{11} &= -(1 + r(R_1))\frac{n^2}{RC}, \\ a_{nn} &= -(1 + r(R_H))\frac{n^2}{RC}, \quad r(\gamma) = \frac{2R}{2n\gamma + R}, \\ a_{i,i-1} &= \frac{n^2}{RC}, \quad i = 2, 3, \dots, n, \\ a_{i,i+1} &= \frac{n^2}{RC}, \quad i = 1, 2, 3, \dots, n-1, \\ B &= \frac{n^2 r(R_1)}{RC}e_1, \quad e_1 = [1\ 0\ 0\ \dots\ 0\ 0]^T, \\ W &= \frac{nr(R_H)R_H}{R}[0\ 0\ \dots\ 0\ 1]. \end{aligned} \tag{7}$$

For fixed n the tridiagonal real Jacobi matrix A has only single real eigenvalues λ_i. The matrix A is diagonalizable. The Jordan canonical form of A is $J = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. From Gershgorin's criterion and the fact that $\det A \neq 0$, we have $\lambda_i \in [-m, 0)$, where $m = \max_i(|a_{i,i-1}| + |a_{i,i+1}|)$. Thus (Mitkowski, 2000, p. 301) the system (6) is asymptotically stable.

3. Problem Formulation and Its Solution

Consider the system (6). Let $x(0) = 0$ and (cf. Eqn. (1)) the cost function be

$$\begin{aligned} J(u) &= \int_0^T u(t)i(t)\,\mathrm{d}t \\ &= \frac{2n}{2nR_1 + R}\int_0^T u(t)[u(t) - x_1(t)]\,\mathrm{d}t, \end{aligned} \tag{8}$$

where $J(u)$ is the energy delivered to the electric RC-network, and T is the time horizon.

Optimal control problem: Let T and E be fixed. Find a control $u_o \in U_d$ such that

$$\begin{aligned} & J(u) \ge J(u_o), \quad \forall\, u \in U_d, \\ & U_d = \left\{u : \frac{1}{R_H}\int_0^T y(t)^2\,\mathrm{d}t \right. \\ & \qquad \left. = \frac{n^2 R_H}{(nR_H + R/2)^2}\int_0^T x_n(t)^2\,\mathrm{d}t = E \right\}, \end{aligned} \tag{9}$$

where E is the energy producing heat on the resistance R_H (see Fig. 3) and U_d is the set of admissible controls.

Remark 4. The set U_d is non-empty. Indeed, examine, e.g. $u(t) = \text{const}$ such that

$$\frac{1}{R_H}\int_0^T y(t)^2\,\mathrm{d}t = \frac{n^2 R_H}{(nR_H + R/2)^2}\int_0^T x_n(t)^2\,\mathrm{d}t = E$$

(cf. (6) and (7) for $x(0) = 0$). Now, we consider the spaces $L^p(0,T)$, $p \in [1,\infty)$ with the norms $\|f\|_p = [\int_0^T |f(t)|^p \, \mathrm{d}t]^{1/p}$. From the Hölder inequality (Musielak, 1976, p. 45; Luenberger, 1974, p. 58) we have $\int_0^T u(t)x_1(t)\,\mathrm{d}t \le \|ux_1\|_1 \le \|u\|_2\|x_1\|_2$. The system (6) is asymptotically stable, controllable and observable (cf. (7); the pair (A,B) is controllable and (W,A) is observable). Consequently,

$$(R_1 + R/2n)J(u) = \|u\|_2^2 - \int_0^T u(t)x_1(t)\,\mathrm{d}t \ge \|u\|_2^2 - \|u\|_2\|x_1\|_2 \ge -\|x_1\|_2^2/4$$

(see (8) and (9)), for every $u \in U_d$ the norm $\|x_1\|_2$ is finite and $J(u) \to \infty$ as $\|u\| \to \infty$. Thus there exists the optimal control u_o, cf. (8). We can notice that $J(u) = J(-u)$. ♦

The Maximum Principle makes it possible to construct an algorithm for determining optimal control. Defining new state variables

$$\dot{x}_{n+1}(t) = x_n(t)^2, \quad x_{n+1}(0) = 0,$$

$$\dot{x}_{n+2}(t) = u(t)[u(t) - x_1(t)], \quad x_{n+2}(0) = 0, \tag{10}$$

we have

$$x_{n+1}(T) = \frac{(nR_H + R/2)^2}{n^2 R_H} E,$$

$$x_{n+2}(T) = \frac{2nR_1 + R}{2n} J(u).$$

Let $\tilde{x}(t) = [x(t)^T \; x_{n+1}(t) \; x_{n+2}(t)]^T$ and $\tilde{\psi}(t) = [\psi(t)^T \; \psi_{n+1}(t) \; \psi_{n+2}(t)]^T$. Then we obtain the Hamiltonian in the form

$$\begin{aligned} &H\big(\tilde{\psi}(t), \tilde{x}(t), u(t)\big) \\ &= \psi(t)^T\big[Ax(t) + Bu(t)\big] + \psi_{n+1}(t)x_n(t)^2 \\ &\quad + \psi_{n+2}(t)u(t)\big[u(t) - x_1(t)\big]. \end{aligned} \tag{11}$$

In this case $\psi_{n+1}(t) = -\rho = \text{const}$, $\psi_{n+2}(t) = -1$, $\psi(T) = 0$ and (the adjoint system)

$$\dot{\psi}(t) = -A^T\psi(t) - b,$$

$$b^T = \big[u(t) \; 0 \; 0 \; \dots \; 0 \; -2\rho x_n(t)\big], \tag{12}$$

where ψ is the adjoint function. Using the Maximum Principle (Pontriagin *et al.*, 1983, Górecki, 1993, p. 393), from (11) we get

$$\begin{aligned} u(t) &= \frac{1}{2}\big[B^T\psi(t) + x_1(t)\big] \\ &= \frac{1}{2}\left[\frac{2n^2}{(2nR_1 + R)C}\psi_1(t) + x_1(t)\right]. \end{aligned} \tag{13}$$

The control (13) depends on the real number ρ and is called the extremal control. The optimal control u_o can exist only among the extremal controls (13).

From (6), (12) and (13), we obtain the canonical system in the following form:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\psi}(t) \end{bmatrix} = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \begin{bmatrix} x(t) \\ \psi(t) \end{bmatrix},$$

$$x(0) = 0, \quad \psi(T) = 0, \tag{14}$$

where the matrices Z_i (depending on ρ) are given by the closed-loop system (6), (12) and (13). Let

$$Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}, \quad e^{Zt} = \begin{bmatrix} \Phi_1(t) & \Phi_2(t) \\ \Phi_3(t) & \Phi_4(t) \end{bmatrix}. \tag{15}$$

Then from (14) and (15) we have

$$x(t) = \Phi_2(t)\psi(0), \quad \psi(t) = \Phi_4(t)\psi(0). \tag{16}$$

If $E \neq 0$, then $x(t) \neq 0$, cf. (9). Thus from (16) we get $\psi(0) \neq 0$. Since $\psi(T) = 0$, cf. (14), from (16) we have

$$\det \Phi_4(T) = 0. \tag{17}$$

The idea of the control algorithm:

- Determine the parameter ρ using Eqn. (17).
- From (9) and (16) calculate $\psi(0)$.
- From (13) and (16) determine

$$\begin{aligned} u(t) &= \frac{1}{2}\big[B^T\psi(t) + x_1(t)\big] \\ &= \frac{1}{2}\big[B^T\Phi_4(t) + e_1^T\Phi_2(t)\big]\psi(0), \end{aligned} \tag{18}$$

where $e_1 = [1 \; 0 \; 0 \; \dots \; 0 \; 0]^T \in \mathbb{R}^n$.

4. RC Ladder Network with $n=1$

A very interesting case corresponds to $n = 1$. This is because closed-form formulae for the optimal trajectories can be obtained, in particular for the optimal control $u_o(t)$, as well as a closed-form formula for the cost function $J(u_o)$.

Now we consider an RC ladder network shown in Fig. 3 with $n = 1$. In this case we obtain the following parameters (see (6)):

$$\begin{aligned} A &= -\frac{R_1 + R_H + R}{C(R_1 + R/2)(R_H + R/2)}, \\ B &= \frac{1}{C(R_1 + R/2)}, \\ W &= \frac{R_H}{R_H + R/2}, \end{aligned} \tag{19}$$

and in (15) we have

$$Z_1 = -\frac{2R_1 + R_H + 3R/2}{2C(R_1 + R/2)(R_H + R/2)},$$

$$Z_2 = \frac{1}{2C^2(R_1 + R/2)^2}, \tag{20}$$

$$Z_3 = 2\rho - \frac{1}{2}, \quad Z_4 = -Z_1.$$

Remark 5. (Górecki, 1993, p. 394, 584; Korytowski, 2001). The matrix Z for $n = 1$, cf. (15), has eigenvalues $\lambda_1 = \lambda$ and $\lambda_2 = -\lambda$, where $\lambda = \sqrt{Z_1^2 + Z_2 Z_3}$. ♦

Assume that $\operatorname{rank} Z > 0$. If $Z_3 = -Z_1^2/Z_2$, then $\lambda = 0$ (only one eigenvector corresponds to $\lambda = 0$, because $\operatorname{rank} Z > 0$) and $\Phi_4(t) = 1 - tZ_1$. In this case (17) cannot be exploited, because $Z_1 < 0$ and $t > 0$.

If $\lambda \neq 0$, then closed-form formulae for the elements $\Phi_2(t)$ and $\Phi_4(t)$ of the matrix e^{Zt} (see (15)) for $n = 1$ are given by

$$\Phi_2(t) = \frac{Z_2(e^{\lambda t} - e^{-\lambda t})}{2\lambda},$$

$$\Phi_4(t) = \frac{(\lambda - Z_1)e^{\lambda t} + (\lambda + Z_1)e^{-\lambda t}}{2\lambda}, \tag{21}$$

$$\lambda = \sqrt{Z_1^2 + Z_2 Z_3},$$

where the Z_i's are given in (20).

From (17) and (21) we have $e^{2\lambda t} = (Z_1 + \lambda)/(Z_1 - \lambda)$. If λ is real, $\lambda > 0$ and $t > 0$, then $e^{2\lambda t} \neq (Z_1 + \lambda)/(Z_1 - \lambda)$.

Now, if $Z_1^2 + Z_2 Z_3 < 0$, then $\lambda_1 = \lambda$, $\lambda_2 = -\lambda$,

$$\lambda = j\omega, \quad j^2 = -1, \quad \omega = \sqrt{|Z_1^2 + Z_2 Z_3|} \tag{22}$$

and consequently, from (21), we obtain

$$\Phi_2(t) = \frac{Z_2}{\omega}\sin\omega t, \quad \Phi_4(t) = \cos\omega t - \frac{Z_1}{\omega}\sin\omega t. \tag{23}$$

Thus from (23) we can notice that (17) holds for the appropriate ϖt.

We can notice that $Z_1^2 + Z_2 Z_3 < 0$ if and only if

$$\rho < -\frac{2R_1 + R}{2R_H + R}\left[\frac{2R_1 + R}{2R_H + R} + 1\right] = \rho_d. \tag{24}$$

From (17) we conclude that $\Phi_4(T) = 0$. Because in this case $\Phi_4(t)$ is given by (23), we have the following equation:

$$\tan z = -Kz, \quad K = -\frac{1}{Z_1 T}, \quad z = \omega T. \tag{25}$$

It has many (positive) solutions:

$$z_i \in \big(\pi/2 + (i-1)\pi,\ \pi + (i-1)\pi\big), \qquad i = 1, 2, 3, \dots. \tag{26}$$

For every z_i there exists

$$\rho_i = -\left[\frac{(2R_1 + R)Cz_i}{2T}\right]^2 + \rho_d, \tag{27}$$

cf. (20) and (22), where ρ_d is given in (24).

From (9) and (16) we have

$$\int_0^T x_1(t)^2\,\mathrm{d}t = \int_0^T \Phi_2(t)^2\,\mathrm{d}t\psi_1(0)^2 = \frac{(R_H + R/2)^2}{R_H}E. \tag{28}$$

In this case ($n = 1$) the number $\psi_1(0)$ is dependent on z_i, cf. (26) (or ρ_i, cf. (27)). From (28) we have

$$\psi_1(0) = \pm(R_H + R/2)\frac{z_i}{Z_2 T}\sqrt{\frac{2E(1 + K^2 z_i^2)}{R_H T(1 + K + K^2 z_i^2)}},$$

$$K = -\frac{1}{Z_1 T}, \tag{29}$$

where the Z_i's are given in (20).

It is easy to show that, using elementary operations (cf. (8) and (16)), we have

$$J(u) = \frac{1}{(R_1 + R/2)}\int_0^T u(t)\big[u(t) - x_1(t)\big]\,\mathrm{d}t = \frac{\psi_1(0)^2}{4(R_1 + R/2)} \times \int_0^T \left[\frac{4}{(2R_1 + R)^2 C^2}\Phi_4(t)^2 - \Phi_2(t)^2\right]\mathrm{d}t. \tag{30}$$

Consequently, from (30), (23) and (29) we obtain

$$J\big(u(\rho_i)\big) = E\left\{1 + \frac{R_1 + R}{R_H} + \frac{(R_H + R/2)^2(R_1 + R/2)C^2}{R_H T^2}z_i^2\right\}. \tag{31}$$

Remark 6. We can notice (cf. (31)), that $J(u(\rho_1)) < J(u(\rho_i))$, $\forall\, i$, where ρ_i is given by (27) and z_i is given by (25) and (26). ♦

Remark 7. From (9) we get

$$\|x_1\|_2^2 = \frac{(R_H + R/2)^2}{R_H}E.$$

Thus from Remark 4 we have

$$(R_1 + R/2)J(u) = \|u\|_2^2 - \int_0^T u(t)x_1(t)\,\mathrm{d}t \geq \|u\|_2^2 - \|u\|_2\|x_1\|_2, \tag{32}$$

and consequently

$$(R_1 + R/2)J(u) \geq \|u\|_2^2 - \|u\|_2\sqrt{\tilde{R}E} \geq -\tilde{R}E/4, \quad \tilde{R} = \frac{(R_H + R/2)^2}{R_H}. \tag{33}$$

Since the function $J(u)$ is continuous and (33) holds, there exists the optimal control u_o, cf. (8). One can notice that $J(u) = J(-u)$. ♦

Using Remarks 6 and 7, we obtain optimal control (for $\rho = \rho_1$) in the following form, cf. (18):

$$u_o(t) = \frac{1}{2}\left[\frac{2}{(2R_1 + R)C}\left(\cos\omega t - \frac{Z_1}{\omega}\sin\omega t\right) + \frac{Z_2}{\omega}\sin\omega t\right]\psi_1(0), \quad \omega = z_1/T, \tag{34}$$

where $\psi_1(0)$ is given by (29). The optimal trajectories are given by the following equalities:

$$x_1(t) = \psi_1(0)\frac{Z_2}{\omega}\sin\omega t, \quad i(t) = \frac{u(t) - x_1(t)}{R_1 + R/2}. \tag{35}$$

Example 1. Let $R_1 = 1$, $R = 1$, $R_H = 2$, $C = 1$, $T = 0.5$, $E = 10$. Then for $n = 1$ we have $K = 2.7272$, cf. (25), $Z_1 = -0.7333$, $Z_2 = 0.2222$, $z_1 = 1.7746$, $\rho_1 = -29.3013$, $\omega = 3.5491$, $\psi_1(0) = 169.3551$ and $J(u_o) = 610.4439$. The optimal control $u_o(t)$, '$\times-$', the optimal electric current $i(t)$, '$+-$', the function $\psi_1(t)$ '$\circ-$', and the optimal trajectory $x_1(t)$, '$*-$', are shown in Fig. 4. ♦

5. RC Ladder Network with n=2

Consider the electric RC ladder network shown in Fig. 3 with $n = 2$. The parameters R, R_1, R_H and C are known. Equations (6) and (7) describe the system. In this case optimal control can be determined by numerical calculations.

Example 2. Let $R_1 = 1$, $R = 1$, $R_H = 2$, $C = 1$, $T = 0.5$ and $E = 10$. Then for $n = 2$ the parameters of the system are given by (7). The function $\rho \mapsto \det\Phi_4(T)$ is shown in Fig. 5. In this case $\det\Phi_4(T) = 0$ for $\rho = -43.757$, $\psi_1(0) = 90.3034$, $\psi_2(0) = -1.2256\psi_1(0)$ and $J(u_o) = 609.7385$. The optimal control $u_o(t)$, '$\times-$', the optimal electric current $i(t)$, '$+-$', the function $\psi_1(t)$, '$\circ-$', $\psi_2(t)$, '$*-$' and the optimal trajectories $x_1(t)$, '$\cdot-$' and $x_2(t)$, '$:-$' are shown in Fig. 6. ♦

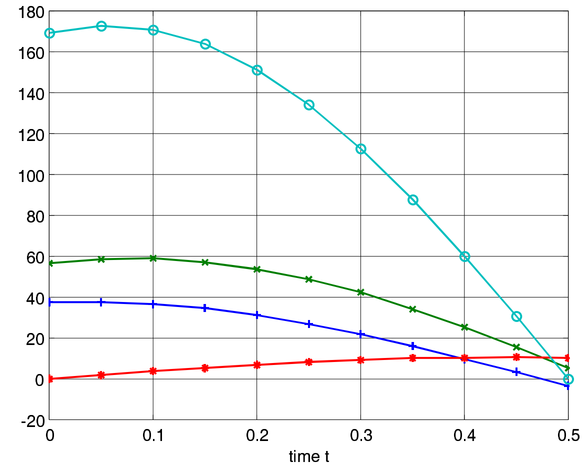

Fig. 4. Optimal trajectories for $n = 1$.

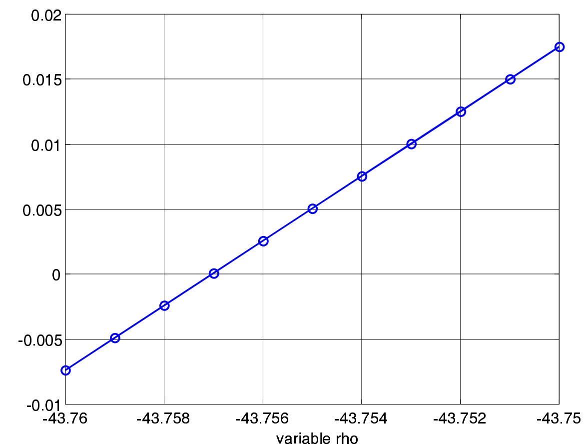

Fig. 5. Function $\rho \mapsto \det\Phi_4(T)$.

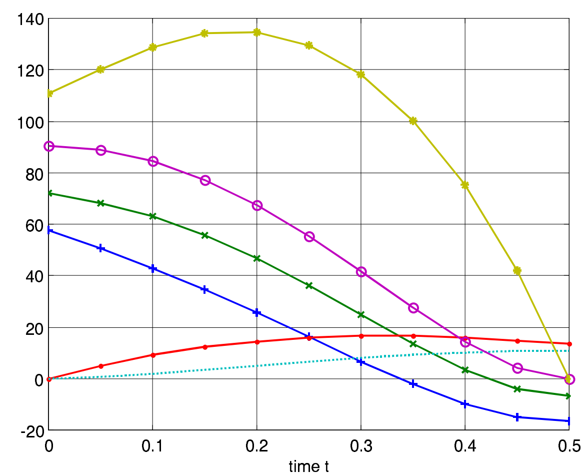

Fig. 6. Optimal trajectories for $n = 2$.

6. Concluding Remarks

In applications, optimal control problems are of paramount importance. Unfortunately, only in few examples we can find closed-form formulae for optimal control. In this paper such an example was studied. The resulting two-point boundary-value problem (14), (9) was analytically solved (for $n = 1$). For large n we have to solve this problem numerically.

RC ladder networks constitute a kind of approximations to RC-long lines (see Remarks 2 and 3). Probably, the results presented in this paper can be applied to distributed systems. A very important problem is the transfer of an energy quantum in a given time with the simultaneous minimization of the energy delivered to the system. For example, it can be used in microelectronics, biology and engineering. Generally, the problem of energy minimization is very important.

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