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A NOTE ON SOME CHARACTERIZATION OF INVARIANT ZEROS IN SINGULAR SYSTEMS AND ALGEBRAIC CRITERIA OF NONDEGENERACY

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The question how the classical definition of the Smith zeros of an LTI continuous-time singular control system $S(E, A, B, C, D)$ can be generalized and related to state-space methods is discussed. The zeros are defined as those complex numbers for which there exists a zero direction with a nonzero state-zero direction. Such a definition allows an infinite number of zeros (then the system is called degenerate). A sufficient and necessary condition for nondegeneracy is formulated. Moreover, some characterization of invariant zeros, based on the Weierstrass-Kronecker canonical form of the system and the first nonzero Markov parameter, is obtained.

Keywords: singular control systems, multivariable zeros, state-space methods, Markov parameters

1. Introduction

During the past two decades considerable attention has been paid to the determination and computation of multivariable zeros of a singular system described by the state-space model $S(E, A, B, C, D)$. The characterization of the zeros of singular systems proposed in this paper is parallel to that for standard linear systems.

The distinct Smith zeros of the system $S(E, A, B, C, D)$ are those points of the complex plane where the system matrix

$$P(s) = \begin{bmatrix} sE - A & -B \\ C & D \end{bmatrix}$$

loses its normal rank. Recall (Callier and Desoer, 1982, pp. 25–26) that the normal rank of a polynomial matrix $M(s)$ is the rank over the ring of all polynomials in one complex variable s with coefficients in the field of real numbers. The Smith zeros of $S(E, A, B, C, D)$ are defined as the roots of the so-called zero polynomial which is the product of diagonal (invariant) polynomials of the Smith canonical form of $P(s)$ (i.e., as the Smith zeros of the pencil $P(s)$) (Misra *et al.*, 1994). The Smith zeros of the pencil $[sE - A, \quad -B]$ are called the input decoupling (i.d.) zeros, whereas the Smith zeros of the pencil $\left[\begin{smallmatrix} sE-A \\ C \end{smallmatrix}\right]$ are called the output decoupling (o.d.) zeros of $S(E, A, B, C, D)$ (Misra *et al.*, 1994). If the system $S(E, A, B, C, D)$ has no input and no output decoupling zeros, then the Smith zeros of the underlying system matrix $P(s)$ are called the transmission zeros of $S(E, A, B, C, D)$ (Misra *et al.*, 1994).

In (Tokarzewski, 1998) it was shown that if the system $S(E, A, B, C, D)$ with the regular pencil $sE - A$ is nondegenerate, then the set of its invariant zeros coincides with the set of invariant zeros of the appropriate standard linear system. In this way, the question of seeking invariant zeros of a nondegenerate singular system can be reduced to such a question for standard systems (suitable procedures for finding invariant zeros in standard linear systems can be found in (Tokarzewski, 2002a)). Unfortunately, no algebraic critera of degeneracy or nondegeneracy for singular systems are accessible at present. This paper constitutes an extended version of the conference paper (Tokarzewski, 2003).

2. Preliminary Results

2.1. Invariant Zeros

Consider a system $S(E, A, B, C, D)$ of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

$t \geq 0$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^r$, where E, A, B, C, D ($D \neq 0$ or $D = 0$) are real matrices of appropriate dimensions and the matrix E is singular but $\det(sE - A) \neq 0$ (i.e., the pencil $sE - A$ is regular). We adopt the following definition of the invariant zeros of (1).

Definition 1. (Tokarzewski, 1998; 2002a; 2002b; 2003): A number $\lambda \in \mathbb{C}$ is an *invariant zero* of (1) if and only if there exist vectors $0 \neq x^0 \in \mathbb{C}^n$ (state-zero direction) and $g \in \mathbb{C}^m$ (input-zero direction) such that the triple $\lambda,\ x^0,\ g$ satisfies

$$\begin{bmatrix} \lambda E - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x^0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2}$$

The system is called *degenerate* if it has an infinite number of invariant zeros.

The set of all invariant zeros of (1) will be denoted by

$$\mathbb{Z}^I = \Big\{ \lambda \in \mathbb{C}: \ \exists\, 0 \neq x^0 \in \mathbb{C}^n,\ \exists\, g \in \mathbb{C}^m, \\ P(\lambda) \begin{bmatrix} x^0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Big\} \tag{3}$$

and the set of all Smith zeros by

$$\mathbb{Z}^S := \big\{ \lambda \in \mathbb{C} : \operatorname{rank} P(\lambda) < \text{normal rank } P(s) \big\}. \tag{4}$$

Remark 1. In the system (1) the set $\mathbb{Z}^I$ has the same invariance properties as $\mathbb{Z}^S$, i.e., it is invariant under the following sets of transformations:

(i) nonsingular coordinate transformations in the state-space,

(ii) nonsingular transformations of the inputs or outputs, and

(iii) constant state or output feedback to the inputs.

This claim follows immediately from Definition 1. The proof is analogous to the proof of (Tokarzewski, 2002a, Lemma 2.3, p. 18) and for this reason it is omitted here.

2.2. Relationship between Invariant Zeros and Smith Zeros

The sets $\mathbb{Z}^S$ and $\mathbb{Z}^I$ are interrelated as follows.

Proposition 1. (Tokarzewski, 2002b)

(i) If $\lambda \in \mathbb{C}$ is a Smith zero of (1), then λ is an invariant zero of (1), i.e., $\mathbb{Z}^S \subseteq \mathbb{Z}^I$.

(ii) The system (1) is nondegenerate if and only if $\mathbb{Z}^S = \mathbb{Z}^I$.

(iii) The system (1) is degenerate if and only if $\mathbb{Z}^I = \mathbb{C}$.

Proof. A full proof of this result can be found in (Tokarzewski, 2002b). ■

Thus, each Smith zero is also an invariant zero. Moreover, $\mathbb{Z}^I$ may be equal to $\mathbb{Z}^S$ (then $\mathbb{Z}^I$ may be empty or finite) or $\mathbb{Z}^I$ may be equal to the whole complex plane. In this way, the set of the invariant zeros may be empty, finite or equal to $\mathbb{C}$, and when the system is nondegenerate, the sets of the Smith zeros and of the invariant zeros coincide. Of course, Proposition 1 tells us also that if in the system (1) there exists at least one invariant zero which is not a Smith zero, then the system is degenerate.

Corollary 1. *If the system matrix $P(s)$ corresponding to the system (1) (with $D \neq 0$ or $D = 0$) has full column normal rank, then the system is nondegenerate, i.e., $\mathbb{Z}^S = \mathbb{Z}^I$.*

Proof. In view of Proposition 1 it is enough to show that any invariant zero is also a Smith zero. However, from Definition 1 it follows that if $\lambda \in \mathbb{Z}^I$, then the columns of $P(\lambda)$ are linearly dependent over $\mathbb{C}$. Thus, we can write the relation

$$\operatorname{rank} P(\lambda) < \text{normal rank } P(s) = n + m,$$

which means that $\lambda \in \mathbb{Z}^S$. ■

Remark 2. Note that Corollary 1 tells us also that if the system (1) is degenerate, then $\text{normal rank } P(s) < n + m$.

Corollary 2. *In a square m-input m-output system (1) let the matrix $\left[\begin{smallmatrix} -B \\ D \end{smallmatrix}\right]$ have full column rank.*

Then

(a) $\lambda \in \mathbb{C}$ is an invariant zero of the system if and only if $\det P(\lambda) = 0$,

(b) the system is degenerate if and only if $\det P(s) \equiv 0$ (or equivalently, $\det G(s) \equiv 0$).

Proof. (a) Let $\det P(\lambda) = 0$. Then there exists a nonzero vector $\left[\begin{smallmatrix} x^0 \\ g \end{smallmatrix}\right]$ satisfying (2). Suppose that in this vector we have $x^0 = 0$. Then from (2) we get $\left[\begin{smallmatrix} -B \\ D \end{smallmatrix}\right] g = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, i.e., $g = 0$. This contradicts the assumption that $\left[\begin{smallmatrix} x^0 \\ g \end{smallmatrix}\right]$ is nonzero. Thus, $x^0 \neq 0$ and, consequently, λ is an invariant zero. Conversely, if λ is an invariant zero, i.e., a triple composed of λ, $x^0 \neq 0$ and g satisfies (2), then the columns of $P(\lambda)$ are linearly dependent and consequently, $\det P(\lambda) = 0$.

(b) Suppose that $\det P(s) \equiv 0$. Then $\det P(\lambda) = 0$ at any $\lambda \in \mathbb{C}$ and, consequently, in view of (a), the system is degenerate. In order to prove the converse, suppose that the system is degenerate and $\det P(s)$ is not identically zero. This means, however, that $\det P(s)$ is a nonzero polynomial in s, and in view of (a) the system cannot be degenerate (its invariant zeros are exactly the roots of $\det P(s)$). The remaining part of (b) follows from the relation $\det P(s) = \det(sE - A)\det G(s)$. ■

2.3. Fundamental Matrices, Markov Parameters and the Weierstrass Canonical Form

It is well known (Kaczorek, 1998; 2000) that for a regular pencil $sE - A$ with the index of nilpotency q there exist matrices Φ_i, $i = -q, -(q-1), \dots, -1, 0, 1, 2, \dots$, (called fundamental matrices) such that

$$(sE - A)^{-1} = \sum_{i=-q}^{\infty} \Phi_i s^{-(i+1)} \tag{5}$$

and

$$E\Phi_i - A\Phi_{i-1} = \Phi_i E - \Phi_{i-1} A = \begin{cases} I & \text{for} \quad i = 0, \\ 0 & \text{for} \quad i \neq 0. \end{cases} \tag{6}$$

The transfer function matrix for the system (1) can then be written in the form

$$\begin{aligned} G(s) &= D + C(sE - A)^{-1}B \\ &= D + \sum_{i=-q}^{\infty} C\Phi_i B s^{-(i+1)}, \end{aligned} \tag{7}$$

where the matrices D and $C\Phi_i B$ are called the Markov parameters for (1).

If a regular pencil $sE - A$ has an index of nilpotency q and $\deg\det(sE - A) = n_1$, then there exist nonsingular matrices P and Q such that (cf. the Weierstrass-Kronecker theorem (Dai, 1989; Kaczorek, 1999; 2000)):

$$P(sE - A)Q = \begin{bmatrix} sI_1 - A_1 & 0 \\ 0 & sN - I_2 \end{bmatrix}. \tag{8}$$

This is a Weierstrass canonical form of $sE - A$. Using (8) we can write

$$Q^{-1}(sE - A)^{-1}P^{-1} = \begin{bmatrix} (sI_1 - A_1)^{-1} & 0 \\ 0 & (sN - I_2)^{-1} \end{bmatrix}. \tag{9}$$

When the matrix $sE - A$ is taken in its Weierstrass canonical form (8), we get

$$\begin{aligned} (sE - A)^{-1} &= \begin{bmatrix} (sI_1 - A_1)^{-1} & 0 \\ 0 & (sN - I_2)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & (sN - I_2)^{-1} \end{bmatrix} \\ &\quad + \begin{bmatrix} (sI_1 - A_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{10}$$

and

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & I_2 \end{bmatrix}, \quad E = \begin{bmatrix} I_1 & 0 \\ 0 & N \end{bmatrix},$$

$$\begin{aligned} \Phi_{-q} &= -\begin{bmatrix} 0 & 0 \\ 0 & N^{q-1} \end{bmatrix}, \dots, \\ \Phi_{-k} &= -\begin{bmatrix} 0 & 0 \\ 0 & N^{k-1} \end{bmatrix}, \dots, \Phi_{-1} = -\begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix}, \end{aligned} \tag{11}$$

$$\Phi_0 = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Phi_1 = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Phi_2 = \begin{bmatrix} A_1^2 & 0 \\ 0 & 0 \end{bmatrix}, \dots, \quad \Phi_k = \begin{bmatrix} A_1^k & 0 \\ 0 & 0 \end{bmatrix}, \dots$$

(recall that $(sN - I_2)^{-1} = -s^{q-1}N^{q-1} - \dots - sN - I_2$ and $(sI_1 - A_1)^{-1} = \sum_{i=0}^{\infty} s^{-(i+1)} A_1^i$).

Remark 3. It is easy to check that the matrices Φ_i in (11) satisfy (6).

Remark 4. The transformation $P(sE - A)Q$ of a regular pencil $sE - A$, where P and Q are arbitrary $n \times n$ nonsingular matrices, does not change the Markov parameters of the system (1) (in consequence, also the transfer function matrix $G(s)$ of (1) remains unchanged).

In fact, under such a transformation the system (1) becomes a new system $S(E', A', B', C', D')$, where $x' = Q^{-1}x$ and $E' = PEQ$, $A' = PAQ$, $B' = PB$, $C' = CQ$, $D' = D$. Moreover, $\Phi_i' = Q^{-1}\Phi_i P^{-1}$ are fundamental matrices for $S(E', A', B', C', D')$ and $C'\Phi_i' B' = C\Phi_i B$.

Furthermore, from the relation

$$\begin{aligned} &\begin{bmatrix} P & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} sE - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I_m \end{bmatrix} \\ &\qquad = \begin{bmatrix} P(sE - A)Q & -PB \\ CQ & D \end{bmatrix} \end{aligned} \tag{12}$$

it is clear that the transformation considered changes neither the zero polynomial, nor the set of the invariant zeros (i.e., a triple $\lambda, x^0 \neq 0,\ g$ satisfies (2) for the system (1) if and only if the triple $\lambda, x'^0 = Q^{-1}x^0 \neq 0,\ g$ satisfies (2) for the system $S(E', A', B', C', D')$).

3. Sufficient and Necessary Condition for Nondegeneracy

Proposition 2. *The system (1) (with $D = 0$ or $D \neq 0$) is degenerate if and only if*

$$\text{normal rank}\, P(s) < n + \text{rank} \begin{bmatrix} -B \\ D \end{bmatrix}. \tag{13}$$

Proof. Let

$$\text{rank} \begin{bmatrix} -B \\ D \end{bmatrix} = m' \quad (m' \leq m).$$

(i) Suppose first that $m' = m$ (i.e., $\left[\begin{smallmatrix} -B \\ D \end{smallmatrix}\right]$ has full column rank). Then necessity of the condition (13) follows from Remark 2.

Conversely, suppose that (13) holds. Then for any $\lambda \in \mathbb{C}$ we have

$$\text{rank}\, P(\lambda) \leq \text{normal rank}\, P(s) < n + m. \tag{14}$$

From (14) and from the assumption

$$\text{rank} \begin{bmatrix} -B \\ D \end{bmatrix} = m$$

it follows that at any given complex number λ the equation

$$P(\lambda) \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{15}$$

with $n+m$ unknowns has a solution $\left[\begin{smallmatrix} x^0 \\ g \end{smallmatrix}\right]$ with $x^0 \neq 0$. This means that the system is degenerate.

(ii) Suppose now that $m' < m$ and assume (without loss of generality) that the first m' columns of $\left[\begin{smallmatrix} -B \\ D \end{smallmatrix}\right]$ are linearly independent. The submatrix of $\left[\begin{smallmatrix} -B \\ D \end{smallmatrix}\right]$ composed of these columns is denoted by $\left[\begin{smallmatrix} -B' \\ D' \end{smallmatrix}\right]$, i.e.,

$$\begin{bmatrix} -B \\ D \end{bmatrix} = \begin{bmatrix} -B' & -B'' \\ D' & D'' \end{bmatrix}$$

and

$$\text{rank} \begin{bmatrix} -B \\ D \end{bmatrix} = \text{rank} \begin{bmatrix} -B' \\ D' \end{bmatrix} = m'.$$

Consider the system $S(E, A, B', C, D')$ and its system matrix

$$P'(s) = \begin{bmatrix} sE - A & -B' \\ C & D' \end{bmatrix}.$$

The sets of the invariant zeros for the systems $S(E, A, B, C, D)$ (1) and $S(E, A, B', C, D')$ coincide, i.e.,

$$\mathbb{Z}^I_{S(E,A,B,C,D)} = \mathbb{Z}^I_{S(E,A,B',C,D')}. \tag{16}$$

The proof of (16) follows from the definition of the invariant zeros and from the relation

$$\text{Im} \begin{bmatrix} -B \\ D \end{bmatrix} = \text{Im} \begin{bmatrix} -B' \\ D' \end{bmatrix},$$

where $\text{Im}\, M$ denotes the subspace spanned by the columns of M. We are to show that $\lambda \in \mathbb{Z}^I_{S(E,A,B,C,D)}$ if and only if $\lambda \in \mathbb{Z}^I_{S(E,A,B',C,D')}$. Suppose first that $\lambda \in \mathbb{Z}^I_{S(E,A,B,C,D)}$, i.e., via Definition 1, there exist $x^0 \neq 0$ and $g \in \mathbb{C}^m$ such that $\lambda E x^0 - A x^0 = Bg$ and $Cx^0 + Dg = 0$. Since

$$\text{Im} \begin{bmatrix} -B \\ D \end{bmatrix} = \text{Im} \begin{bmatrix} -B' \\ D' \end{bmatrix},$$

we can find a $g' \in \mathbb{C}^{m'}$ such that

$$\begin{bmatrix} -B \\ D \end{bmatrix} g = \begin{bmatrix} -B' \\ D' \end{bmatrix} g'.$$

Consequently, at the same λ and x^0 we get the relations $\lambda E x^0 - A x^0 = B'g'$ and $Cx^0 + D'g' = 0$, i.e., $\lambda \in \mathbb{Z}^I_{S(E,A,B',C,D')}$. The proof of the converse implication proceeds along the same lines.

For $P(s)$ and $P'(s)$ the following relations hold:

$$\text{normal rank}\, P(s) = \text{normal rank}\, P'(s),$$

and

$$\text{rank}\, P(\lambda) = \text{rank}\, P'(\lambda) \quad \text{for any } \lambda \in \mathbb{C}.$$

Now, from the first part of the proof which considers the system $S(E, A, B', C, D')$ it follows that $S(E, A, B', C, D')$ is degenerate if and only if normal rank $P'(s) < n + m'$.

Finally, the following sequence of equivalent conditions holds: The system $S(E, A, B, C, D)$ in (1) is degenerate $\Leftrightarrow$ $S(E, A, B', C, D')$ is degenerate $\Leftrightarrow$ $\text{normal rank}\, P(s) = \text{normal rank}\, P'(s) < n + m'$. ∎

Corollary 3. *If in the system (1) (with $D = 0$ or $D \neq 0$) we have $r < \text{rank} \left[\begin{smallmatrix} -B \\ D \end{smallmatrix}\right]$, then the system is degenerate.*

Proof. The claim follows from Proposition 2 and from the following relations:

$$\text{normal rank}\, P(s) \le \min\left\{n + \text{rank}\begin{bmatrix}-B\\ D\end{bmatrix}, n+r\right\}$$

$$< n + \text{rank}\begin{bmatrix}-B\\ D\end{bmatrix}.$$

■

The desired criterion of the nondegeneracy of the system (1) takes the following form:

Proposition 3. *The system $S(E,A,B,C,D)$ in (1) (with $D=0$ or $D\neq 0$) is nondegenerate if and only if*

$$\text{normal rank}\, P(s) = n + \text{rank}\begin{bmatrix}-B\\ D\end{bmatrix}.$$

Proof. The claim follows from Proposition 2 and from the fact that the normal rank of $P(s)$ cannot be greater than $n + \text{rank}\left[\begin{smallmatrix}-B\\ D\end{smallmatrix}\right]$. ■

4. Characterization of Invariant Zeros via the First Nonzero Markov Parameter

In this section we consider the system (1) in its Weierstrass canonical form (moreover, we assume $D=0$) (a suitable procedure for finding a Weierstrass canonical form of (1) can be found in (Kaczorek, 2000, p. 332)):

$$\begin{bmatrix}I_1 & 0\\ 0 & N\end{bmatrix}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}A_1 & 0\\ 0 & I_2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}B_1\\ B_2\end{bmatrix}u(t), \tag{1'}$$

$$y(t) = \begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix},$$

i.e., the matrices E and A and the fundamental matrices are as in (11) and $B = \left[\begin{smallmatrix}B_1\\ B_2\end{smallmatrix}\right]$, $C = [C_1 \quad C_2\ ]$ (the system (1') may be viewed as a parallel connection of the subsystems $S_1(A_1, B_1, C_1)$ and $S_2(N, I_2, B_2, C_2)$).

Moreover, we assume that the first nonzero Markov parameter for (1') has a negative index i (see (7)) and we denote this parameter by $C\Phi_{-k}B$, $1 \le k \le q$, i.e.,

$$C\Phi_{-q}B = C\Phi_{-(q-1)}B = \cdots = C\Phi_{-(k+1)}B = 0, \qquad C\Phi_{-k}B \neq 0, \tag{17}$$

and $\text{rank}\, C\Phi_{-k}B = p \le \min\{m, r\}$.

Note that using (7), (11) and (17), we can write the transfer function matrix for (1') as

$$G(s) = C(sE-A)^{-1}B = -C_2N^{k-1}B_2s^{k-1} - \dots$$
$$- C_2NB_2s - C_2B_2 + C_1(sI_1 - A_1)^{-1}B_1, \tag{18}$$

i.e., $C\Phi_{-k}B = -C_2N^{k-1}B_2$.

Define the $n\times n$ matrix

$$K_{-k} := I - B(C\Phi_{-k}B)^{+}C\Phi_{-k}, \tag{19}$$

where '$+$' means the operation of taking the Moore-Penrose pseudoinverse (Ben-Israel and Greville, 2002; Gantmacher, 1988). Recall (Ben-Israel and Greville, 2002; Gantmacher, 1988) that if the matrices H_1 and H_2, where H_1 is $r\times p$ and H_2 is $p\times m$, give a skeleton factorization of $C\Phi_{-k}B$, i.e., $C\Phi_{-k}B = H_1H_2$, then $(C\Phi_{-k}B)^{+} = H_2^{+}H_1^{+}$, where $H_1^{+} = (H_1^TH_1)^{-1}H_1^T$ and $H_2^{+} = H_2^T(H_2H_2^T)^{-1}$.

Lemma 1. *The matrix K_{-k} in (19) has the following properties:*

(i) $K_{-k}^2 = K_{-k}$,

(ii) $\Sigma_{-k} := \{x : K_{-k}x = x\} = \text{Ker}(H_1^TC\Phi_{-k})$, $\dim \Sigma_{-k} = n - p$,

(iii) $\Omega_{-k} := \{x : K_{-k}x = 0\} = \text{Im}(BH_2^T)$, $\dim\Omega_{-k} = p$,

(iv) $\mathbb{C}^n(\mathbb{R}^n) = \Sigma_{-k} \oplus \Omega_{-k}$,

(v) $K_{-k}BH_2^T = 0, \quad H_1^TC\Phi_{-k}K_{-k} = 0.$

Proof. Set $C' = H_1^TC$ and $B' = BH_2^T$. Note that the $p\times p$ matrix $C'\Phi_{-k}B' = H_1^TH_1H_2H_2^T$ is nonsingular. Define $K'_{-k} := I - B'(C'\Phi_{-k}B')^{-1}C'\Phi_{-k}$. Then $K'_{-k} = K_{-k}$. In fact, it is enough to observe that

$$\begin{aligned}
&B'(C'\Phi_{-k}B')^{-1}C'\Phi_{-k}\\
&\quad = BH_2^T(H_1^TC\Phi_{-k}BH_2^T)^{-1}H_1^TC\Phi_{-k}\\
&\quad = BH_2^T(H_1^TH_1H_2H_2^T)^{-1}H_1^TC\Phi_{-k}\\
&\quad = BH_2^T(H_2H_2^T)^{-1}(H_1^TH_1)^{-1}H_1^TC\Phi_{-k}\\
&\quad = BH_2^{+}H_1^{+}C\Phi_{-k} = B(C\Phi_{-k}B)^{+}C\Phi_{-k}.
\end{aligned}$$

The remaining part of the proof proceeds for K'_{-k}. It follows the same lines as the proof of (Tokarzewski, 2002a, Lemma 3.1, p. 42) and for this reason is omitted here. ■

Remark 5. Using (11) and (19), the matrix K_{-k} can be written in the form

$$K_{-k} = \begin{bmatrix}I_1 & -B_1(C_2N^{k-1}B_2)^{+}C_2N^{k-1}\\ 0 & I_2 - B_2(C_2N^{k-1}B_2)^{+}C_2N^{k-1}\end{bmatrix}, \tag{20}$$

where $K_{-k,2} := I_2 - B_2(C_2N^{k-1}B_2)^+C_2N^{k-1}$ is projective (idempotent).

Lemma 2. *If in the system (1') a triple* $\lambda, x^0 \neq 0$, g *satisfies (2), then*

$$\begin{aligned} C\Phi_{-q}x^0 &= 0 \\ &\vdots \\ C\Phi_{-(k+1)}x^0 &= 0, \\ Cx_0 &= 0, \end{aligned} \tag{21}$$

and

$$C\Phi_{-k}Bg = -C\Phi_{-k}x^0.$$

Moreover,

$$K_{-k}Ex^0 = Ex^0 \tag{22}$$

Proof. The equality $\lambda Ex^0 - Ax^0 = Bg$ is multiplied successively from the left by $C\Phi_{-q}, \dots, C\Phi_{-(k+1)}$, and we use the relations $\Phi_{-l}E = \Phi_{-(l+1)}$ and $\Phi_{-l}A = \Phi_{-l}$ (cf. (11)) as well as (17). In this way we get $C\Phi_{-q}x^0 = 0, \dots, C\Phi_{-(k+1)}x^0 = 0$. Premultiplying $\lambda Ex^0 - Ax^0 = Bg$ by $C\Phi_{-k}$ we get $C\Phi_{-k}Bg = -C\Phi_{-k}x^0$. Finally, (22) follows from (19) and from the relations $\Phi_{-k}E = \Phi_{-(k+1)}$ and $C\Phi_{-(k+1)}x^0 = 0$. ■

Lemma 3. *If in the system (1') a triple* $\lambda, x^0 \neq 0$, g *satisfies (2), then*

(i) $\lambda Ex^0 - K_{-k}Ax^0 = Bg_1, \quad K_{-k}Ax^0 - Ax^0 = Bg_2,$ $Cx^0 = 0,$

where $g = g_1 + g_2$, $g_1 \in \operatorname{Ker}(C\Phi_{-k}B)$, $g_2 \in \operatorname{Im}(C\Phi_{-k}B)^T$ *and* g_1, g_2 *are uniquely determined by* g. *Moreover,*

(ii) $Bg_1 \in \Sigma_{-k}, \quad Bg_2 \in \Omega_{-k}$ *and* $g_2 = -(C\Phi_{-k}B)^+C\Phi_{-k}x^0$.

Proof. Let $g = g_1 + g_2$ with g_1, g_2 defined as $g_1 := (I_m - (C\Phi_{-k}B)^+C\Phi_{-k}B)g$ and $g_2 := (C\Phi_{-k}B)^+C\Phi_{-k}Bg$. Then $Bg_1 = K_{-k}Bg$ and $Bg_2 = (I - K_{-k})Bg$. Thus, $K_{-k}Bg_1 = Bg_1$ and $K_{-k}Bg_2 = 0$ (i.e., $Bg_1 \in \Xi_{-k}$ and $Bg_2 \in \Omega_{-k}$). Now, the equality $\lambda Ex^0 - Ax^0 = Bg$ may be written as

(iii) $(\lambda E - K_{-k}A)x^0 + (K_{-k} - I)Ax^0 = Bg_1 + Bg_2$

with the vectors $(\lambda E - K_{-k}A)x^0$ and Bg_1 in Σ_{-k} and $(K_{-k} - I)Ax^0$ and Bg_2 in Ω_{-k}. Note that, in view of (22), we have $K_{-k}(\lambda Ex^0 - K_{-k}Ax^0) = (\lambda Ex^0 - K_{-k}Ax^0)$. Moreover, $K_{-k}(K_{-k} - I)Ax^0 = 0$. Now, from Lemma 1 (iv) it follows that the decomposition (iii) is unique. This proves the first two equalities in (i). The expression for g_2 in (ii) follows from the definition of g_2 and from the relation $C\Phi_{-k}Bg = -C\Phi_{-k}x^0$ in (21). Finally, the relation $C\Phi_{-k}Bg_1 = 0$ follows from the definition of g_1. ■

Remark 6. The pencil $sE - K_{-k}A$ is not regular, i.e., $\det(sE - K_{-k}A) \equiv 0$. We can verify this claim by using the relation $K_k = K'_{-k}$ (see the proof of Lemma 1).

$$\begin{aligned} &\det(sE - K_{-k}A) \\ &\quad = \det(sE - K'_{-k}A) \\ &\quad = \det\big((sE - A) + B'(C'\Phi_{-k}B')^{-1}C'\Phi_{-k}\big) \\ &\quad = \det(sE - A) \\ &\qquad \times \det\big[I_n + (sE - A)^{-1}B'(C'\Phi_{-k}B')^{-1}C'\Phi_{-k}\big] \\ &\quad = \det(sE - A) \\ &\qquad \times \det\big[I_p + C'\Phi_{-k}(sE - A)^{-1}B'(C'\Phi_{-k}B')^{-1}\big]. \end{aligned}$$

Now we show the equality $C'\Phi_{-k}(sE - A)^{-1}B' = -C'\Phi_{-k}B'$, which will give the desired result. For this purpose observe first that $\Phi_{-k}\Phi_i = 0$ for all $i \geq 0$ (see (11)) and $\Phi_{-k}\Phi_i = -\Phi_{-(k-i-1)}$ for $i = -q, \dots, -1$ (in particular, $\Phi_{-k}\Phi_{-1} = -\Phi_{-k}$ and $\Phi_{-k}\Phi_{-2} = -\Phi_{-(k+1)}$). Thus, we can write

$$\begin{aligned} \Phi_{-k}(sE - A)^{-1} &= \sum_{i=-q}^{-1} \Phi_{-k}\Phi_i s^{-(i+1)} \\ &= \Phi_{-k}\Phi_{-q}s^{q-1} + \dots + \Phi_{-k}\Phi_{-2}s \\ &\quad + \Phi_{-k}\Phi_{-1}. \end{aligned}$$

Premultiply the right-hand side of the above relation by C' and postmultiply the result by B'. Now, in view of the relation $\Phi_{-i} = 0$ for all $i \geq q + 1$ and the assumption $C'\Phi_{-q}B' = \dots = C'\Phi_{-(k+1)}B' = 0$, we get the desired equality.

Finally, note that (cf. (19))

$$K_{-k}A = A + BF,$$

where $F = -(C\Phi_{-k}B)^+C\Phi_{-k}$ (since $\Phi_{-k}A = \Phi_{-k}$).

4.1. First Nonzero Markov Parameter of a Full Column Rank

Lemma 4. *If in the system (1') the first nonzero Markov parameter* $C\Phi_{-k}B$ *has full column rank, then so does the system matrix* $P(s)$ *of (1').*

Proof. We consider separately two cases.

In the first case we assume that (1') is square ($m = r$) and the $m \times m$ matrix $C\Phi_{-k}B = -C_2N^{k-1}B_2$ is nonsingular. Since $\det P(s) = \det(sE - A)\det G(s)$, we only need to show that $\det G(s) \neq 0$. Using (18) we can write

$$G(s) = -C_2N^{k-1}B_2s^{k-1}\big(I_m + H(s)\big),$$

where

$$\begin{aligned} H(s) &= (C_2N^{k-1}B_2)^{-1}C_2N^{k-2}B_2s^{-1} \\ &\quad + \cdots + (C_2N^{k-1}B_2)^{-1}C_2B_2s^{-(k-1)} \\ &\quad - (C_2N^{k-1}B_2)^{-1}s^{-(k-1)}C_1(sI_1 - A_1)^{-1}B_1 \end{aligned}$$

and $\lim_{s\to\infty} H(s) = 0$. Thus, $\det(I_m + H(s)) \neq 0$ and, consequently, $\det G(s) \neq 0$, i.e., $P(s)$ is invertible.

In the second case it is assumed that $m < r$ and the $r \times m$ matrix $C\Phi_{-k}B = -C_2N^{k-1}B_2$ has the full column rank m. To $C\Phi_{-k}B$ we apply the singular value decomposition (SVD) (Callier and Desoer, 1982, pp. 2–10):

$$C\Phi_{-k}B = U\Lambda V^T,$$

where the $r \times r$ matrix U and the $m \times m$ matrix V are orthogonal and $\Lambda = \left[\begin{smallmatrix} M_m \\ 0 \end{smallmatrix}\right]$ with an $m \times m$ diagonal and nonsingular matrix M_m. Set $\bar{B} = BV = \bar{B}_m$ and $\bar{C} = U^TC = \left[\begin{smallmatrix} \bar{C}_m \\ \bar{C}_{r-m} \end{smallmatrix}\right]$, where $\bar{C}_m$ consists of the first m rows of $\bar{C}$, and observe that $M_m = \bar{C}_m\Phi_{-k}\bar{B}_m$. Now we can write

$$\begin{aligned} \bar{P}(s) &= \begin{bmatrix} sE - A & -\bar{B} \\ \bar{C} & 0 \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & U^T \end{bmatrix} \begin{bmatrix} sE - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & V \end{bmatrix}. \end{aligned}$$

On the other side, $\bar{P}(s)$ can be written as

$$\bar{P}(s) = \begin{bmatrix} sE - A & -\bar{B}_m \\ \bar{C}_m & 0 \\ \bar{C}_{r-m} & 0 \end{bmatrix},$$

where

$$\bar{P}'(s) = \begin{bmatrix} sE - A & -\bar{B}_m \\ \bar{C}_m & 0 \end{bmatrix}$$

is square.

In order to show that $P(s)$ has full column normal rank, it is enough to observe that $\det \bar{P}'(s)$ is a nonzero polynomial. For this purpose we consider the square system $\bar{S}'(E, A, \bar{B}_m, \bar{C}_m)$ in which the first nonzero Markov parameter $M_m = \bar{C}_m\Phi_{-k}\bar{B}_m$ is nonsingular. Decomposing $\bar{C}_m = [\bar{C}_{m,1} \quad \bar{C}_{m,2}]$ with an $m \times n_1$ matrix $\bar{C}_{m,1}$ and an $m \times n_2$ matrix $\bar{C}_{m,2}$ as well as $\bar{B}_m = \left[\begin{smallmatrix} \bar{B}_{m,1} \\ \bar{B}_{m,2} \end{smallmatrix}\right]$ with an $n_1 \times m$ matrix $\bar{B}_{m,1}$ and an $n_2 \times m$ matrix $\bar{B}_{m,2}$, we get $M_m = \bar{C}_m\Phi_{-k}\bar{B}_m = -\bar{C}_{m,2}N^{k-1}\bar{B}_{m,2}$. For the transfer function matrix of $\bar{S}'(E, A, \bar{B}_m, \bar{C}_m)$ we now have

$$\begin{aligned} \bar{G}'(s) &= \bar{C}_m(sE - A)^{-1}\bar{B}_m \\ &= -\bar{C}_{m,2}N^{k-1}\bar{B}_{m,2}s^{k-1} \\ &\quad - \cdots - \bar{C}_{m,2}\bar{B}_{m,2} + \bar{C}_{m,1}(sI_1 - A_1)^{-1}\bar{B}_{m,1}. \end{aligned}$$

Proceeding analogously as in the first case, we get $\det \bar{G}'(s) \neq 0$ and, consequently, $\det \bar{P}'(s) = \det(sE - A) \det \bar{G}'(s) \neq 0$. ■

Proposition 4. *If in the system (1') the first nonzero Markov parameter $C\Phi_{-k}B = -C_2N^{k-1}B_2$ has full column rank, then the system is nondegenerate, i.e., $\mathbb{Z}^S = \mathbb{Z}^I$. Moreover, $\lambda \in \mathbb{C}$ is an invariant zero of the system if and only if there exists $x^0 \neq 0$ such that*

$$\begin{bmatrix} \lambda E - K_{-k}A \\ C \end{bmatrix} x^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{23}$$

Proof. The first claim follows directly from Corollary 1 and Lemma 4. The proof of the second claim is as follows:

(⇐) If (23) is satisfied for some $\lambda \in \mathbb{C}$ and $x^0 \neq 0$, then taking into account the definition of K_{-k} (19) and setting $g = -(C\Phi_{-k}B)^+C\Phi_{-k}x^0$, we can transform (23) into the form of (2).

(⇒) From Lemma 3 it follows that if $C\Phi_{-k}B$ has full column rank, then $g_1 = 0$ and, consequently, $\lambda Ex^0 - K_{-k}Ax^0 = 0,\ Cx^0 = 0$. ■

Remark 7. If in (1') the matrix $C\Phi_{-k}B$ has full column rank, then the pencil $\left[\begin{smallmatrix} sE - K_{-k}A \\ C \end{smallmatrix}\right]$ has the full column normal rank n. In fact, suppose that

$$\text{normal rank} \begin{bmatrix} sE - K_{-k}A \\ C \end{bmatrix} = \rho < n.$$

This means that at any fixed $\lambda \in \mathbb{C}$ we have

$$\text{rank} \begin{bmatrix} \lambda E - K_{-k}A \\ C \end{bmatrix} \leq \rho < n,$$

i.e., the columns of $\left[\begin{smallmatrix} \lambda E - K_{-k}A \\ C \end{smallmatrix}\right]$ are linearly dependent (over $\mathbb{C}$). In consequence, there exists a vector $x^0 \neq 0$ such that (23) holds. Thus the system is degenerate. This, however, contradicts Proposition 4.

From the above and from Proposition 4 we infer that if in the system (1') the first nonzero Markov parameter $C\Phi_{-k}B$ has full column rank, then the invariant zeros of the system are exactly those points of the complex plane where the pencil $\left[\begin{smallmatrix} sE - K_{-k}A \\ C \end{smallmatrix}\right]$ loses its normal column rank n.

4.2. SVD of the First Nonzero Markov Parameter

In this subsection we apply SVD to the first nonzero Markov parameter of $S(E, A, B, C)$ in (1') (see (17)), i.e., we write (recall that $0 < \operatorname{rank} C\Phi_{-k}B = p \leq \min\{m, r\}$):

$$C\Phi_{-k}B = U\Lambda V^T, \tag{24}$$

where

$$\Lambda = \begin{bmatrix} M_p & 0 \\ 0 & 0 \end{bmatrix}$$

is $r \times m$-dimensional, M_p is a $p \times p$ diagonal matrix with positive singular values of $C\Phi_{-k}B$ and U and V are $r \times r$ and $m \times m$ orthogonal matrices, respectively, (i.e., $U^TU = I_r = UU^T, V^TV = I_m = VV^T$). Introducing the matrices V and U^T to the system $S(E, A, B, C)$ as a pre- and a postcompensator, respectively, we obtain an auxiliary system $S(E, A, \bar{B}, \bar{C})$ of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + \bar{B}\bar{u}(t), \\ \bar{y}(t) &= \bar{C}x(t), \end{aligned} \tag{25}$$

where

$$\bar{B} = BV, \quad \bar{C} = U^TC$$

and

$$\bar{u} = V^Tu, \quad \bar{y} = U^Ty \tag{26}$$

are decomposed as follows:

$$\begin{aligned} \bar{B} &= \begin{bmatrix} \bar{B}_p & \bar{B}_{m-p} \end{bmatrix}, & \bar{C} &= \begin{bmatrix} \bar{C}_p \\ \bar{C}_{r-p} \end{bmatrix}, \\ \bar{u} &= \begin{bmatrix} \bar{u}_p \\ \bar{u}_{m-p} \end{bmatrix}, & \bar{y} &= \begin{bmatrix} \bar{y}_p \\ \bar{y}_{r-p} \end{bmatrix}, \end{aligned} \tag{27}$$

and $\bar{B}_p$ consists of the first p columns of $\bar{B}$, while $\bar{C}_p$ consists of the first p rows of $\bar{C}$. Similarly, $\bar{u}_p$ consists of the first p rows of vector $\bar{u}$ and $\bar{y}_p$ consists of the first p components of vector $\bar{y}$.

It is clear (cf. (17), (24), (26) and (27)) that $\bar{C}\Phi_{-k}\bar{B}$ is the first nonzero Markov parameter for the system (25), as well as that

$$\bar{C}\Phi_{-k}\bar{B} = \begin{bmatrix} \bar{C}_p\Phi_{-k}\bar{B}_p & \bar{C}_p\Phi_{-k}\bar{B}_{m-p} \\ \bar{C}_{r-p}\Phi_{-k}\bar{B}_p & \bar{C}_{r-p}\Phi_{-k}\bar{B}_{m-p} \end{bmatrix} = \begin{bmatrix} M_p & 0 \\ 0 & 0 \end{bmatrix}, \tag{28}$$

i.e.,

$$\begin{aligned} \bar{C}_p\Phi_{-k}\bar{B}_p &= M_p, & \bar{C}_p\Phi_{-k}\bar{B}_{m-p} &= 0, \\ \bar{C}_{r-p}\Phi_{-k}\bar{B}_p &= 0, & \bar{C}_{r-p}\Phi_{-k}\bar{B}_{m-p} &= 0. \end{aligned} \tag{29}$$

Lemma 5. *The sets of the invariant zeros of the systems $S(E, A, B, C)$ in (1') and $S(E, A, \bar{B}, \bar{C})$ in (25) coincide.*

Proof. The claim follows directly from Remark 1(ii). ■

For the system $S(E, A, \bar{B}, \bar{C})$ in (25) we form the projection matrix

$$\bar{K}_{-k} := I - \bar{B}(\bar{C}\Phi_{-k}\bar{B})^+\bar{C}\Phi_{-k} \tag{30}$$

which, in view of (24) and (27), can be evaluated as

$$\begin{aligned} \bar{K}_{-k} &= I - \begin{bmatrix} \bar{B}_p & \bar{B}_{m-p} \end{bmatrix} \begin{bmatrix} M_p & 0 \\ 0 & 0 \end{bmatrix}^+ \begin{bmatrix} \bar{C}_p \\ \bar{C}_{r-p} \end{bmatrix} \Phi_{-k} \\ &= I - \bar{B}_pM_p^{-1}\bar{C}_p\Phi_{-k}. \end{aligned} \tag{31}$$

Remark 8. The matrices K_{-k} in (19) and $\bar{K}_{-k}$ in (30) satisfy the relation $K_{-k} = \bar{K}_{-k}$. From (24) it follows that $(C\Phi_{-k}B)^+ = V\Lambda^+U^T$ (Ben-Israel and Greville, 2002). Moreover, from (24) and (26) we have $\bar{C}\Phi_{-k}\bar{B} = \Lambda$. Now, we can write

$$\begin{aligned} \bar{K}_{-k} &= I - \bar{B}(\bar{C}\Phi_{-k}\bar{B})^+\bar{C}\Phi_{-k} \\ &= I - BV\Lambda^+U^TC\Phi_{-k} \\ &= I - B(C\Phi_{-k}B)^+C\Phi_{-k} = K_{-k}. \end{aligned} \tag{32}$$

The relations (31) and (29) imply

$$\bar{K}_{-k}\bar{B}_p = 0, \quad \bar{K}_{-k}\bar{B}_{m-p} = \bar{B}_{m-p}. \tag{33}$$

Lemma 6. *Suppose that the system $S(E, A, B, C)$ in (1') is such that in the corresponding system $S(E, A, \bar{B}, \bar{C})$ in (25) is $\bar{B}_{m-p} = 0$. Then the following sets of the invariant zeros (for appropriate systems) coincide:*

$$\mathbb{Z}^I_{S(E,A,B,C)} = \mathbb{Z}^I_{S(E,A,\bar{B},\bar{C})} = \mathbb{Z}^I_{S(E,A,\bar{B}_p,\bar{C})}, \tag{34}$$

where $S(E, A, \bar{B}_p, \bar{C})$ is obtained from $S(E, A, \bar{B}, \bar{C})$ by neglecting the input $\bar{u}_{m-p}$.

Proof. The system (25) has the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + \bar{B}_p\bar{u}_p(t) + \bar{B}_{m-p}\bar{u}_{m-p}(t), \\ \bar{y}(t) &= \bar{C}x(t). \end{aligned} \tag{35}$$

When $\bar{B}_{m-p} = 0$ and a triple $\lambda, x^0 \neq 0$, $\bar{g} = \begin{bmatrix} \bar{g}_p \\ \bar{g}_{m-p} \end{bmatrix}$ satisfies (2) (when applied to the system (25)), the triple $\lambda, x^0 \neq 0$, $\bar{g}_p$ satisfies (2) when applied to the system $S(E, A, \bar{B}_p, \bar{C})$ of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + \bar{B}_p\bar{u}_p(t), \\ \bar{y}(t) &= \bar{C}x(t). \end{aligned} \tag{36}$$

In this way we have shown that if λ is an invariant zero of $S(E, A, \bar{B}, \bar{C})$, then this λ is also an invariant zero of $S(E, A, \bar{B}_p, \bar{C})$. Conversely, if a triple $\lambda, x^0 \neq 0$, $\bar{g}_p$ satisfies (2) (when applied to the system $S(E, A, \bar{B}_p, \bar{C})$) then the triple $\lambda, x^0 \neq 0$, $\bar{g} = \begin{bmatrix} \bar{g}_p \\ \bar{g}_{m-p} \end{bmatrix}$, where $\bar{g}_{m-p}$ is arbitrary (since in (35) we have $\bar{B}_{m-p} = 0$), represents an invariant zero λ for (25). The first equality in (34) follows from Lemma 5. ■

Proposition 5. *Suppose that the system* $S(E,A,B,C)$ *in (1') is such that in the system* $S(E,A,\bar{B},\bar{C})$ *in (25) corresponding to it is* $\bar{B}_{m-p}=0$. *Then the system (1') is nondegenerate. Moreover,* $\lambda\in\mathbb{C}$ *is an invariant zero of (1') if and only if there exists an* $x^0\neq 0$ *such that*

$$\begin{bmatrix}\lambda E-\bar{K}_{-k}A\\ \bar{C}\end{bmatrix}x^0=\begin{bmatrix}0\\0\end{bmatrix}. \tag{37}$$

Proof. In view of Lemma 6, we can consider invariant zeros of the system $S(E,A,\bar{B}_p,\bar{C})$. The first nonzero Markov parameter in $S(E,A,\bar{B}_p,\bar{C})$ is equal to $\bar{C}\Phi_{-k}\bar{B}_p=\left[\begin{smallmatrix}M_p\\0\end{smallmatrix}\right]$ and it has full column rank. Now, the nondegeneracy of the system (1') follows from Proposition 4 (when applied to the system $S(E,A,\bar{B}_p,\bar{C})$) as well as from Lemma 6. The proof of the second claim follows the same lines as the proof of the second claim in Proposition 4 (when applied to the system $S(E,A,\bar{B}_p,\bar{C})$). ■

Remark 9. Under the assumptions of Proposition 5, the pencil $\left[\begin{smallmatrix}sE-\bar{K}_{-k}A\\ \bar{C}\end{smallmatrix}\right]$ has the full column normal rank n. The proof of this claim is analogous to that given in Remark 7. Thus, under the assumption of Proposition 5, the invariant zeros of the system (1') are those points of the complex plane where this pencil loses its full column normal rank.

5. Examples

Example 1. In the system (1), let

$$E=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\quad A=\begin{bmatrix}-1&0&-3\\0&-2&0\\0&0&-3\end{bmatrix},$$

$$B=\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix},\quad C=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}.$$

The Smith form of $P(s)$ is

$$\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&0\end{bmatrix},$$

i.e., the system has no Smith zeros.

Since the condition (13) in Proposition 2 is fulfilled, the system in question is degenerate. For instance, at any given $\omega\neq 0$ the triple

$$\lambda=j\omega,\quad x^0=\begin{bmatrix}0\\0\\1\end{bmatrix},\quad g=\begin{bmatrix}1\\0\\2\end{bmatrix}$$

satisfies (2) and generates the following output-zeroing inputs. The input

$$u(t)=\begin{bmatrix}\cos\omega t\\0\\2\cos\omega t\end{bmatrix}$$

applied to the system subject to the initial condition

$$x(0)=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

yields

$$x(t)=\begin{bmatrix}0\\0\\\cos\omega t\end{bmatrix}$$

and $y(t)\equiv 0$. The input

$$u(t)=\begin{bmatrix}\sin\omega t\\0\\2\sin\omega t\end{bmatrix}$$

applied to the system subject to the initial condition

$$x(0)=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

yields

$$x(t)=\begin{bmatrix}0\\0\\\sin\omega t\end{bmatrix}$$

and $y(t)\equiv 0$. ♦

Example 2. Consider the system (1') with the matrices

$$E=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\quad A=\begin{bmatrix}-1&0&0\\0&-2&0\\0&0&1\end{bmatrix},$$
$$B=\begin{bmatrix}1&0\\0&1\\1&0\end{bmatrix},\quad C=\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}. \tag{38}$$

$$N=0,\quad q=1,\quad A_1=\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix},$$

$$\Phi_{-1}=-\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}.$$

The first nonzero Markov parameter is

$$C\Phi_{-1}B=\begin{bmatrix}-1&0\\0&0\end{bmatrix},$$

$\operatorname{rank} C\Phi_{-1}B=p=1$. In (24) we take $U=-I_2$, $V=I_2$ and

$$\Lambda=\begin{bmatrix}1&0\\0&0\end{bmatrix}.$$

In (27) we have $\bar{B}=B$ and $\bar{C}=-C$. Moreover,

$$\bar{B}_{m-p}=\begin{bmatrix}0\\1\\0\end{bmatrix}.$$

On ther other hand, for the system (38) we have $\det P(s)=s$ and, by virtue of Corollary 2, the system is nondegenerate and it has exactly one invariant zero $\lambda=0$.

This example shows that in Proposition 5 the condition $\bar{B}_{m-p}=0$ is merely a sufficient condition of nondegeneracy. ♦

Example 3. Consider the system (1') with the matrices

$$A_1=\begin{bmatrix}2&-1&0\\0&0&0\\-1&0&0\end{bmatrix},\quad B_1=\begin{bmatrix}0\\0\\1\end{bmatrix},$$

$$C_1=\begin{bmatrix}0&-1&0\end{bmatrix},$$

$$N=\begin{bmatrix}0&1&1\\0&0&1\\0&0&0\end{bmatrix},\qquad B_2=\begin{bmatrix}1\\1\\1\end{bmatrix},$$

$$C_2=\begin{bmatrix}1&0&1\end{bmatrix},\quad q=3.$$

The first nonzero Markov parameter is

$$C\Phi_{-q}B=-\begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}0&0\\0&N^2\end{bmatrix}\begin{bmatrix}B_1\\B_2\end{bmatrix}$$
$$=-C_2N^2B_2=-1.$$

The system is nondegenerate (cf. Proposition 4) although the subsystem $S(A_1,B_1,C_1)$ is degenerate (its transfer function equals zero identically (cf. Tokarzewski, 2002a, Lemma 2.9, p. 28)).

From Corollary 2 we infer that the invariant zeros of the system are the roots of the polynomial $\det P(s)=s^2(s^2+2s+2)(s-2)$ (which is equal to the zero polynomial of the system). Thus, $\mathbb{Z}^I=\mathbb{Z}^S=\{0,2,-1+j1,-1-j1\}$. The same result is obtained employing Proposition 4 (or, more precisely, Remark 7). Calculating K_{-q} in accordance with (20), we get the matrix $\left[\begin{smallmatrix}sE-K_{-q}A\\ C\end{smallmatrix}\right]$ in the form

$$\begin{bmatrix}s-2&1&0&0&0&0\\0&s&0&0&0&0\\1&0&s&0&0&1\\0&0&0&-1&s&s+1\\0&0&0&0&-1&s+1\\0&0&0&0&0&0\\0&-1&0&1&0&1\end{bmatrix}. \tag{39}$$

Now, it is easy to check that (39) loses its full column normal rank exactly at the roots of the polynomial $s^2(s^2+2s+2)(s-2)$. ♦

Example 4. Consider the system (1) with the matrices

$$E=\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad A=\begin{bmatrix}-1&0&-6\\0&-1&3\\0&0&-3\end{bmatrix},$$

$$B=\begin{bmatrix}1&1\\1&0\\0&0\end{bmatrix},\quad C=\begin{bmatrix}0&1&2\end{bmatrix},\quad D=\begin{bmatrix}1&0\end{bmatrix}.$$

The Smith form of the system matrix is

$$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&(s+2)(s+3)&0\end{bmatrix},$$

i.e., $\mathbb{Z}^S=\{-2,-3\}$. Since the condition (13) of Proposition 2 is fulfilled, the system is degenerate, i.e., $\mathbb{Z}^I=\mathbb{C}$.

6. Concluding Remarks

Sufficient and necessary conditions of degeneracy/nondegeneracy for singular control systems with a regular pencil $sE-A$ have been formulated (resp. Propositions 2 and 3). Clearly, these conditions apply to standard linear systems as well.

Moreover, it has been shown that if a singular system is taken in its Weierstrass canonical form, then, under some additional assumptions, its invariant zeros can be characterized as output-decoupling zeros of a closed-loop state feedback system (Propositions 4 and 5).

Further research can be focused on characterization of individual kinds of decoupling zeros in the context of the four-fold Kalman decomposition of a singular system (cf. Kaczorek, 2003).

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