OPTIMIZATION OF THRUST ALLOCATION IN THE PROPULSION SYSTEM OF AN UNDERWATER VEHICLE

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The paper addresses methods of thrust distribution in a propulsion system for an unmanned underwater vehicle. It concentrates on finding an optimal thrust allocation for desired values of forces and moments acting on the vehicle. Special attention is paid to the unconstrained thrust allocation. The proposed methods are developed using a configuration matrix describing the layout of thrusters in the propulsion system. The paper includes algorithms of thrust distribution for both faultless work of the propulsion system and a failure of one of the thrusters. Illustrative examples are provided to demonstrate the effectiveness and correctness of the proposed methods.

Keywords: underwater vehicle, unconstrained control, propulsion system, thrust allocation

1. Introduction

There are various categories of unmanned underwater vehicles (UUVs). The ones most often used are remotely operated vehicles (ROVs). They are equipped with propulsion systems and controlled only by thrusters. An ROV is usually connected to a surface ship by a tether, by which all communication is wired.

The general motion of marine vessels in 6 degrees of freedom (DOF) can be described by the following vectors (Fossen, 1994):

$$\begin{aligned} \boldsymbol{\eta} &= [x, y, z, \phi, \theta, \psi]^T, \\ \boldsymbol{v} &= [u, v, w, p, q, r]^T, \\ \boldsymbol{\tau} &= [X, Y, Z, K, M, N]^T, \end{aligned} \tag{1}$$

where

$\boldsymbol{\eta}$ – the vector of the position and orientation in the earth-fixed frame,

$x,\ y,\ z$ – position coordinates,

$\phi,\ \theta,\ \psi$ – orientation coordinates (Euler angles),

$\boldsymbol{v}$ – the vector of linear and angular velocities in the body-fixed frame,

$u,\ v,\ w$ – linear velocities along longitudinal, transversal and vertical axes,

$p,\ q,\ r$ – angular velocities about longitudinal, transversal and vertical axes,

$\boldsymbol{\tau}$ – the vector of forces and moments acting on the vehicle in the body-fixed frame,

$X,\ Y,\ Z$ – the forces along longitudinal, transversal and vertical axes,

$K,\ M,\ N$ – the moments about longitudinal, transversal and vertical axes.

Modern ROVs are often equipped with control systems in order to execute complex manoeuvres without constant human intervention. The basic modules of the control system are depicted in Fig. 1. The autopilot computes demanded propulsion forces and moments $\boldsymbol{\tau}_d$ by comparing the vehicle's desired position, orientation and velocities with their current estimates. The corresponding values of propeller thrust $\boldsymbol{f}$ are calculated in the thrust distribution module and transmitted as a control input to the propulsion system.

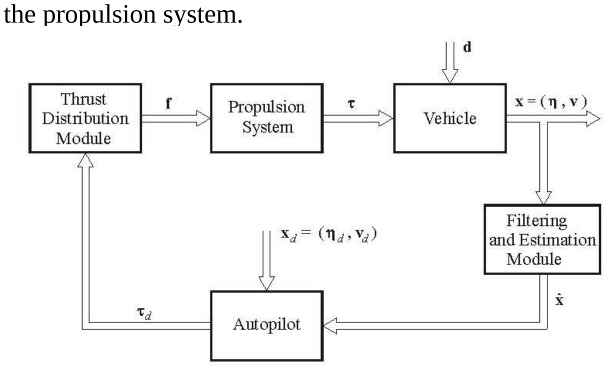

Fig. 1. A block diagram of the control system ($\boldsymbol{d}$ denotes the vector of environmental disturbances).

Moreover, the ROV rarely moves in underwater space without any interaction with environmental disturbances. The most significant influence is exerted by the sea current. Due to limited driving power of the propulsion system, its linear velocities are often comparable with the current speed. Therefore, to improve the control quality in extreme conditions, current-induced disturbances should be taken into account in the control system.

Until now, most of the literature in the field of control of the underwater vehicles has been focused on designing control laws (Canudas *et al.*, 1998; Craven *et al.*, 1998; Fossen, 1994; Garus and Kitowski, 1999; Katebi and Grimble, 1999), and little is known about thrust allocation (Berge and Fossen, 1997; Fossen, 2002; Garus, 2004; Sordelen, 1997; Whitcomb and Yoerger, 1999). Almost no attention has been paid to the important case when one or more thrusters are off in the propulsion system.

The objective of this work is to present methods of thrust allocation for the plane motion of a vehicle. Generally, it is an overactuated control problem since the number of thrusters is greater than the number of the DOF of the vehicle. The paper includes algorithms of thrust distribution for faultless work of the propulsion system and a case of a failure of one of the thrusters. Illustrative examples are also presented.

2. Description of the Propulsion System

For conventional ROVs the basic motion is the movement in a horizontal plane with some variation due to diving. They operate in a crab-wise manner in 4 DOF with small roll and pitch angles that can be neglected during normal operations. Therefore, it is purposeful to regard the vehicle's spatial motion as a superposition of two displacements: the motion in the vertical plane and the motion in the horizontal plane. It allows us to divide the vehicle's propulsion system into two independent subsystems responsible for movements in these planes, respectively. The most often applied configuration of thrusters in the propulsion system is shown in Fig. 2.

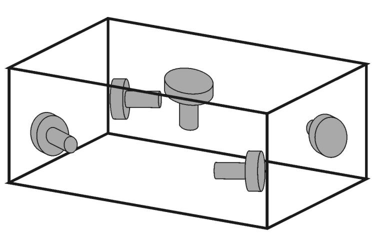

Fig. 2. Configuration of thrusters in the propulsion system.

The first subsystem permits the motion in heave and consists of 1 or 2 thrusters generating a propulsion force Z acting in the vertical axis. The thrust distribution is performed in such a way that the propeller thrust, or the sum of propellers thrusts, is equal to the demanded force Z_d.

The other subsystem assures the motion in surge, sway and yaw and it is usually composed of 4 thrusters mounted askew in relation to the vehicle's main symmetry axes (see Fig. 3). The forces X and Y acting in the longitudinal and transversal axes and the moment N about the vertical axis are a combination of thrusts produced by the propellers of the subsystem. Hence, from an operating point of view, the control system should include a procedure of thrust distribution determining thrust allocation such that the produced propulsion forces and moment are equal to the desired ones.

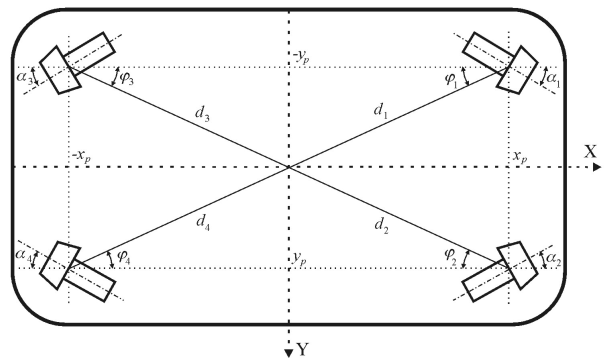

Fig. 3. Layout of thrusters in the subsystem responsible for the horizontal motion.

The relationship between the forces and moments and the propeller thrust is a complicated function that depends on the vehicle's velocity, the density of water, the tunnel length and cross-sectional area, the propeller's diameter and revolutions. A detailed analysis of thruster dynamics can be found, e.g., in (Charchalis, 2001; Healey *et al.*, 1995). In practical applications the vector of propulsion forces and the moment $\boldsymbol{\tau}$ acting on the vehicle in the horizontal plane can be described as a function of the thrust vector $\boldsymbol{f}$ by the following expression (Fossen, 1994; Garus, 2003):

$$\boldsymbol{\tau} = \boldsymbol{T}(\boldsymbol{\alpha})\boldsymbol{P}\boldsymbol{f}, \tag{2}$$

where

$$\boldsymbol{\tau} = [\tau_1, \tau_2, \tau_3]^T,$$

τ_1 – force in the longitudinal axis,

τ_2 – force in the transversal axis,

τ_3 – moment about the vertical axis,

$\boldsymbol{T}$ – thruster configuration matrix,

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{t}_1 \\ \boldsymbol{t}_2 \\ \boldsymbol{t}_3 \end{bmatrix} = \begin{bmatrix} \cos\alpha_1 & \cos\alpha_2 & \dots & \cos\alpha_n \\ \sin\alpha_1 & \sin\alpha_2 & \dots & \sin\alpha_n \\ d_1\sin(\gamma_1) & d_2\sin(\gamma_2) & \dots & d_n\sin(\gamma_n) \end{bmatrix}, \tag{3}$$

where $\gamma_1 = \alpha_1 - \varphi_1,\ \gamma_2 = \alpha_2 - \varphi_2,\ \gamma_n = \alpha_n - \varphi_n,$

$\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_n]^T$ – vector of thrust angles,

α_i – angle between the longitudinal axis and direction of the propeller thrust f_i,

d_i – distance of the i-th thruster from the centre of gravity,

φ_i – angle between the longitudinal axis and the line connecting the centre of gravity with the symmetry centre of the i-th thruster,

$\boldsymbol{f} = [f_1, f_2, \dots, f_n]^T$ – thrust vector,

$\boldsymbol{P}$ – diagonal matrix of the readiness of the thrusters:

$$p_{ii} = \begin{cases} 0 & \text{if the } i\text{-th thruster is off,} \\ 1 & \text{if the } i\text{-th thruster is active.} \end{cases}$$

The computation of $\boldsymbol{f}$ from $\boldsymbol{\tau}$ is a model-based optimisation problem and it is regarded below for two cases, namely, constrained and unconstrained thrust allocations. It will be assumed that the allocation problem is constrained if there are bounds on the thrust vector elements f_i. They are caused by thruster limitations like saturation or tear and wear. If those constraints are not taken into account, it will lead to unconstrained thrust allocation.

3. Constrained Thrust Allocation

In this case the values of the vector $\boldsymbol{f}$ are bounded and the task of finding optimal thrust allocation can be solved by means of quadratic programming (QP), which is formulated as follows:

$$\min_{\boldsymbol{f}} \frac{1}{2}\boldsymbol{f}^T\boldsymbol{H}\boldsymbol{f}, \tag{4}$$

subject to

$$\begin{gathered} \boldsymbol{T}\boldsymbol{f} = \boldsymbol{\tau}_d, \\ \boldsymbol{f}_{\min} \le \boldsymbol{f} \le \boldsymbol{f}_{\max}, \end{gathered} \tag{5}$$

where

$\boldsymbol{H}$ – symmetric positive definite matrix (usually diagonal),

$\boldsymbol{\tau}_d = [\tau_{d1}, \tau_{d2}, \tau_{d3}]^T$,

$\boldsymbol{f}_{\min} = [f_{1\min}, f_{2\min}, \dots, f_{n\min}]^T$,

$\boldsymbol{f}_{\max} = [f_{1\max}, f_{2\max}, \dots, f_{n\max}]^T$.

The solution of the constrained optimisation problem can be obtained by using any of the well-known QP methods, e.g., Zoutendijk's algorithm, Beale's algorithm or Rosen's algorithm. A basic disadvantage of the above approach is that it is computationally time consuming. Due to the real-time implementation of the procedure of the thrust allocation, a practical application of the method is restricted by the computational power of an on-board computer.

4. Unconstrained Thrust Allocation

Assume that the vector $\boldsymbol{\tau}_d$ is bounded in such a way that the calculated elements of the vector f can never exceed the boundary values $\boldsymbol{f}_{\min}$ and $\boldsymbol{f}_{\max}$. Then the unconstrained thrust allocation problem can be formulated as the following least-squares optimisation problem:

$$\min_{\boldsymbol{f}} \frac{1}{2}\boldsymbol{f}^T\boldsymbol{H}\boldsymbol{f}, \tag{6}$$

subject to

$$\boldsymbol{\tau}_d - \boldsymbol{T}\boldsymbol{f} = \boldsymbol{0}, \tag{7}$$

where $\boldsymbol{H}$ is a positive definite matrix.

The solution of the above problem using Lagrange multipliers is shown in (Fossen, 1994) as

$$\boldsymbol{f} = \boldsymbol{T}^*\boldsymbol{\tau}_d, \tag{8}$$

where

$$\boldsymbol{T}^* = \boldsymbol{H}^{-1}\boldsymbol{T}^T\left(\boldsymbol{T}\boldsymbol{H}^{-1}\boldsymbol{T}^T\right)^{-1} \tag{9}$$

is recognized as the generalized inverse. For the case $\boldsymbol{H} = \boldsymbol{I}$, the expression (9) reduces to the Moore-Penrose pseudoinverse:

$$\boldsymbol{T}^* = \boldsymbol{T}^T\left(\boldsymbol{T}\boldsymbol{T}^T\right)^{-1}. \tag{10}$$

The above approach assures a proper solution only for faultless work of the propulsion system and cannot be directly used in the case of a thruster damage. To increase its applicability and overcome this difficulty, two alternative algorithms are proposed.

4.1. Solution Using Singular Value Decomposition

Singular value decomposition (SVD) is an eigenvalue-like decomposition for rectangular matrices (Kiełbasinski and Schwetlich, 1992). SVD has the following form for the thruster configuration matrix (3):

$$\boldsymbol{T} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^T, \tag{11}$$

where

$\boldsymbol{U}, \boldsymbol{V}$ – orthogonal matrices of dimensions 3×3 and $n \times n$, respectively,

$$\boldsymbol{S} = \left[\begin{array}{c|c} \boldsymbol{S}_T & \boldsymbol{0} \end{array}\right] = \left[\begin{array}{ccc|c} \sigma_1 & 0 & 0 & \\ 0 & \sigma_2 & 0 & \boldsymbol{0} \\ 0 & 0 & \sigma_3 & \end{array}\right],$$

$\boldsymbol{S}_\tau$ – diagonal matrix of dimensions 3×3,

$\boldsymbol{0}$ – null matrix of dimensions $3 \times (n-3)$.

The diagonal entries σ_i are called the singular values of $\boldsymbol{T}$. They are positive and ordered so that $\sigma_1 \geq \sigma_2 \geq \sigma_3$.

This decomposition of the matrix $\boldsymbol{T}$ allows us to work out a computationally convenient procedure to calculate the thrust vector $\boldsymbol{f}$ being a minimum-norm solution to (8). The procedure is analysed for two cases:

1. All thrusters are operational ($\boldsymbol{P} = \boldsymbol{I}$).
2. One of the thrusters is off due to a fault ($\boldsymbol{P} \neq \boldsymbol{I}$).

4.1.1. Algorithm for All Thrusters Active

Set $\boldsymbol{\tau}_d = [\tau_{d1}, \tau_{d2}, \tau_{d3}]^T$ as the required input vector, $\boldsymbol{f} = [f_1, f_2, \ldots, f_n]^T$ as the thrust vector necessary to generate the vector τ_d, and n as the number of thrusters.

A direct substitution of (11) shows that the vector $\boldsymbol{f}$ determined by (8) and (10) can be written in the form

$$\boldsymbol{f} = \boldsymbol{V}\boldsymbol{S}^*\boldsymbol{U}^T\boldsymbol{\tau}_d = \boldsymbol{V}\begin{bmatrix} \boldsymbol{S}_T^{-1} \\ \boldsymbol{0} \end{bmatrix}\boldsymbol{U}^T\boldsymbol{\tau}_d. \tag{12}$$

4.1.2. Algorithm for One Non-Operational Thruster

Assume that the k-th thruster is off. This means that $f_k = 0$ and $p_{kk} = 0$. The substitution of (11) into (2) leads to the following dependence:

$$\boldsymbol{\tau}_d = \boldsymbol{T}\boldsymbol{P}\boldsymbol{f} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^T\boldsymbol{P}\boldsymbol{f}. \tag{13}$$

Defining

$$\boldsymbol{f}' = [f_1, \ldots, f_{k-1}, f_{k+1}, \ldots f_n]^T,$$

$$\boldsymbol{V}^* = \boldsymbol{V}^T\boldsymbol{P} = \left[\boldsymbol{v}_1^*, \ldots, \boldsymbol{v}_{k-1}^*, \boldsymbol{0}, \boldsymbol{v}_{k+1}^*, \ldots, \boldsymbol{v}_n^*\right],$$

$$\boldsymbol{V}_{\boldsymbol{f}}^* = \left[\boldsymbol{v}_1^*, \ldots, \boldsymbol{v}_{k-1}^*, \boldsymbol{v}_{k+1}^*, \ldots, \boldsymbol{v}_n^*\right],$$

the expression (13) can be written as

$$\boldsymbol{\tau}_d = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}_{\boldsymbol{f}}^*\boldsymbol{f}'. \tag{14}$$

The matrices $\boldsymbol{U}$ and $\boldsymbol{S}\boldsymbol{V}_{\boldsymbol{f}}^*$ have dimensions 3×3 and $3 \times m$, where $m = n - 1$, so the vector $\boldsymbol{f}'$ can be computed as

$$\boldsymbol{f}' = \left(\left(\boldsymbol{S}\boldsymbol{V}_{\boldsymbol{f}}^*\right)^T\boldsymbol{S}\boldsymbol{V}_{\boldsymbol{f}}^*\right)^{-1}\left(\boldsymbol{S}\boldsymbol{V}_{\boldsymbol{f}}^*\right)^T\boldsymbol{U}^T\boldsymbol{\tau}_d. \tag{15}$$

Hence, the value of the thrust vector $\boldsymbol{f}$ can be obtained as follows:

$$\boldsymbol{f} = \left[f_1', \ldots, f_{k-1}', 0, f_k', \ldots f_m'\right]^T. \tag{16}$$

Note that if $n = 4$ then (15) can be simplified to the form

$$\boldsymbol{f}' = \left(\boldsymbol{S}\boldsymbol{V}_{\boldsymbol{f}}^*\right)^{-1}\boldsymbol{U}^T\boldsymbol{\tau}_d. \tag{17}$$

4.1.3. Numerical Example

Calculations have been conducted for the following data:

$$\boldsymbol{\tau}_d = \begin{bmatrix} 300 & -50 & 10 \end{bmatrix}^T, \quad \boldsymbol{P} = \boldsymbol{I}_{(4\times 4)},$$

$$\boldsymbol{T} = \begin{bmatrix} 0.875 & 0.875 & -0.875 & -0.875 \\ 0.485 & -0.485 & 0.485 & -0.485 \\ 0.332 & -0.332 & -0.332 & 0.332 \end{bmatrix},$$

$$\boldsymbol{S}_T = \mathrm{diag}\left(\ 1.749, \quad 0.967, \quad 0.664\ \right),$$

$$\boldsymbol{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{V} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix}.$$

The values of the thruster configuration matrix $\boldsymbol{T}$ are taken for the ROV called “Ukwiał” designed and built for the Polish Navy (Garus, 2004).

The vector $\boldsymbol{f}$ is computed using the expression (12):

$$\boldsymbol{f} = \boldsymbol{V}S^*\boldsymbol{U}^T\boldsymbol{\tau}_d = \boldsymbol{V}\begin{bmatrix} \boldsymbol{S}_T^{-1} \\ \boldsymbol{0} \end{bmatrix}\boldsymbol{U}^T\boldsymbol{\tau}_d$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{1.749} & 0 & 0 \\ 0 & \frac{1}{0.967} & 0 \\ 0 & 0 & \frac{1}{0.664} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T\begin{bmatrix} 300 \\ -50 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 67.5 \\ 104.0 \\ -119.1 \\ -52.4 \end{bmatrix}.$$

4.2. Solution Using the Walsh Matrix

The solution proposed below is restricted to ROVs having the configuration of thrusters exactly as shown in Fig. 3, i.e., the propulsion system consists of four identical thrusters located symmetrically around the centre of gravity. In such a case $d_j = d_k = d$, $\alpha_j \bmod (\Pi/2) = \alpha_k \bmod (\Pi/2) = \alpha$, $\varphi_j \bmod (\Pi/2) = \varphi_k \bmod (\Pi/2) = \varphi$ for $j,k = 1,\dots,4$ and the thrusters configuration matrix $\boldsymbol{T}$ can be written in the form

$$\boldsymbol{T} = \begin{bmatrix} \cos\alpha & \cos\alpha & -\cos\alpha & -\cos\alpha \\ \sin\alpha & -\sin\alpha & \sin\alpha & -\sin\alpha \\ d\sin(\gamma) & -d\sin(\gamma) & -d\sin(\gamma) & d\sin(\gamma) \end{bmatrix}, \tag{18}$$

where $\gamma = \alpha - \varphi$. Then the matrix $\boldsymbol{T}$ has the following properties:

(a) it is a row-orthogonal matrix,

(b) $|t_{ij}| = |t_{ik}|$ for $i = 1,2,3$ and $j,k = 1,\dots,4$,

(c) it can be written as a product of two matrices: a diagonal matrix $\boldsymbol{Q}$ and a row-orthogonal matrix $\boldsymbol{W}_f$ having values ± 1:

$$\boldsymbol{T} = \boldsymbol{Q}\boldsymbol{W}_{\boldsymbol{f}} = \begin{bmatrix} t_{11} & 0 & 0 \\ 0 & t_{21} & 0 \\ 0 & 0 & t_{31} \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}. \tag{19}$$

It allows us to work out a simple and fast procedure to compute the thrust vector $\boldsymbol{f}$ by applying an orthogonal Walsh matrix (see Appendix A). It should be emphasized that the use of this method does not require calculations of any additional matrices. This is the main advantage of the proposed solution in comparison with the previous one.

The procedure is considered for the same two cases:

1. all thrusters are operational ($\boldsymbol{P} = \boldsymbol{I}$),
2. one of the thrusters is off due to a fault ($\boldsymbol{P} \neq \boldsymbol{I}$).

4.2.1. Algorithm for All Thrusters Active

As in Section 4.1.1, set $\boldsymbol{\tau}_d = [\tau_{d1}, \tau_{d2}, \tau_{d3}]^T$ as the required input vector and $\boldsymbol{f} = [f_1, f_2, \dots, f_n]^T$ as the thrust vector necessary to generate the input vector $\boldsymbol{\tau}_d$.

The substitution of (19) into (2) gives

$$\boldsymbol{\tau}_d = \boldsymbol{Q}\boldsymbol{W}_{\boldsymbol{f}}\boldsymbol{P}\boldsymbol{f}. \tag{20}$$

By multiplying both sides of (20) by $\boldsymbol{Q}^{-1}$, the following expression is obtained:

$$\boldsymbol{Q}^{-1}\boldsymbol{\tau}_d = \boldsymbol{W}_{\boldsymbol{f}}\boldsymbol{P}\boldsymbol{f}. \tag{21}$$

Substituting

$$\boldsymbol{S} = \begin{bmatrix} 0 \\ \boldsymbol{Q}^{-1}\boldsymbol{\tau}_d \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\tau_{d1}}{t_{11}} \\ \frac{\tau_{d2}}{t_{22}} \\ \frac{\tau_{d3}}{t_{33}} \end{bmatrix}, \tag{22}$$

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{w}_0 \\ \boldsymbol{W}_{\boldsymbol{f}} \end{bmatrix}, \tag{23}$$

where $\boldsymbol{w}_0 = [\ 1 \quad 1 \quad 1 \quad 1\ ]$, and assuming that $\boldsymbol{P} = \boldsymbol{I}$, (21) can be transformed to the form

$$\boldsymbol{S} = \boldsymbol{W}\boldsymbol{f}. \tag{24}$$

The matrix $\boldsymbol{W}$ is the Walsh matrix having the following properties: $\boldsymbol{W} = \boldsymbol{W}^T$ and $\boldsymbol{W}\boldsymbol{W}^T = n\boldsymbol{I}$, where $n = \dim \boldsymbol{W}$.

Hence, the thrust vector $\boldsymbol{f}$ can be expressed as follows:

$$\boldsymbol{f} = \frac{1}{4}\boldsymbol{W}^T\boldsymbol{S} = \frac{1}{4}\boldsymbol{W} \begin{bmatrix} 0 \\ \boldsymbol{Q}^{-1}\boldsymbol{\tau}_d \end{bmatrix}. \tag{25}$$

4.2.2. Algorithm for One Non-Operational Thruster

Assume that the k-th thruster is off so that $f_k = 0$ and $p_{kk} = 0$. Defining

$$\boldsymbol{f}' = [f_1, \dots, f_{k-1}, f_{k+1}, \dots f_4]^T,$$

$$\boldsymbol{P}' = \operatorname{diag}\{p_{11}, \dots, p_{k-1k-1}, p_{k+1k+1}, \dots, p_{44}\},$$

$$\boldsymbol{W}_{\boldsymbol{f}'} = [\boldsymbol{w}_{\boldsymbol{f}1}, \dots, \boldsymbol{w}_{\boldsymbol{f}k-1}, \boldsymbol{w}_{\boldsymbol{f}k+1}, \dots, \boldsymbol{w}_{\boldsymbol{f}4}]$$

the expression (20) can be written in the form

$$\boldsymbol{\tau}_d = \boldsymbol{Q}\boldsymbol{W}_{\boldsymbol{f}'}\boldsymbol{P}'\boldsymbol{f}'. \tag{26}$$

Since $\boldsymbol{Q}$, $\boldsymbol{W}'_{\boldsymbol{f}}$ and $\boldsymbol{P}'$ are square matrices having dimensions 3×3 and $\boldsymbol{P} \neq \boldsymbol{I}$, the thrust vector $\boldsymbol{f}'$ can be calculated as

$$\boldsymbol{f}' = \boldsymbol{W}_{\boldsymbol{f}'}^{-1}\boldsymbol{Q}^{-1}\boldsymbol{\tau}_d. \tag{27}$$

Hence the value of the thrust vector $\boldsymbol{f}$ is obtained as follows:

$$\boldsymbol{f} = [f'_1, \dots, f'_{k-1}, 0, f'_k, \dots f'_m]^T. \tag{28}$$

4.2.3. Numerical Example

Calculations were performed for the same data $\boldsymbol{T}$, $\boldsymbol{P}$ and $\boldsymbol{\tau}_d$ as in Section 4.1.3 and the Walsh matrix $\boldsymbol{W}$ of the form

$$
\boldsymbol{W} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.
$$

Using (25), the following value of the thrust vector $\boldsymbol{f}$ is obtained:

$$
\begin{aligned}
\boldsymbol{f} &= \frac{1}{4}\boldsymbol{W}\begin{bmatrix} 0 \\ \boldsymbol{Q}^{-1}\boldsymbol{\tau}_d \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \\
&\quad \times \left[\begin{array}{ccc:c} \multicolumn{4}{c}{0} \\ \hdashline \frac{1}{0.875} & & & 300 \\ & \frac{1}{0.485} & & -50 \\ & & \frac{1}{0.332} & 10 \end{array}\right] \\
&= \begin{bmatrix} 67.5 \\ 104.0 \\ -119.1 \\ -52.4 \end{bmatrix}.
\end{aligned}
$$

It is equal to that computed in the previous example.

5. Conclusions

The paper presents methods of thrust distribution for an unmanned underwater vehicle. To avoid a significant amount of computations, the problem of thrust allocation has been regarded as an unconstrained optimisation problem. The described algorithms are based on the decomposition of the thruster configuration matrix. This allows us to obtain minimum Euclidean norm solutions. The main advantage of the approach is its computational simplicity and flexibility with respect to the construction of the propulsion system and the number of thrusters. Moreover, the proposed techniques can be used for both faultless work of the propulsion system and a failure of one of the thrusters.

The developed algorithms of thrust distribution are of a general character and can be successfully applied to all types of ROVs.

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Appendix A

A Walsh matrix $\boldsymbol{W}$ is a square, orthogonal matrix having all elements equal to ± 1. Examples of Walsh matrices of dimensions 2, 4 and 8 are given below:

$$\boldsymbol{W}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \boldsymbol{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix},$$

$$\boldsymbol{W}_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}.$$

It is convenient to generate a Walsh matrix $\boldsymbol{W}$ using its connection with the Hadamard matrix $\boldsymbol{H}$. The main advantage of the Hadamard matrix is its possibility to be generated in a recursive way by means of the following dependence:

$$\boldsymbol{H}_{2N} = \begin{bmatrix} \boldsymbol{H}_N & \boldsymbol{H}_N \\ \boldsymbol{H}_N & -\boldsymbol{H}_N \end{bmatrix},$$

where $\boldsymbol{H}_1 = [1]$. Detailed algorithms of the transformation of the $\boldsymbol{H}_N$ matrix to the $\boldsymbol{W}_N$ matrix can be found in (Ahmed and Rao, 1975; Kulesza, 1984).