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ON A REGULARIZATION METHOD FOR VARIATIONAL INEQUALITIES WITH P_0 MAPPINGS

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We consider partial Browder-Tikhonov regularization techniques for variational inequality problems with P_0 cost mappings and box-constrained feasible sets. We present classes of economic equilibrium problems which satisfy such assumptions and propose a regularization method for these problems.

Keywords: variational inequalities, partial regularization approach, P_0-mappings

1. Introduction

Let K and V be nonempty convex sets in the real Euclidean space $\mathbb{R}^n$, $K \subseteq V$, and let $G : V \to \mathbb{R}^n$ be a mapping. Denote by $\langle a, b\rangle$ the scalar product of elements a and b in $\mathbb{R}^n$. The variational inequality problem (VI for brevity) is the problem of finding $x^* \in K$ such that

$$\langle G(x^*), x - x^*\rangle \geq 0, \quad \forall x \in K. \tag{1}$$

Variational inequalities are known to be a very useful tool for formulating and investigating various equilibrium problems arising in mathematical physics, economics, engineering, and operations research (Baiocchi and Capelo, 1984; Facchinei and Pang, 2003; Nagurney, 1999). However, many problems arising in applications possess a special structure of constraints, in which the feasible set K is a box-constrained set. Such VIs extend the usual complementarity problems and are traditionally considered in the case where G satisfies P type properties (Cottle *et al.*, 1992; Facchinei and Pang, 2003; Moré and Rheinboldt, 1973). These properties yield various existence and uniqueness results for the problem (1) and suggest effective solution methods. However, they seem too restrictive for applications where P_0 conditions are used. For such problems, various regularization approaches become very popular, most works in this field being concentrated on the full Browder-Tikhonov regularization, see, e.g. (Cottle *et al.*, 1992; Facchinei and Kanzow, 1999; Facchinei and Pang, 2003; Qi, 1999) and references therein. In this paper, we consider a more general scheme which admits partial regularization of the initial problem since it appears to be sufficient for any auxiliary problem to have a unique solution. More precisely, we employ a minimal number of regularization terms for each problem under consideration and establish sufficient conditions for the convergence of solutions of perturbed problems. Then the perturbed problems become closer to the initial problem. However, even the full regularization method does not guarantee the convergence of the sequence of solutions of perturbed problems to a solution of the initial problem if the cost mapping is not monotone (Facchinei and Kanzow, 1999; Facchinei and Pang, 2003, Sec. 12.2). We first consider the case when the feasible set is bounded and afterwards present some additional conditions which enable us to apply the method in the unbounded case. We describe two rather broad classes of perfectly and non-perfectly competitive economic equilibrium models which are involved in this class of VIs and outline regularization approaches for these problems.

In what follows, for a vector $x \in \mathbb{R}^n$, $x \geq 0$ (resp., $x > 0$) means $x_i \geq 0$ (resp. $x_i > 0$) for all $i = 1, \dots, n$, and we set

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x \geq 0\}$$

and

$$\mathbb{R}^n_> = \{x \in \mathbb{R}^n \mid x > 0\}.$$

Also, we denote by K^* the solution set of the problem (1). Let L be any subset of $N = \{1, \dots, n\}$. We denote by A_L the $n \times n$ diagonal matrix whose diagonal entries are given by

$$a_{ii} = \begin{cases} > 0 & \text{if} \quad i \in L, \\ = 0 & \text{if} \quad i \notin L. \end{cases}$$

Then A_N is a diagonal positive definite matrix. Next, if $a_{ii} = 1$ for all i, then $A_N = I_n$, i.e., it is the identity matrix in $\mathbb{R}^n$.

2. Technical Preliminaries

In this section, we recall some definitions and give some properties which will be used in our further deliberations. We shall consider the VI (1) under the following standing assumptions:

(A1) *$G : V \to \mathbb{R}^n$ is a continuous mapping, and V is a nonempty convex subset of $\mathbb{R}^n$.*

(A2) *K is a box constrained set, that is,*

$$K = \prod_{i=1}^{n} K_i \subseteq V,$$

where $K_i = [\alpha_i, \beta_i] \subseteq [-\infty, +\infty]$ for every $i = 1, \dots, n$.

Note that K is obviously a nonempty convex and closed set. If, in addition, $\beta_i < +\infty$ for $i \in N$, then K is also a bounded set. First, we recall definitions of several properties of matrices, cf. (Fiedler and Pták, 1962; Ortega and Rheinboldt, 1970).

Definition 1. An $n \times n$ matrix A is said to be

(a) a *P-matrix* if it has positive principal minors;

(b) a *P_0-matrix* if it has nonnegative principal minors;

(c) a *Z-matrix* if it has nonpositive off-diagonal entries;

(d) an *M-matrix* if it has nonpositive off-diagonal entries and its inverse A^{-1} exists and has nonnegative entries;

(e) an *M_0-matrix*if it is both a P_0- and a Z-matrix.

It is well known that A is M if and only if $A \in P \cap Z$ (Fiedler and Pták, 1962; Ortega and Rheinboldt, 1970). Hence, each M-matrix is a P-matrix, but the reverse assertion is not true in general.

The following proposition gives a criterion for a matrix A to be an M- or an M_0-matrix.

Proposition 1. (Fiedler and Pták, 1962) *Suppose that a matrix A is a Z-matrix. If there exists a vector $x > 0$ such that $Ax > 0$ (resp. $Ax \geq 0$), then A is an M-matrix (resp. an M_0-matrix).*

Now we recall extensions of these properties for mappings which were introduced in (Konnov, 2000; Moré and Rheinboldt, 1973).

Definition 2. Let U be a convex subset of $\mathbb{R}^n$. A mapping $F : U \to \mathbb{R}^n$ is said to be

(a) a *P-mapping*, if $\max_{1 \leq i \leq n} (x_i - y_i)(F_i(x) - F_i(y)) > 0$ for all $x, y \in U, \quad x \neq y$;

(b) a *strict P-mapping*, if there exists $\gamma > 0$ such that $F - \gamma I_n$ is a P-mapping;

(c) a *uniform P-mapping*, if there exists $\tau > 0$ such that

$$\max_{1 \leq i \leq n} (x_i - y_i)(F_i(x) - F_i(y)) \geq \tau \|x - y\|^2$$

for all $x, y \in U$;

(d) a *P_0-mapping*, if for all $x, y \in U$, $x \neq y$, there exists an index i such that $x_i \neq y_i$ and $(x_i - y_i)(F_i(x) - F_i(y)) \geq 0$.

In fact, if F is affine, that is, $F(x) = Ax + b$, then F is a P-mapping (P_0-mapping) if and only if its Jacobian $\nabla F(x) = A$ is a P-matrix (P_0-matrix). In the general nonlinear case, if the Jacobian $\nabla F(x)$ is a P-matrix, then F is a P-mapping, but the reverse assertion is not true in general. At the same time, F is a P_0-mapping if and only if its Jacobian $\nabla F(x)$ is a P_0-matrix. Next, if F is a strict P-mapping, then its Jacobian is a P-matrix (Facchinei and Kanzow, 1999; Konnov, 2000; Moré and Rheinboldt, 1973).

We recall an additional relationship between P_0 and strict P-mappings.

Lemma 1. (Konnov and Volotskaya, 2002, Lem. 3.6) *If $F : U \to \mathbb{R}^n$ is a P_0-mapping and $\varepsilon > 0$, then $F + \varepsilon I_n$ is a strict P-mapping.*

Note that each uniform P-mapping is a strict P-mapping, but the reverse assertion is not true in general. Thus, although most existence and uniqueness results for VIs were established for uniform P-mappings (e.g., see Moré and Rheinboldt, 1973; Ortega and Rheinboldt, 1970), this concept is not convenient for various Tikhonov regularization procedures which involve mappings of the form $F + \varepsilon I_n$ (Facchinei and Kanzow, 1999; Facchinei and Pang, 2003). At the same time, such mappings are strict P if F is P_0 because of Lemma 1, and this fact can serve as a motivation for developing the theory of VIs with strict P-mappings. Also, this concept is very useful in the investigation of VIs arising from economic applications. Moreover, it appears to be sufficient for obtaining existence and uniqueness results.

Proposition 2. (Facchinei and Kanzow, 1999, Thm. 3.5; Konnov, 2000, Prop. 3) *Let (A1) and (A2) hold, and let G be a strict P-mapping. Then the VI (1) has a unique solution.*

We can even somewhat strengthen this result for bounded sets.

Proposition 3. (Konnov and Volotskaya, 2002, Cor. 4.3) *Let (A1) and (A2) hold. If G is a P-mapping and K is a bounded box-constrained set, then the VI (1) has a unique solution.*

In the unbounded case we can also replace the strict P property by a coercivity condition. We can consider a somewhat extended version of this condition.

(A3) *Suppose that there exist sets $\tilde{D} \subseteq D \subseteq \mathbb{R}^n$ such that, for each point $y \in K \backslash D$, there exists a point $x \in \tilde{D} \bigcap K$ such that*

$$\max_{i=1,\dots,n} G_i(y)(y_i - x_i) > 0. \tag{2}$$

From the definition we obtain immediately the following characterization of the solution set:

Lemma 2. *If (A1)–(A3) are satisfied and $K^* \neq \emptyset$, then $K^* \subseteq K \bigcap D$.*

Moreover, it follows that it is possible to describe changes in the solution set after some reductions of the feasible set.

Proposition 4. *Suppose that (A1)–(A3) are satisfied with $D = K^*$, $\tilde{K}^*$ being the solution set of the VI of the form (1), where K is replaced by a set $\tilde{K} = \prod_{i=1}^{n} \tilde{K}_i$, $\tilde{K}_i$ is a nonempty convex closed set for each $i = 1, \dots, n$. If $\tilde{D} \bigcap K \subseteq \tilde{K} \subseteq K$, then $\tilde{K}^* = \tilde{K} \bigcap K^*$.*

Proof. Clearly, $\tilde{K} \bigcap K^* \subseteq \tilde{K}^*$. Suppose that there is a point $y \in \tilde{K}^* \backslash K^*$. Then $y \in \tilde{K} \backslash D$. Applying (A3), we see that there exists a point $x \in \tilde{D} \bigcap K \subseteq \tilde{K}$ such that (2) holds, i.e., $y \notin \tilde{K}^*$, so we get a contradiction, and the result follows. ■

If the set D in (A3) is bounded, we obtain a modification of the other known coercivity conditions (Facchinei and Pang, 2003, Vol. 1, pp. 227–293).

Proposition 5. *Suppose that (A1)–(A3) are satisfied, and D in (A3) is bounded. Then*

(i) the VI (1) is solvable, and $K^* \subseteq K \bigcap D$;

(ii) if, additionally, G is a P-mapping, K^* is a singleton.

Proof. Since D is bounded, choose a closed Euclidean ball B with the center at 0 such that $\mathrm{int}B \supseteq D$. Then the VI of the form (1) with the feasible set $B \bigcap K$ will be solvable (see, e.g., Facchinei and Pang, 2003, Cor. 2.2.5). Moreover, all these solutions will belong to $\mathrm{int}B$. It follows now from (Facchinei and Pang, 2003, Prop. 2.2.8) that the VI (1) is also solvable. The second part of (i) follows from Lemma 2, whereas (ii) follows from the definition of the P-property. ■

The properties above appear to be very useful for regularization methods.

3. Regularization Approach

We shall approximate the VI (1) with the following problem: Find $x^\varepsilon \in K$ such that

$$\langle G(x^\varepsilon) + \varepsilon A_L x^\varepsilon, x - x^\varepsilon \rangle \geq 0, \quad \forall x \in K, \tag{3}$$

where $\varepsilon > 0$ is a parameter, and L is a nonempty subset of N.

We first consider the convergence of the sequence $\{x^\varepsilon\}$ in the bounded case.

Theorem 1. *Suppose that (A1) and (A2) are fulfilled. Let the problem (3) have a unique solution x^ε, and let $[\alpha_i, \beta_i] \subset (-\infty, +\infty)$ for every $i \in N$. Then the sequence $\{x^{\varepsilon_k}\}$, where $\{\varepsilon_k\} \searrow 0$, has some limit points, and all these points are contained in the solution set of the VI (1).*

Proof. Since the sequence $\{x^\varepsilon\}$ is contained in the bounded set K, it has some limit points. If x^* is an arbitrary limit point of $\{x^\varepsilon\}$, then taking the limit in (3) gives

$$\langle G(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K,$$

i.e., x^* solves the VI (1). ■

We now give additional examples of sufficient conditions for the nonemptiness of the solution set of the auxiliary VI (3).

Proposition 6. *Let (A1) and (A2) hold, G be a P_0-mapping, and $L = \{1, \dots, n\}$. Then the problem (3) has a unique solution.*

Proof. Due to Lemma 1 it means that $G + \varepsilon I_n$ is a strict P-mapping. By Proposition 2 it follows that the problem (3) has a unique solution. ■

For an index set L, we shall write $x_L = (x_i)_{i\in L}$ and $Q_L(x) = \nabla_{x_L} G(x_L)$. Hence $Q_N(x) = \nabla G(x)$.

Proposition 7. *Let (A1) and (A2) hold, G be a P_0-mapping, and $[\alpha_i, \beta_i] \subset (-\infty, +\infty)$ for $i \in N$. Suppose that, for every $x \in K$, $\nabla G(x)$ is a Z-matrix and there is a $J \subseteq N$ such that $Q_J(x)$ is a P-matrix. If we set $L = N\backslash J$, then VI (3) has a unique solution.*

Proof. Without loss of generality we can suppose that $J = \{1, \dots, k\}$. Hence $L = \{k+1, \dots, n\}$. Then

$$\nabla G(x) = \begin{pmatrix} Q_J(x) & B_k^{'} \\ B_k^{''} & C_k \end{pmatrix},$$

where $B_k^{'}$ is a rectangular matrix which has k rows and $n-k$ columns, $B_k^{''}$ is a rectangular matrix which has $n-k$ rows and k columns, and C_k is an $(n-k)\times(n-k)$ matrix. Since $\nabla G(x)$ is a Z-matrix and by assumption, $Q_J(x)$ is an M-matrix. Let us consider the mapping $\tilde{G} : V \to \mathbb{R}^n$, whose components are defined by

$$\tilde{G}_i(x) = \begin{cases} G_i(x) & \text{if } 1 \le i \le k, \\ G_i(x) + \varepsilon a_{ii} x_i & \text{if } k < i \le n, \end{cases}$$

with $\varepsilon > 0$. Clearly, its Jacobian

$$\nabla \tilde{G}(x) = \begin{pmatrix} Q_J(x) & B_k^{'} \\ B_k^{''} & C_k \end{pmatrix} + \varepsilon A_L$$

is an M-matrix (Fiedler and Pták, 1962). By definition, $\tilde{G}$ is a P-mapping. Due to Proposition 3, it follows that the problem: Find $x^* \in K$ such that

$$\langle \tilde{G}(x^*), x - x^* \rangle \ge 0, \quad \forall x \in K,$$

has a unique solution. However, this problem is clearly equivalent to the VI (3) and the result follows. ■

Let us now turn to the case when K is an arbitrary set satisfying (A2), i.e., it may be unbounded. It is known (see Facchinei and Kanzow, 1999, Ex. 4.6) that even full regularization applied to a VI with a P_0 cost mapping does not guarantee the convergence of the sequence $\{x^\varepsilon\}$ to a solution. In (Facchinei and Kanzow, 1999), such convergence is proved for complementarity problems with bounded solution sets. We now consider another approach which is based on introducing an auxiliary bounded VI. Namely, let us define the set

$$\tilde{K} = \prod_{i=1}^{n} \tilde{K}_i, \quad \tilde{K}_i = [\tilde{\alpha}_i, \tilde{\beta}_i] \subset (-\infty, +\infty), \qquad (4)$$

where $\tilde{\alpha}_i < \tilde{\beta}_i$,

$$\tilde{\alpha}_i = \alpha_i \quad \text{if} \quad \alpha_i > -\infty \quad \text{and}$$
$$\tilde{\beta}_i = \beta_i \quad \text{if} \quad \beta_i < +\infty \quad \text{for} \quad i = 1, \dots, n. \qquad (5)$$

From the definition it follows that $\tilde{K} \subseteq K$ and that $\tilde{K}$ is bounded. Then we can consider the reduced VI: Find $\tilde{x} \in \tilde{K}$ such that

$$\langle G(\tilde{x}), x - \tilde{x} \rangle \ge 0, \quad \forall x \in \tilde{K}, \qquad (6)$$

and the corresponding regularized VI: Find $z^\varepsilon \in \tilde{K}$ such that

$$\langle G(z^\varepsilon) + \varepsilon A_L z^\varepsilon, x - z^\varepsilon \rangle \ge 0, \quad \forall x \in \tilde{K}. \qquad (7)$$

Due to Theorem 1, all the limit points of the sequence $\{z^\varepsilon\}$ will belong to the solution set $\tilde{K}^*$ of the VI (6) under the corresponding assumptions. However, the strict inclusion $K^* \bigcap \tilde{K} \subset \tilde{K}^*$ may prevent convergence to a solution of the initial problem. We now give two sufficient conditions, which are based on (A3), for such convergence.

Theorem 2. *Suppose that (A1)–(A3) are satisfied, where $D = K^*$ and $\tilde{D} \subseteq K^* \bigcap \tilde{K}$ is such that $\tilde{D}$ is nonempty. If the problem (7) has a unique solution z^ε, then the sequence $\{z^{\varepsilon_k}\}$, where $\{\varepsilon_k\} \searrow 0$, has some limit points and all these points are contained in the solution set of the VI (1).*

Proof. Applying Theorem 1 to the VI (6), we see that $\{z^{\varepsilon_k}\}$ has some limit points and all these points belong to $\tilde{K}^*$. Since all the assumptions of Proposition 4 hold, we obtain $\tilde{K}^* = K^* \bigcap \tilde{K}$ and the result follows. ■

Observe that the solution set K^* need not be bounded in the above theorem. However, we can adjust (A3) to such a condition.

Theorem 3. *Suppose that (A1)–(A3) are satisfied, where $\tilde{D} = D$ and D is bounded. Let $\tilde{K}$ be chosen so that*

$$\forall d \in D \bigcap K, \quad d_i \begin{cases} \ge \tilde{\alpha}_i & \textit{if} \quad \alpha_i > -\infty, \\ > \tilde{\alpha}_i & \textit{if} \quad \alpha_i = -\infty, \\ \le \tilde{\beta}_i & \textit{if} \quad \beta_i < +\infty, \\ < \tilde{\beta}_i & \textit{if} \quad \beta_i = +\infty, \end{cases}$$
$$\textit{for} \quad i = 1, \dots, n.$$

If the problem (7) has a unique solution z^ε, then the sequence $\{z^{\varepsilon_k}\}$, where $\{\varepsilon_k\} \searrow 0$, has some limit points and all these points are contained in the solution set of the VI (1).

Proof. Again, applying Theorem 1 to VI (6), we see that $\{z^{\varepsilon_k}\}$ has some limit points and all these points belong to $\tilde{K}^*$. By Lemma 2, $K^* \subseteq K \bigcap D \subseteq \tilde{K}$, and hence $K^* \subseteq \tilde{K}^*$. Suppose that there exists a point $\tilde{x} \in \tilde{K}^* \backslash K^*$. Since $K \bigcap D \subseteq \tilde{K}$, we have $\tilde{x} \in K \bigcap D$. Moreover, there is a point $y \in K \backslash \tilde{K}$ and an index i such that

$$G_i(\tilde{x})(y_i - \tilde{x}_i) < 0.$$

It follows that either $y_i < \tilde{\alpha}_i < \tilde{x}_i$ with $\alpha_i = -\infty$ or $y_i > \tilde{\beta}_i > \tilde{x}_i$ with $\beta_i = +\infty$ since $\tilde{x} \in D \bigcap K$. Then there exists a number $\lambda \in (0, 1)$ such that $\lambda y_i + (1 - \lambda)\tilde{x}_i \in \tilde{K}_i$, and hence

$$G_i(\tilde{x})[\lambda y_i + (1 - \lambda)\tilde{x}_i - \tilde{x}_i] \geq 0,$$

i.e.,

$$\lambda G_i(\tilde{x})(y_i - \tilde{x}_i) \geq 0,$$

so we get a contradiction. Therefore, $K^* = \tilde{K}^*$, and the result follows. ■

Thus, replacing the unbounded VI (1) with a suitably bounded VI (6), which has the same solution set, we can obtain convergence for partial regularization methods.

4. Application to the Walrasian Equilibrium Model

In this section, we apply the results above to a class of general economic equilibrium problems. We now consider a market structure with perfect competition. The model deals with n commodities. Then, given a price vector $p \in \mathbb{R}^n_+$, we can define the value $E(p)$ of the excess demand mapping $E : \mathbb{R}^n_+ \to \mathbb{R}^n$, which is supposed to be single valued. Traditionally (see, e.g., Nikaïdo, 1968), a vector $p^* \in \mathbb{R}^n$ is said to be an equilibrium price vector if it solves the following complementarity problem:

$$p^* \geq 0, \quad E(p^*) \leq 0, \quad \langle p^*, E(p^*) \rangle = 0,$$

or, equivalently, the following VI: Find $p^* \geq 0$ such that

$$\langle -E(p^*), p - p^* \rangle \geq 0, \quad \forall p \geq 0. \tag{8}$$

We denote by E^* the solution set of this problem and determine our model from this standard one. First, unlike the classical Walrasian models, we suppose that each price of a commodity which is involved in the market structure has a lower positive bound and may, in principle, have an upper bound. It follows that the feasible prices are assumed to be contained in the box-constrained set

$$K = \prod_{i=1}^{n} K_i,$$

$$K_i = \{t \in \mathbb{R} \mid 0 < \tau_i^{'} \leq t \leq \tau_i^{''} \leq +\infty, \quad i = 1, \dots, n\}. \tag{9}$$

Next, as usual, the excess demand mapping is represented as follows: $E(p) = B(p) - S(p)$, where B and S are the demand and supply mappings, respectively. Clearly, both of these mappings are also single-valued.

Then the problem of finding an *equilibrium price* can be formulated as the box-constrained VI: Find $p^* \in K$ such that

$$\langle G(p^*), p - p^* \rangle \geq 0, \quad \forall p \in K, \tag{10}$$

where $G = -E$. We denote by K^* the solution set of this problem and recall definitions of some known properties of demand mappings (see, e.g. Nikaïdo, 1968).

Definition 3. A mapping $Q : V \to \mathbb{R}^n$ is said to

(a) satisfy the *gross substitutability* property, if $\partial Q_j / \partial p_i \geq 0$, $j \neq i$;

(b) be *positive homogeneous of the degree* m, if $Q(\alpha x) = \alpha^m Q(x)$ for every $\alpha \geq 0$.

We first consider the following set of assumptions, which are rather usual (e.g., see Nikaïdo, 1968).

(B1) *The excess demand mapping $E : \mathbb{R}^n_> \to \mathbb{R}^n$ is continuously differentiable on $V = \mathbb{R}^n_>$, positive homogeneous of the degree 0, and possesses the gross substitutability property.*

From the gross substitutability of E it follows that

$$\frac{\partial G_i(p)}{\partial p_j} \leq 0, \quad i \neq j.$$

Hence $\nabla G(p)$ is a Z-matrix. Next, since $G_i(p)$ is homogeneous of the degree 0(zero), it follows from the Euler theorem (see, e.g., Nikaïdo, 1968, Lem. 18.4) that

$$\sum_{j=1}^{n} \frac{\partial G_i(p)}{\partial p_j} p_j = 0 \quad \text{for all } i = 1, \dots, n. \tag{11}$$

Applying now Proposition 1, we conclude that $\nabla G(p)$ is an M_0-matrix, and hence G is also a P_0-mapping and we thus have obtained the following assertions:

Lemma 3. *If (B1) holds, then G is a P_0-mapping and $\nabla G(p)$ is an M_0-matrix for each $p \in \mathbb{R}^n_>$.*

On account of (11), the mapping G cannot be a P-mapping. Following the approach of Section 3, we approximate the VI (10) with the perturbed VI: Find $p^\varepsilon \in K$ such that

$$\langle G(p^\varepsilon) + \varepsilon A_L p^\varepsilon, p - p^\varepsilon \rangle \geq 0, \quad \forall p \in K, \tag{12}$$

where $\varepsilon > 0$ is a small parameter, L is a subset of N.

If $\tau_i^{''} < +\infty$ for all $i = 1, \dots, n$ and $L = N$, then, on account of Lemma 3 and Proposition 6, each perturbed VI (12) has a unique solution. Moreover, by Theorem 1, $\{p^\varepsilon\}$ then has some limit points and all these points solve the VI (9), (10).

However, Lemma 3 and Proposition 7 show that the partial regularization approach is also applicable to the initial problem.

Theorem 4. *Suppose that $\tau_i^{''} < +\infty$ for each $i = 1, \dots, n$ and that (B1) holds and that there exists an index set $J \subseteq N$ such that, for each $p \in K$,*

$$\sum_{j \in N \setminus J} \frac{\partial G_i(p)}{\partial p_j} < 0 \quad \text{for } i \in J. \tag{13}$$

Then the problem (12) with $L = N \setminus J$ has a unique solution p^ε, so that the sequence $\{p^{\varepsilon_k}\}$ with $\{\varepsilon_k\} \searrow 0$ has some limit points and all these points solve the VI (9), (10).

Proof. Due to Proposition 1, combining (11) and (13), we conclude that $Q_J(p)$ is an M-matrix. The result follows now from Proposition 7 and Theorem 1. ■

The simplest case corresponds to the singleton, i.e., when $J = N \setminus \{j\}$. This means that the j-th column of the Jacobian $\nabla G(p)$ contains only negative entries with the exception of the diagonal entry. Then $L = \{j\}$, and we can employ the minimal regularization terms.

Under some additional assumptions, the regularization approach can be applied to unbounded equilibrium problems. We introduce the following set of additional assumptions:

(B1′) *(a) For each $i = 1, \dots, n$, the function $E_i : \mathbb{R}^n_> \to \mathbb{R}$ is bounded from below;*

(b) if $\{p_k\} \to p \in \mathbb{R}^n_+ \setminus \mathbb{R}^n_>$, then there exists an index i such that $\lim_{k\to\infty} E_i(p^k) = +\infty$;

(c) (Walras law) for each $p \in \mathbb{R}^n_>$, we have

$$\langle p, E(p) \rangle = 0.$$

These assumptions are also rather standard. Nevertheless, they enable us to obtain existence results and the revealed preference property for solutions of the VI (8).

Proposition 8. *(see, e.g., Nikaïdo, 1968, Sections 18.2 and 18.3) If (B1) and (B1′) are satisfied, then the VI (8) is solvable. Moreover,*

$$\langle p^*, E(p) \rangle > 0,$$

where p^* is a solution to the VI (8) and p is an arbitrary point in $\mathbb{R}^n_> \setminus E^*$.

We now consider the reduced VI: Find $\tilde{p} \in \tilde{K}$ such that

$$\langle G(\tilde{p}), p - \tilde{p} \rangle \geq 0, \quad \forall p \in \tilde{K}, \tag{14}$$

where

$$\tilde{K} = \prod_{i=1}^{n} \tilde{K}_i, \quad \tilde{K}_i = [\tau_i^{'}, \tilde{\tau}_i] \subset (0, +\infty) \tag{15}$$

with $\tau_i^{'} < \tilde{\tau}_i$ and $\tilde{\tau}_i = \tau_i^{''}$ if $\tau_i^{''} < +\infty$ for $i = 1, \dots, n$.

Clearly, the VI (14), (15) is an analogue of VI (4)–(6). Similarly, we can define the regularized VI (7) where $\tilde{K}$ is defined in (15).

Theorem 5. *Suppose that (B1) and (B1′) are satisfied and that there exists an index set $J \subseteq N$ such that, for each $p \in \tilde{K}$, (13) holds. Then the problem (7), (15) with $L = N \setminus J$ has a unique solution z^ε, so that the sequence $\{z^{\varepsilon_k}\}$ with $\{\varepsilon_k\} \searrow 0$ has some limit points, and all these points are solutions to the VI (9), (10).*

Proof. Using an argument similar to that in the proof of Theorem 4, we see that $\{z^{\varepsilon_k}\}$ has some limit points and all these points are solutions to the VI (14), (15). Denote by $\tilde{K}^*$ the solution set of this VI. Since E is positive homogeneous of the degree 0, E^* is a nonempty convex cone due to Proposition 8. Moreover, $E^* \bigcap \tilde{K} \neq \emptyset$ and the condition (A3) is satisfied for the VI (8), where $D = E^*$. Hence, it is satisfied for the VI (9), (10) with $D = E^*$ and $\tilde{D} = K^*$ and for the VI (14), (15) with $D = E^*$ and $\tilde{D} = \tilde{K}^*$. It follows now from Proposition 4 that $K^* = E^* \bigcap K$ and $\tilde{K}^* = E^* \bigcap \tilde{K} = K^* \bigcap \tilde{K}$. The proof is complete. ■

The gross substitutability of demand is also one of the most popular conditions on market structures; see, e.g., (Nikaïdo, 1968) and the references therein. This means that all the commodities in the market are *substitutable* for consumers in the sense that if the price of the i-th commodity increases, then the demand of other commodities does not decrease. Next, the positive homogeneity of the degree 0 of demand is also rather a standard condition. It follows usually from the insatiability of consumers (Manne, 1985; Nikaïdo, 1968). However, we need some other assumptions about supply. We will consider the case where each producer supplies a single commodity. It is possible to consider a more general market structure where there exist consumers with a single commodity demand mapping. Then each S_i may be interpreted as a partial excess supply mapping for the i-th commodity (see Konnov and Volotskaya, 2002). Under these assumptions, there is no loss of generality in supposing that the i-th producer supplies only the i-th commodity. Then the second set of assumptions can be formulated as follows:

(B2) *The demand mapping* $B : \mathbb{R}^n_> \to \mathbb{R}^n$ *is continuously differentiable, positive homogeneous of the degree 0, and possesses the gross substitutability property. The supply mappings* $S_i : \mathbb{R}_> \to \mathbb{R}_+$, $i = 1, \dots, n$, *are monotone and continuously differentiable.*

Then following the proof of Lemma 3, we see that B maintains the properties of E, and hence the assertion of Lemma 3 remains true.

Lemma 4. *If (B2) holds, then* $-\nabla B(p)$ *and* $\nabla G(p)$ *are* M_0*-matrices for each* $p \in \mathbb{R}^n_>$.

It follows that we can apply the same partial regularization approach and that Theorem 4 also remains true.

Corollary 1. *Suppose that* $\tau_i^{''} < +\infty$ *for* $i = 1, \dots, n$ *and that (B2) holds and that there exists an index set* $J \subseteq N$ *such that, for each* $p \in K$,

$$\sum_{j \in N \setminus J} \frac{\partial B_i(p)}{\partial p_j} > 0 \quad \text{for } i \in J.$$

Then the problem (12) with $L = N \setminus J$ *has a unique solution* p^ε, *so that the sequence* $\{p^{\varepsilon_k}\}$ *with* $\{\varepsilon_k\} \searrow 0$ *has some limit points and all these points solve the VI (9), (10).*

The assumptions above can be slightly weakened under additional assumptions about S.

Theorem 6. *Suppose that* $\tau_i^{''} < +\infty$ *for* $i = 1, \dots, n$ *and that (B2) holds, there exists an index set* $J^{'} \subseteq N$ *such that, for each* $p \in K$,

$$\sum_{j \in N \setminus J'} \frac{\partial B_i(p)}{\partial p_j} > 0 \quad \text{for } i \in J', \tag{16}$$

and that there exists an index set $J^{''} \subseteq N$ *such that* $S_i^{'}(p_i) > 0$ *for* $p_i \in [\tau_i^{'}, \tau_i^{''}]$, $i \in J^{''}$. *Then the problem (12) with* $L = N \setminus J$, $J = J^{'} \bigcup J^{''}$ *has a unique solution* p^ε, *so that the sequence* $\{p^{\varepsilon_k}\}$ *with* $\{\varepsilon_k\} \searrow 0$ *has some limit points and all these points solve the VI (9), (10).*

Proof. By Lemma 4, $-\nabla B(p)$ is an M_0-matrix for each $p \in \mathbb{R}^n_>$. With no loss of generality, we suppose that $J = \{1, \dots, k\}$. Then

$$\sum_{j=1}^{n} \frac{\partial B_i(p)}{\partial p_j} p_j = 0 \quad \text{for } i = 1, \dots, k.$$

Hence

$$\sum_{j=1}^{k} \frac{\partial G_i(p)}{\partial p_j} p_j > 0 \quad \text{for } i = 1, \dots, k.$$

Due to Proposition 1, we conclude that $Q_J(p)$ is an M-matrix and the result follows now from Proposition 7 and Theorem 1. ■

Based on Theorem 3, we can also apply the regularization method in the unbounded case.

Theorem 7. *Suppose that (B2) is satisfied, there exists a bounded set* $W \subseteq K$ *such that, for each* $p \in K \setminus W$, *we have*

$$\max_{i=1,\dots,n} \big[S_i(p_i) - B_i(p)\big](p_i - \tau_i^{'}) > 0. \tag{17}$$

Suppose that $\tilde{K}$ *in (15) is chosen so that* $\forall i = 1, \dots, n$, $\forall w \in W$, $w_i < \tilde{\tau}_i$ *if* $\tau_i^{''} = +\infty$, *and there exists an index set* $J^{'} \subseteq N$ *such that, for each* $p \in \tilde{K}$, *(16) holds. Moreover, there exists an index set* $J^{''} \subseteq N$ *such that* $S_i^{'}(p_i) > 0$ *for* $p_i \in [\tau_i^{'}, \tilde{\tau}_i]$, $i \in J^{''}$. *Then the problem (7), (15) with* $L = N \setminus J$, $J = J^{'} \bigcup J^{''}$ *has a unique solution* z^ε, *so that the sequence* $\{z^{\varepsilon_k}\}$ *with* $\{\varepsilon_k\} \searrow 0$ *has some limit points, and all these points are solutions to the VI (9), (10).*

Proof. Following an argument similar to that in Theorem 6, we see that the VI (7), (15) has a unique solution, and that the sequence $\{z^{\varepsilon_k}\}$ has some limit points, and all these points solve the VI (14), (15). Following the proof of Theorem 3, we obtain $\tilde{K}^* = K^*$, i.e., the assertion is true. ■

Condition (17) seems rather natural. It means that for each price vector p with sufficiently large components there exists at least one commodity among the corresponding indices such that its supply exceeds its demand. Note that Theorem 7 also states the existence result of the source equilibrium problem.

5. Application to the Oligopolistic Equilibrium Model

In this section, we consider an oligopolistic market structure in which n firms supply a homogeneous product. Let $p(\sigma)$ denote the inverse demand function, that is, the price at which consumers will purchase a quantity σ. If each i-th firm supplies q_i units of the product, then the total supply in the market is defined by

$$\sigma_q = \sum_{i=1}^{n} q_i.$$

If we denote by $f_i(q_i)$ the i-th firm's total cost of supplying q_i units of the product, then the i-th firm's profit is defined by

$$\varphi_i(q) = q_i p(\sigma_q) - f_i(q_i). \tag{18}$$

As usual, each output level is nonnegative, i.e., $q_i \geq 0$ for $i = 1, \dots, n$. In addition to that, we suppose that it can be in principle bounded from above, i.e., there exist numbers $\beta_i \in (0, +\infty]$ such that $q_i \leq \beta_i$ for $i = 1, \dots, n$. In order to define a solution in this market structure, we use the Nash-Cournot equilibrium concept for noncooperative games (Okuguchi and Szidarovszky, 1990).

Definition 4. A feasible vector of output levels $q^* = (q_1^*, q_2^*, \dots, q_n^*)$ for firms $1, \dots, n$ is said to constitute a *Nash-Cournot equilibrium* solution for the oligopolistic market, provided that q_i^* maximizes the profit function φ_i of the i-th firm over $[0, \beta_i]$ given that the other firms produce quantities $q_j^*, j \neq i$, for each $j = 1, \dots, n$.

That is, for $q^* = (q_1^*, q_2^*, \dots, q_n^*)$ to be a Nash-Cournot equilibrium, q_i^* must be an optimal solution to the problem

$$\max_{0 \leq q_i \leq \beta_i} \to \{q_i p(q_i + \sigma_i^*) - f_i(q_i)\}, \tag{19}$$

where $\sigma_i^* = \sum_{j=1, j \neq i}^{n} q_j^*$ for each $i = 1, \dots, n$. This problem can be transformed into an equivalent VI of the form (1) if each profit function φ_i in (18) is concave in q_i. This assumption conforms to the usually accepted economic behavior, and implies that (19) is a concave maximization problem. More precisely, throughout this section we suppose that *the price function $p(\sigma)$ is nonincreasing and twice continuously differentiable and that the industry revenue function $\mu(\sigma) = \sigma p(\sigma)$ is concave for $\sigma \geq 0$, $f_i(q_i)$ is convex and twice continuously differentiable for $i = 1, \dots, n$.* These assumptions imply concavity in q_i of each profit function $q_i p(\sigma_q) - f_i(q_i)$. Next, we set $V = \mathbb{R}_+^n$,

$$K = \prod_{i=1}^{n} K_i, K_i = \{t \in \mathbb{R} \mid 0 \leq t \leq \beta_i \leq +\infty\},$$
$$i = 1, \dots, n. \tag{20}$$

Under the assumptions above, we can define the single-valued mappings $G : \mathbb{R}_+^n \to \mathbb{R}^n$ and $F : \mathbb{R}_+^n \to \mathbb{R}^n$ with components $G_i(q) = -p(\sigma_q) - q_i p'(\sigma_q)$ and $F_i(q_i) = f_i'(q_i)$, respectively. Then (see, e.g., Okuguchi and Szidarovszky, 1990), the problem of finding the Nash-Cournot equilibrium in the oligopolistic market can be rewritten as the following VI: Find $q^* \in K$ such that

$$\langle G(q^*) + F(q^*), q - q^* \rangle \geq 0, \quad \forall q \in K. \tag{21}$$

This problem is nothing but a VI of the form (1). We denote by K^* the solution set of the problem (21), (20).

Lemma 5. *There holds* $\det Q_L(q) = [-(k+1)p'(\sigma_q) - (\sum_{i=1}^{k} q_i) p''(\sigma_q)](-p'(\sigma_q))^{k-1}$ *for* $L = \{1, \dots, k\}$.

Proof. For brevity, we set $\alpha_i = -p'(\sigma_q) - q_i p''(\sigma_q)$ and $\beta = -p'(\sigma_q)$. Thus

$$\det Q_L(q) = \begin{vmatrix} \beta + \alpha_1 & \alpha_1 & \alpha_1 & \dots & \alpha_1 \\ \alpha_2 & \beta + \alpha_2 & \alpha_2 & \dots & \alpha_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_k & \alpha_k & \alpha_k & \dots & \beta + \alpha_k \end{vmatrix}$$

for $L = \{1, \dots, k\}$.

Adding all the rows to the first one and subtracting the first column from the others yields

$$\det Q_L(q) = \begin{vmatrix} \beta + \sum_{i=1}^{k} \alpha_i & 0 & 0 & \dots & 0 \\ \alpha_2 & \beta & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_k & 0 & 0 & \dots & \beta \end{vmatrix} x$$

$$= \beta^{k-1}(\beta + \sum_{i=1}^{k} \alpha_i) \quad \text{for } L = \{1, \dots, k\}.$$

Hence

$$\det Q_L(q) = \Big[-(k+1)p'(\sigma_q) - \Big(\sum_{i=1}^{k} q_i\Big) p''(\sigma_q) \Big]$$
$$\times (-p'(\sigma_q))^{k-1} \quad \text{for} \quad L = \{1, \dots, k\}.$$

■

Proposition 9. $\nabla G(q)$ *is a* P_0*-matrix for every* $q \in V$.

Proof. By assumption, $p'(\sigma) \leq 0$. Fix $q \in K$. If $p''(\sigma_q) \leq 0$, then from Lemma 5 it follows that $\det Q_L(q) \geq 0$. Otherwise, if $p''(\sigma_q) \geq 0$, we see that $\det Q_L(q) = (-p'(\sigma_q))^{k-1}[-(k-1)p'(\sigma_q) - \mu''(\sigma_q) + (\sum_{i=k+1}^{n} q_i) p''(\sigma_q)]$. Since $\mu''(\sigma_q) \leq 0$, we obtain $\det Q_L(q) \geq 0$ and the result follows. ■

Thus, the problem of finding the *Nash-Cournot equilibrium* can be approximated with the regularized VI: Find $q^* \in K$ such that

$$\langle G(q^\varepsilon) + F(q^\varepsilon) + \varepsilon A_L q^\varepsilon, q - q^\varepsilon \rangle \geq 0, \quad \forall q \in K, \tag{22}$$

where $\varepsilon > 0$ is a parameter.

Theorem 8. *Suppose that $\beta_i < +\infty$ for $i = 1, \dots, n$ and that there exists an index set $J \subseteq N$ such that $f_i''(q_i) > 0$ for $q_i \in [0, \beta_i]$ and $i \in J$. Then the problem (22) with $L = N \backslash J$ has a unique solution q^ε, so that the sequence $\{q^{\varepsilon_k}\}$ with $\{\varepsilon_k\} \searrow 0$ has some limit points and all these points solve the VI (21), (20).*

Proof. By Proposition 9, G is a P_0-mapping. Since $\nabla F(q) + \varepsilon A_L$ is now a diagonal positive definite matrix, $G + F + \varepsilon A_L$ is a P-mapping and the VI (22) has a unique solution on account of Proposition 3. The result now follows from Theorem 1. ∎

By utilizing the additional coercivity condition (Kolstad and Mathiesen, 1987, Def. 4), we can apply the same regularization approach to the VI (21) with the unbounded feasible set K defined in (20).

Definition 5. An industry output is said to be *bounded* if there exists a compact subset P of $\mathbb{R}^n_+$ such that for $\tilde{q} \in \mathbb{R}^n_+ \backslash P$ we have

$$G_i(\tilde{q}) + F_i(\tilde{q}) = f_i^{'}(\tilde{q}_i) - p(\sigma_{\tilde{q}}) - \tilde{q}_i p^{'}(\sigma_{\tilde{q}}) > 0, \quad i = 1, \dots, n. \tag{23}$$

Without loss of generality we suppose that $0 \in P$. Let us now consider the reduced VI: Find $\tilde{p} \in \tilde{K}$ such that

$$\langle G(\tilde{p}) + F(\tilde{p}), p - \tilde{p} \rangle \geq 0, \quad \forall p \in \tilde{K}, \tag{24}$$

where

$$\tilde{K} = \prod_{i=1}^{n} [0, \tilde{\beta}_i], \ 0 < \tilde{\beta}_i < +\infty \quad \text{and } \tilde{\beta}_i = \beta_i \quad \text{if } \beta_i < +\infty \tag{25}$$

for $i = 1, \dots, n$. We denote by $\tilde{K}^*$ the solution set of the VI (24), (25). Similarly, we consider the corresponding regularized problem: Find $z^\varepsilon \in \tilde{K}$ such that

$$\langle G(z^\varepsilon) + F(z^\varepsilon) + \varepsilon A_L z^\varepsilon, p - z^\varepsilon \rangle \geq 0, \quad \forall p \in \tilde{K}. \tag{26}$$

Now we can establish the convergence result for the regularization method based on the VI (26), (25).

Theorem 9. *Suppose that an industry output is bounded so that* $\forall i = 1, \dots, n$, $\forall p \in K \bigcap P$, $p_i < \tilde{\beta}_i$ *if* $\beta_i = +\infty$, *and that there exists an index set* $J \subseteq N$ *such that* $f_i^{''}(q_i) > 0$ *for* $q_i \in [0, \tilde{\beta}_i]$ *and* $i \in J$. *Then the problem (25), (26) with* $L = N \backslash J$ *has a unique solution* z^ε, *so that the sequence* $\{z^{\varepsilon_k}\}$ *with* $\{\varepsilon_k\} \searrow 0$ *has some limit points and all these points solve the VI (21), (20).*

Proof. Again, similarly to the proof of Theorem 8, we conclude that the VI (25), (26) has a unique solution and that $\{z^{\varepsilon_k}\}$ has some limit points so that all these points solve the VI (24), (25). Since the industry output is bounded, (23) implies (A3) with $D = P$ and $\tilde{D} = \{0\}$. Applying now an argument similar to that in the proof of Theorem 3, we obtain $K^* = \tilde{K}^*$, and hence the assertion is true. ∎

Observe that Theorem 9 also establishes an existence result for the VI (21), (20).

6. Concluding Remarks

In this paper, we have considered partial Browder-Tikhonov type regularization techniques for variational inequality problems with a P_0 cost mapping and a box-constrained feasible set. We have presented perfectly and nonperfectly competitive economic equilibrium models which are involved in this class of VIs and specialize regularization methods for these problems.

The general P_0 properties are not sufficient for providing rapid convergence of iterative solution methods. If the cost mapping does not possess strengthened P-type properties, it is possible to apply the regularization approach to these problems and obtain such properties for perturbed VIs. Therefore, one can solve various economic equilibrium problems with the help of the usual iterative methods.

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