OPTIMAL RANDOM SAMPLING FOR SPECTRUM ESTIMATION IN DASP APPLICATIONS

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In this paper we analyse a class of DASP (Digital Alias-free Signal Processing) methods for spectrum estimation of sampled signals. These methods consist in sampling the processed signals at randomly selected time instants. We construct estimators of Fourier transforms of the analysed signals. The estimators are unbiased inside arbitrarily wide frequency ranges, regardless of how sparsely the signal samples are collected. In order to facilitate quality assessment of the estimators, we calculate their standard deviations. The optimal sampling scheme that minimises the variance of the resulting estimator is derived. The further analysis presented in this paper shows how sampling instant jitter deteriorates the quality of spectrum estimation. A couple of numerical examples illustrate the main thesis of the paper.

Keywords: digital alias-free signal processing, random sampling, spectral analysis, optimal sampling

1. Introduction

Traditional Digital Signal Processing (DSP) consists in sampling the processed signals at evenly distributed time instants. Nonuniform sampling is often perceived as unwelcome nuisance that appears when signals are measured on “when available” rather than “when needed” bases as it frequently happens in astronomy, medicine or geosciences. It also has to be dealt with when, e.g., owing to some imperfections in the data acquisition system, jitter is added to sampling instants, or when samples are lost from otherwise perfectly uniformly distributed sequences. In all these and other similar cases nonuniform sampling is seen as a problem that needs to be tackled. There are a number of papers that deal with adverse effects of nonuniform sampling. The classical Lomb periodogram (Lomb, 1976) and its modifications (Scargle, 1982; 1989) are efficient tools for finding a sinusoidal component in signals sampled at arbitrary time instants. A more general problem of reconstructing waveforms and/or spectra of nonuniformly sampled signals has been addressed by many authors over the last forty years. Interesting examples comprise (Chen and Allebach, 1987; Kida and Mochizuki, 1992; Wingham, 1992; Yen, 1956), where various aspects of that topic, including a search for optimal solutions, are considered. A method of measuring point-wise quality of signal reconstruction from arbitrarily distributed finite sets of samples was formulated in (Tarczyński, 1997; Tarczyński and Cain, 1997). Algorithms for recovering signals when some samples from uniform sequences are missing were proposed in (Ferreira, 1992). Topics related to FIR and to some extent IIR filtering of nonuniformly sampled signals have been investigated. The question of optimal weighted-least-squares-type filtering of signals in which some of the uniformly distributed samples are lost is tackled in (Tarczyński and Valimaki, 1996). FIR filtering of nonuniformly sampled signals is investigated in (Tarczyński *et al.*, 1997).

The theory of nonuniformly sampled signals is by no means confined to the fundamental signal processing topics mentioned above. It is being developed in a wider context attracting research activities not only from the signal processing community, but also mathematics, information systems, natural sciences and others. A few monographs (Benedetto and Ferreira, 2001; Marvasti, 1987; 2001) give overviews of the topic.

Nonuniform sampling does not have to be merely a source of adversity in DSP. It turns out that, if properly used, it may help to resolve a range of engineering problems which, because of either technical or economic constraints, cannot be tackled with the use of the classical DSP techniques. One of the features of nonuniform sampling is that for many signal processing problems it facilitates DSP solutions in much wider frequency ranges than would be possible when similar-rate uniform sampling were used. Here we refer to the techniques exploiting this effect as Digital Alias-free Signal Processing (DASP). This paper investigates some DASP methodologies of estimating the spectrum of the sampled signals.

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2. Digital Alias-free Signal Processing

Aliasing is one of the main factors that limit the bandwidth of uniform-sampling-based DSP. For any two frequencies whose sum or difference is an integer multiple of the sampling frequency, it is possible to construct a pair of sinusoids that, after sampling, are not distinguishable from each other since their samples are identical. Such pairs of signals are called aliases. This term is sometimes extended to signal frequencies. In order to avoid the ambiguity associated with aliasing, the sampling rate is normally chosen so that there are no aliases in the set of frequencies within which the signals are processed. The traditional solution is to sample the signals at or above the Nyquist rate, i.e., at least twice the highest frequency present in their spectra. This approach is recommendable for baseband signals. However, for bandpass and multiband signals it leads to excessive sampling well above the Landau rate (Landau, 1967)—the theoretically lowest sampling rate that still permits perfect reconstruction of the sampled signal. The Landau rate equals twice the effective bandwidth of the signal, which is defined as the total width of bandpass components. In many cases sampling at the Landau rate requires that the samples be taken at nonuniformly distributed time instants. If the Spectrum Support Function (SSF) of the signal is known fairly accurately before sampling starts, then it is possible to construct a Periodic Nonuniform Sampling (PNS) scheme that permits perfect reconstruction of the signal while the average sampling rate is maintained arbitrarily close to the Landau rate (Herley and Wong, 1999; Venkataramani and Bresler, 2001). A more challenging task is to deal with signals whose exact SSF is not known. The construction of sub-Nyquist sampling schemes for such cases is the domain of DASP. Of course, complex sampling schemes are not very useful if they are not accompanied by suitable signal processing algorithms. Therefore, DASP is not confined to sampling techniques. It provides solutions that encompass all practical problems associated with this approach.

Although DASP can be described as an emerging methodology, its main ideas are not entirely new for DSP. One of the first DASP-type approaches was reported in (Shapiro and Silverman, 1960). In their paper, as well as in (Beutler, 1970; Masry, 1978), the authors argued that using a suitably chosen but arbitrarily slow random sampling it is possible to obtain Power Spectral Density (PSD) of ergodic signals. In these cases sampling rates could be not only sub-Nyquist, but also sub-Landau. This result does not contradict what we said earlier about the Landau rate. PSD does not provide full information about the signal. In particular, it does not allow reconstructing the signal (strictly speaking—signal realisation). This observation conveniently indicates that not all signal processing tasks need full reconstructability of the processed signals.

More practice-orientated approaches towards DASP, which include successful implementations, were initiated in (Bilinskis and Mikelsons, 1992). Their approaches are based mostly on additive random sampling. They target spectrum estimation and waveform reconstruction problems for signals with discrete spectra. A possibility of reconstructing the spectra of multi-band signals whose exact SSF is unknown is indicated in (Feng and Bresler, 1996). Unlike Bilinskis, they use deterministic PNS schemes, rather than random sampling. They refer to their method as blind-spectrum estimation.

In this paper we further refine the ideas that were originally presented in (Tarczyński and Allay, 2004) on DASP methods for estimating the spectrum of the processed signals. We consider here a class of methods for estimating the spectrum from finite sets of randomly distributed samples. In the previous paper only two specific sampling schemes were considered. Here we investigate the whole class of similar sampling schemes and look for the optimal one, which provides the most accurate estimate of the signal spectrum.

3. Unbiased Spectrum Estimators

In this paper we define the spectrum of a deterministic signal $x(t)$ as its Fourier transform

$$X(f) = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi f t)\,\mathrm{d}t. \tag{1}$$

Since the practical use of (1) is obstructed by the need of the availability of the analysed signal over an infinitely long interval, the expression (1) is often replaced with a windowed version:

$$X_W(f) = \int_0^T w(t)x(t)\exp(-j2\pi f t)\,\mathrm{d}t, \tag{2}$$

where T is the length and $w(t)$ the shape of the window. Our task is to use samples of the signal $x(t)$ to estimate the spectrum (2).

As we have mentioned earlier, we are going to collect signal samples at randomly selected time instants. There are a few methods of generating random sampling schemes. Jitter and additive random sampling are among the most popular ones (Bilinskis and Mikelsons, 1992). The sampling schemes exploited in this paper belong to a class that we call totally random sampling schemes. The sampling instants t_n are identically distributed random variables independent of each other. Their probability density function (p.d.f.) $p(t)$ takes nonzero values only inside the interval $[0, T]$. Therefore,

$$\int_0^T p(t)\,\mathrm{d}t = 1. \tag{3}$$

The spectrum (2) of $x(t)$ is estimated using the following formula:

$$X_e(f) = \frac{1}{N} \sum_{n=1}^{N} x(t_n) v(t_n) \exp(-j2\pi f t_n), \tag{4}$$

where N is the number of signal samples. The function $v(t)$ is selected in such a way that $X_e(f)$ is an unbiased estimate of $X(f)$. The expected value of $X_e(f)$ can be calculated as follows:

$$\begin{aligned} E\left\{X_e(f)\right\} &= \frac{1}{N} \sum_{n=1}^{N} \int_0^T x(t) v(t) p(t) \exp(-j2\pi f t)\, \mathrm{d}t \\ &= \int_0^T x(t) v(t) p(t) \exp(-j2\pi f t)\, \mathrm{d}t. \end{aligned} \tag{5}$$

By imposing

$$v(t) = \frac{w(t)}{p(t)} \tag{6}$$

we get

$$\begin{aligned} E\left\{X_e(f)\right\} &= \int_0^T x(t) w(t) \exp(-j2\pi f t)\, \mathrm{d}t \\ &= X_W(f). \end{aligned} \tag{7}$$

Therefore, when (6) is satisfied, $X_e(f)$ is an unbiased estimator of $X_W(f)$. This observation is true for all frequencies f and is not affected in any way by the average density of samples (sampling frequency) $f_S = N/T$.

Now we are going to investigate the accuracy of the estimator (4). Since $X_e(f)$ is an unbiased estimator, its standard deviation $\sigma_{X_e}(f)$ is a good indicator of the accuracy of spectrum estimation. The variance can be calculated using $\sigma^2_{X_e}(f) = E\{|X_e(f)|^2\} - |X_W(f)|^2$. The first term on the right-hand side is given by

$$\begin{aligned} E\left\{|X_e(f)|^2\right\} &= \frac{1}{N^2} \sum_{n=1}^{N} \sum_{k=1}^{N} E\Big\{x(t_n) v(t_n) \exp(-j2\pi f t_n) \\ &\quad \times x(t_k) v(t_k) \exp(j2\pi f t_k)\Big\}. \end{aligned}$$

Treating separately the cases $n = k$ and $n \neq k$, we get

$$\begin{aligned} E\left\{|X_e(f)|^2\right\} &= \frac{1}{N^2} \Bigg[\sum_{n=1}^{N} E\Big\{x^2(t_n) v^2(t_n)\Big\} \\ &\quad + \sum_{n=1}^{N} \sum_{\substack{k=1 \\ k\neq n}}^{N} E\Big\{x(t_n) v(t_n) \exp(-j2\pi f t_n)\Big\} \\ &\quad \times E\Big\{x(t_k) v(t_k) \exp(j2\pi f t_k)\Big\} \Bigg]. \end{aligned}$$

This leads to

$$\begin{aligned} E\left\{|X_e(f)|^2\right\} &= \frac{1}{N} \int_0^T x^2(t) v^2(t) p(t)\, \mathrm{d}t + \frac{N-1}{N} |X_W(f)|^2. \end{aligned} \tag{8}$$

Substituting (6) in to (8), we get

$$\begin{aligned} E\left\{|X_e(f)|^2\right\} &= \frac{1}{N} \int_0^T x^2(t) \frac{w^2(t)}{p(t)}\, \mathrm{d}t + \frac{N-1}{N} |X_W(f)|^2. \end{aligned}$$

Let

$$E_S = \int_0^T x^2(t) \frac{w^2(t)}{p(t)}\, \mathrm{d}t \tag{9}$$

be the weighted energy of the signal. The variance of the estimator can be expressed as

$$\sigma^2_{X_e}(f) = \frac{E_S - |X_W(f)|^2}{N}, \tag{10}$$

and its standard deviation as

$$\sigma_{X_e}(f) = \sqrt{\frac{E_S - |X_W(f)|^2}{N}}. \tag{11}$$

An interesting observation resulting from this analysis is that the spectrum estimator is more accurate at frequencies where the signal is stronger. The largest errors are likely to occur at frequencies where the spectrum of the signal is zero. The standard deviation of the spectrum estimator never exceeds

$$\sigma_{X_e,\max} = \sqrt{\frac{E_S}{N}}. \tag{12}$$

4. Optimal Sampling Scheme

It follows from (4) and (6) that for a given window $w(t)$ and a totally random sampling scheme characterised by $p(t)$, the unbiased estimator of the signal spectrum is

$$X_e(f) = \frac{1}{N} \sum_{n=1}^{N} x(t_n) \frac{w(t_n)}{p(t_n)} \exp(-j2\pi f t_n). \tag{13}$$

Here, $p(t)$ can be selected arbitrarily except that it should not be zero inside the interval $[0, T]$. Two special cases were discussed in (Tarczyński and Allay, 2004) when $p(t) = 1/T$ and $p(t) = w(t) \Big/ \int_0^T w(\tau)\, \mathrm{d}\tau$. Here we consider all admissible shapes of $p(t)$ and look for the one that minimises the standard deviation $\sigma_{X_e}(f)$. Note that out of the three variables, which according to (11) influence $\sigma_{X_e}(f)$: E_S, N and $X_W(f)$, only E_S is affected by the selection of $p(t)$. Therefore, when choosing $p(t)$, we can aim at minimising E_S, rather than $\sigma_{X_e}(f)$. Clearly, we accept only those solutions to the optimisation problem that satisfy $p(t) \geq 0$. Using (9) as the cost and (3) as the equality constraint, we construct the Lagrange function

$$L(p(t),\lambda) = \int_0^T x^2(t)\frac{w^2(t)}{p(t)}\,\mathrm{d}t + \lambda\left(\int_0^T p(t)\,\mathrm{d}t - 1\right). \tag{14}$$

By calculating the functional derivative (Weisstein, 2003) of (14) with respect to $p(\tau)$ and equating it to zero, or alternatively by applying to (14) the Euler-Lagrange condition, we get

$$\frac{\partial L(p(t),\lambda)}{\partial p(\tau)} = -\frac{x^2(\tau)w^2(\tau)}{p^2(\tau)} + \lambda = 0.$$

The nonnegative solution to this equation is given by

$$p(\tau) = \frac{|x(\tau)w(\tau)|}{\sqrt{\lambda}}.$$

Substituting this expression into (3) and solving this for λ, we find out that

$$\lambda = \left[\int_0^T |x(t)w(t)|\,\mathrm{d}t\right]^2.$$

Hence

$$p(t) = \frac{|x(t)w(t)|}{\int_0^T |x(\tau)w(\tau)|\,\mathrm{d}\tau}. \tag{15}$$

The formula (15) gives the optimal p.d.f. of the sampling instants. The standard deviation of the optimal estimator is given by

$$\sigma_{X_e,\mathrm{optim}}(f) = \sqrt{\frac{1}{N}\left\{\left[\int_0^T |x(t)w(t)|\,\mathrm{d}t\right]^2 - |X_W(f)|^2\right\}}. \tag{16}$$

It is quite unfortunate, though not completely unexpected, that the optimal p.d.f. (15) is a function of the analysed signal itself. Since in practical applications $x(t)$ is rarely known at the stage of designing the sampling sequence, we suggest that approximated versions of (15) be used. For example, in some applications (like radar or digital communications), signals are transmitted in a form of pulses for which the shapes of bounding envelopes are known. In such cases the envelopes, rather than the signals itself, could be used to obtain $p(t)$. A numerical example in Section 6 of this paper shows a use of an envelope instead of a signal. If no prior information about the processed signal is available, then we recommend that in (15) $x(t)$ be replaced with a constant value. Then only the shape of the window $w(t)$ is used to calculate $p(t)$.

5. Effect of Input Clock Jitter on the Quality of Spectrum Estimation

So far we have assumed that when using (13) we know exactly the time instants at which the signal samples were collected. In many practical applications, particularly when processing high frequency signals, this assumption may not be satisfied with sufficient accuracy. Let t_n be a nominal sampling instant and τ_n an actual sampling instant. The difference between both sequences $\varepsilon_n = \tau_n - t_n$ is referred to as input clock jitter or simply jitter. We assume that ε_n are identically distributed random variables independent of each other and of the nominal sampling instants t_n. The probability density function of the jitter is denoted by $p_\varepsilon(\varepsilon)$. As we will see later, in the presence of jitter it is impossible to find a weight function $v(t)$ such that any estimator in the like of (4) is unbiased. In order to avoid the bias, we have to use a weight that is a function of time and frequency $\hat{v}(t,f)$. This gives us a new estimator:

$$\hat{X}_e(f) = \frac{1}{N}\sum_{n=1}^{N} x(\tau_n)\hat{v}(t_n,f)\exp(-j2\pi f t_n). \tag{17}$$

Its expected value is

$$E\left[\hat{X}_e(f)\right] = \int_{-\infty}^{\infty}\int_0^T x(t+\varepsilon)\hat{v}(t,f) \times \exp(-j2\pi f t)p(t)p_\varepsilon(\varepsilon)\,\mathrm{d}t\,\mathrm{d}\varepsilon.$$

Let us substitute $t = \tau - \varepsilon$. Then we have

$$E\left[\hat{X}_e(f)\right] = \int_{-\infty}^{\infty} p_\varepsilon(\varepsilon)\exp(j2\pi f\varepsilon)\int_\varepsilon^{T+\varepsilon} x(\tau)\exp(-j2\pi f\tau) \times p(\tau-\varepsilon)\hat{v}(\tau-\varepsilon,f)\,\mathrm{d}\tau\,\mathrm{d}\varepsilon.$$

When the jitter is sufficiently small, we get

$$\int_\varepsilon^{T+\varepsilon} x(\tau)\exp(-j2\pi f\tau)p(\tau-\varepsilon)\hat{v}(\tau-\varepsilon,f)\,\mathrm{d}\tau \cong \int_0^T x(\tau)\exp(-j2\pi f\tau)p(\tau)\hat{v}(\tau,f)\,\mathrm{d}\tau.$$

Therefore,

$$E\left[\hat{X}_e(f)\right] = \int_0^T x(\tau)\exp(-j2\pi f\tau)\hat{v}(\tau,f)p(\tau)\chi_\varepsilon(2\pi f)\,\mathrm{d}\tau, \tag{18}$$

where

$$\chi_\varepsilon(\omega) = \int_{-\infty}^{\infty} p_\varepsilon(\varepsilon) \exp(j\omega\varepsilon)\,\mathrm{d}\varepsilon$$

is the characteristic function of the jitter. The estimator (18) is unbiased if $w(t) = \hat{v}(t,f)p(t)\chi_\varepsilon(2\pi f)$. Therefore the weight function should be

$$\hat{v}(t,f) = \frac{w(t)}{p(t)\chi_\varepsilon(2\pi f)}. \tag{19}$$

Substituting (19) into (17), we get

$$\hat{X}_e(f) = \frac{1}{N\chi_\varepsilon(2\pi f)} \sum_{n=1}^{N} x(\tau_n)\frac{w(t_n)}{p(t_n)} \exp(-j2\pi f t_n). \tag{20}$$

To calculate the standard deviation of $\hat{X}_e(f)$, we start with deriving the expected value

$$E\left\{\left|\hat{X}_e(f)\right|^2\right\} = \frac{\sum\limits_{n=1}^{N} E\left\{x^2(\tau_n)w^2(t_n)/p^2(t_n)\right\}}{N^2\left|\chi_\varepsilon(2\pi f)\right|^2} - \frac{N-1}{N}\left|X_W(f)\right|^2.$$

For sufficiently small jitter, we can use the following approximation:

$$\int_0^T x^2(t+\varepsilon)\frac{w^2(t)}{p(t)}\,\mathrm{d}t \cong \int_0^T x^2(t)\frac{w^2(t)}{p(t)}\,\mathrm{d}t.$$

Hence

$$\begin{aligned} E\left\{x^2(\tau_n)\frac{w^2(t_n)}{p^2(t_n)}\right\} &= \int_{-\infty}^{\infty} p_\varepsilon(\varepsilon) \int_0^T x^2(t+\varepsilon)\frac{w^2(t)}{p(t)}\,\mathrm{d}t\,\mathrm{d}\varepsilon \\ &\cong \int_{-\infty}^{\infty} p_\varepsilon(\varepsilon)\,\mathrm{d}\varepsilon \int_0^T x^2(t)\frac{w^2(t)}{p(t)}\,\mathrm{d}t \\ &= E_S. \end{aligned}$$

Finally,

$$\begin{aligned} \sigma^2_{\hat{X}_e}(f) &= E\left\{\left|\hat{X}_e(f)\right|^2\right\} - \left|X(f)\right|^2 \\ &= \frac{E_S\big/\left|\chi_\varepsilon(2\pi f)\right|^2 - \left|X_W(f)\right|^2}{N}. \end{aligned} \tag{21}$$

Similarly to the jitter-free case, out of the few terms that shape the variance (21), only E_S is affected by $p(t)$. The optimal $p(t)$ is given by (15). Consequently, the variance of the optimal estimator is

$$\hat{\sigma}_{X_e,\mathrm{optim}}(f) = \sqrt{\frac{1}{N}\left\{\frac{\left[\int\limits_0^T |x(t)w(t)|\,\mathrm{d}t\right]^2}{\left|\chi_\varepsilon(2\pi f)\right|^2} - \left|X_W(f)\right|^2\right\}}. \tag{22}$$

6. Numerical Examples

In this section we present two numerical examples that illustrate the main concepts advocated in this paper. We estimate the spectrum of a signal whose waveform is described by $x(t) = \exp(-10^6 t)c(t)$. Here $c(t)$ is a narrow-band signal whose instantaneous amplitude is approximately constant inside the observation window of the length $T = 10\mu s$. We assume that the spectrum (2) is to be calculated with the use of a rectangular window $w(t) = 1$. If we were to use (15) to select $p(t)$, we would need to know the exact shape of $x(t)$. In practical analyses this is rarely the case. Therefore we use an approximate version of the formula that exploits the fact that we know the envelope of the signal waveform:

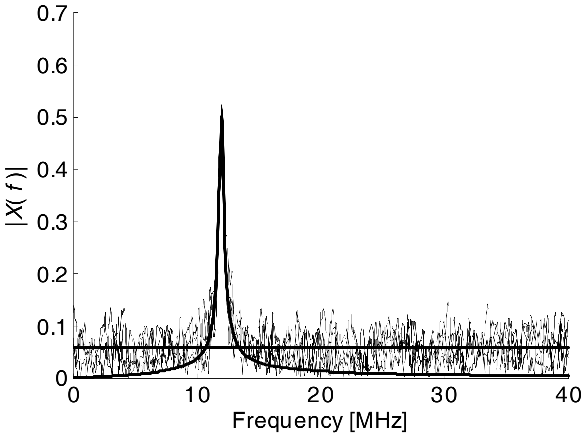
$$p_1(t) = 10^6 \frac{\exp(-10^6 t)}{1-\exp(-10)}. \tag{23}$$

We compare the results of spectrum estimation obtained with the use of (23) with the case when the p.d.f. of sampling instants is distributed uniformly inside the observation window:

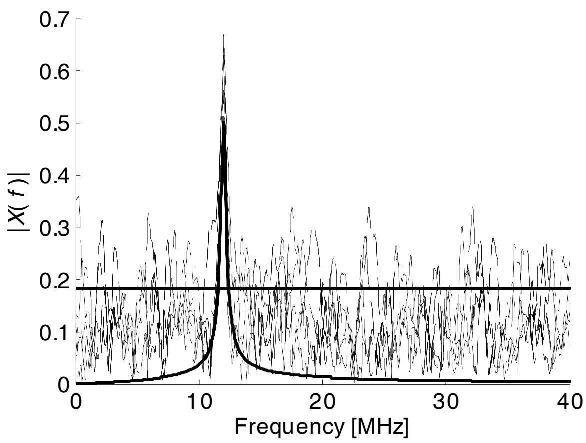
$$p_2(t) = 10^5. \tag{24}$$

In each of the two experiments we collect 150 samples of the signal. The resultant average sample rate is $f_S = 15\,\mathrm{MHz}$. In uniform-sampling-based DSP such a rate would allow performing spectral analysis only up to 7.5 MHz. In our experiment we can set the bandwidth as wide as we wish. There is no natural limit. In the presented simulations we have arbitrarily chosen the bandwidth of 40 MHz. Figure 1 shows the magnitudes of the estimated spectra for both approaches. We also show the magnitude of the target spectrum $X_W(f)$ and the level of the maximum standard deviation $\sigma_{X_e,\max}$ (12). It is visible from the plots that by using information about the distribution of the signal power inside the observation window we can design a system that provides more accurate estimates of the signal spectrum. Additionally, in Fig. 2, we present the results of estimating the spectrum of signal using uniform sampling with $f_S = 15\,\mathrm{MHz}$. The results are clearly affected by aliasing, and hence then are of little practical value for the user.

In the next example we show how input clock jitter affects the quality of spectral analysis. We use the same scenario as before except that now we impose jitter on sampling instants. The jitter has a triangular p.d.f., zero mean and the standard deviation $0.01\,\mu$s. We estimate the spectrum twice. In the first case we ignore the jitter, letting the estimate be biased. In the second case we correct the results as recommended by (17) and (19). Figure 3 shows simulation results for both cases. For the sake of better resolution, we show the plots only for the frequencies between 10 and 20 MHz. It can be seen that the use



(a)



(b)

Fig. 1. DASP Spectrum estimation: (a) p.d.f. described by (23) and (b) by (24). In each case five independent experiments have been simulated (dashed lines). Thick continuous lines show the target spectrum (2) and the maximum standard deviation (12).

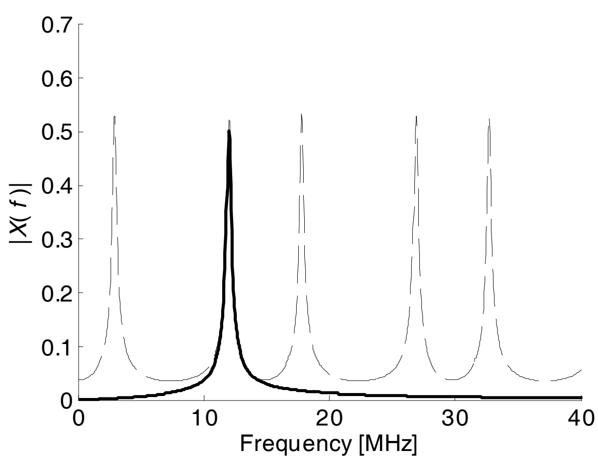
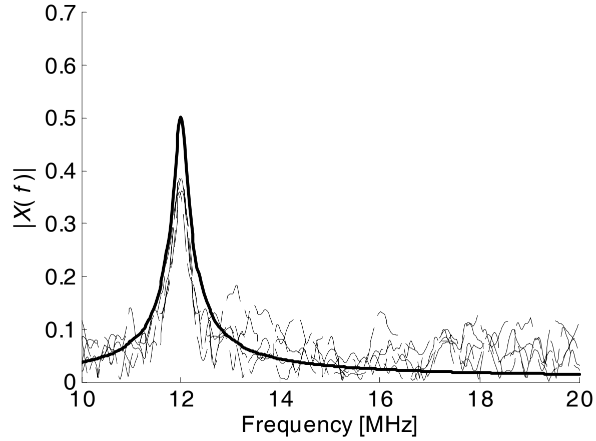
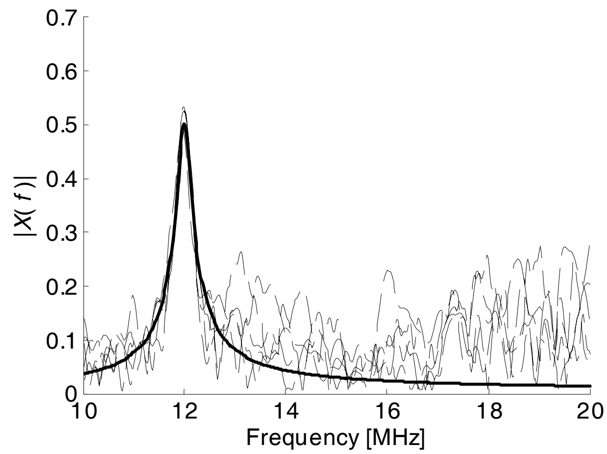


Fig. 2. Spectrum estimation from low-rate uniformly sampled data.



(a)



(b)

Fig. 3. Spectrum estimation from jittered data. In each case five independent experiments have been simulated (dashed lines). Thick continuous lines show the target spectrum (2). (a) Biased estimation when the jitter is ignored, (b) unbiased estimation when the jitter is compensated for.

of the weight (19) removes the bias, but at the price of increased estimation errors, particularly at higher frequencies. This observation is not surprising. If we compare the weights (6) and (19) we note that in the jittered case we divide the weight by the characteristic function $\chi(2\pi f)$. Since $|\chi(2\pi f)| \leq 1$, the estimator takes "inflated" values. This hardly affects the results at low frequencies where $\chi(2\pi f) \approx 1$. However, the situation deteriorates at higher frequencies when the value of $\chi(2\pi f)$ decreases.

7. Conclusions

We have presented new results for alias-free spectrum estimation from data collected using low-rate random sampling schemes. The results provide an extension of the earlier work (Tarczyński and Allay, 2004). We have derived the optimal p.d.f. of sampling instants that allows minimising spectrum estimation errors. We have also derived a closed-form formula for the standard deviation of the estimator. Even a simpler, frequency-independent expression provides an upper limit for the standard deviation. We have shown that the quality of spectrum estimation deteriorates when sampling instants are subjected to jitter. When the jitter is simply ignored, the estimator becomes biased. It is possible to modify the estimator so that the bias is removed nearly completely. The undesired side effect of such an action, which still diminishes the quality of spectrum estimation, is an increased standard deviation of the estimator and, consequently, larger estimation errors.

References

Beutler F. (1970): *Alias-free randomly timed sampling of stochastic processes*. — IEEE Trans. Inf. Theory, Vol. IT-16, No. 2, pp. 147–152.

Benedetto J. and Ferreira P.J.S.G. (2001): *Modern Sampling Theory: Mathematics and Applications*. — Boston, MA: Birkhaeuser.

Bilinskis I. and Mikelsons M. (1992): *Randomized Signal Processing.* — London: Prentice Hall.

Chen D. Shi and Allebach J.P. (1987): *Analysis of error in reconstruction of two-dimensional signal from irregularly spaced samples*. — Vol. ASSP-35, No.2, pp. 173–180.

Feng P. and Bresler Y. (1996): *Spectrum-blind minimum-rate sampling and reconstruction of multiband signals*. — IEEE Int. Conf. *Acoustics, Speech, and Signal Processing, ICASSP-96*, Atlanta, GA, Vol. 3, pp. 1688–1691.

Ferreira P.J.S.G. (1992): *Incomplete sampling and the recovery of missing samples from oversampled band-limited signals*. — IEEE Trans. Signal Process., Vol. 40, No.1, pp. 225–227.

Herley C. and Wong P.W. (1999): *Minimum rate sampling and reconstruction of signals with arbitrary frequency support*. — IEEE Trans. Inf. Theory, Vol. 45, No. 5, pp. 1555–1564.

Kida T. and Mochizuki H. (1992): *Generalized interpolatory approximation of multi-dimensional signals having the minimum measure of error*. — IEICE Trans. Fundamentals, Vol. E75-A, No. 7, pp. 794–805.

Landau H.J. (1967): *Necessary density conditions for sampling and interpolation of certain entire functions*. — Acta Math., Vol. 117, pp. 37–52.

Lomb N.R. (1976): *Least-squares frequency analysis of unequally spaced data*. — Astroph. Space Sc., Vol. 39, No. 2, pp. 447–462.

Marvasti F. (1987): *A Unified Approach to Zero-Crossing and Nonuniform Sampling of Single and Multidimensional Signals and Systems*. — Oak Park IL: Nonuniform Publications.

Marvasti F. (2001): *Nonuniform Sampling, Theory and Practice*. — New York: Kluwer.

Masry E. (1978): *Alias-free sampling: An alternative conceptualization and its applications*. — IEEE Trans. Inf. Theory, Vol. 24, No. 3, pp. 317–324.

Scargle J.D. (1982): *Studies in astronomical time series analysis. II. Statistical aspects of spectral analysis of unevenly spaced data*. — Astronom. J., Vol. 263, No. 1, pp. 835–853.

Scargle J.D. (1989): *Studies in astronomical time series analysis. III. Fourier transforms, autocorrelation functions, and gross-correlation functions of unevenly spaced data*. — Astronom. J., Vol. 343, No. 1, pp. 874–887.

Shapiro H.S. and Silverman R.A. (1960): *Alias-free sampling of random noise*. — SIAM J. Appl. Math. Vol. 8, No. 2, pp. 225–236.

Tarczyński A. (1997): *Sensitivity of signal reconstruction*. — IEEE Signal Process. Lett., Vol. 4, No. 7, pp. 192–194.

Tarczyński A. and Allay N. (2004): *Spectral analysis Of randomly sampled signals: Suppression of aliasing and sampler jitter*. — IEEE Trans. Signal Process., Vol. 52, No. 12, pp. 3324–3334.

Tarczyński A. and Cain G.D. (1997): *Reliability of signal reconstruction from finite sets of samples*. — 1997 Workshop *Sampling Theory and Applications, SAMPTA'97*, Aveiro, Portugal, Vol. 1, pp. 181–186.

Tarczyński A. and Valimaki V. (1996): *Modifying FIR and IIR filters for processing signals with lost samples*. — Proc. *IEEE Nordic Signal Processing Symposium, NORSIG'96*, Helsinki, Finland, Vol. 1, pp. 359–362.

Tarczyński A., Valimaki V. and Cain G.D. (1997): *FIR filtering of nonuniformly sampled signals*. — IEEE Int. Conf. *Acoustics, Speech and Signal Processing, ICASSP'97*, Munich, Germany, Vol. 3, pp. 2237–2240.

Venkataramani R. and Bresler Y. (2001): *Optimal sub-Nyquist nonuniform sampling and reconstruction for multiband signals*. — IEEE Trans. Signal Process., Vol. 49, No. 10, pp. 2301–2313.

Weisstein E.W. (2003): *CRC Concise Encyclopedia of Mathematics*. — London, UK, Chapman Hall/CRC.

Wingham D.J. (1992): *The reconstruction of a band-limited function and its Fourier transform from a finite number of samples at arbitrary locations by singular value decomposition*. — IEEE Trans. Signal Process., Vol. 40, No. 3, pp. 559–570.

Yen J.L. (1956): *On nonuniform sampling of bandwidth-limited signals*. — IRE Trans. Circ. Theory, Vol. 3, No. 4, pp. 251–257.