TIME–OPTIMAL BOUNDARY CONTROL OF A PARABOLIC SYSTEM WITH TIME LAGS GIVEN IN INTEGRAL FORM

ADAM KOWALEWSKI*, ANNA KRAKOWIAK**

* Institute of Automatics
AGH University of Science and Technology
al. Mickiewicza 30, 30–059 Cracow, Poland
e-mail: ako@ia.agh.edu.pl

** Institute of Mathematics
Cracow University of Technology
ul. Warszawska 24, 31–155 Cracow, Poland
e-mail: skrakowi@riad.usk.pl.edu.pl

In this paper, the time-optimal boundary control problem for a distributed parabolic system in which time lags appear in integral form in both the state equation and the boundary condition is presented. Some particular properties of optimal control are discussed.

Keywords: time-optimal boundary control, parabolic system, time lags

1. Introduction

Various optimization problems associated with the optimal control of distributed parabolic systems with time delays appearing in boundary conditions were studied recently by Wang (1975), Knowles (1978), Kowalewski (1988; 1990a;1990b; 1998; 1999; 2001), Kowalewski and Duda (1992) and Kowalewski and Krakowiak (2000).

In (Wang, 1975), optimal control problems for parabolic systems with Neumann boundary conditions involving constant time delays were considered. Such systems constitute, in a linear approximation, a universal mathematical model for many diffusion processes in which time-delayed feedback signals are introduced at the boundary of a system's spatial domain. For example, in the area of plasma control, it is of interest to confine the plasma in a given bounded spatial domain Ω by introducing a finite electric potential barrier or a "magnetic mirror" surrounding Ω.

For a collision-dominated plasma, its particle density is describable by a parabolic equation. Due to the particle inertia and finiteness of the electric potential barrier or the magnetic mirror field strength, the particle reflection at the domain boundary is not instantaneous. Consequently, the particle flux at the boundary of Ω at any time depends on the flux of particles which escaped earlier and reflected back into Ω at a later time. This leads to Neumann boundary conditions involving time delays.

Necessary and sufficient conditions which optimal control must satisfy were derived. Estimates and a sufficient condition for the boundedness of solutions were obtained for parabolic systems with specified forms of feedback control.

Subsequently, in (Knowles, 1978), time-optimal control problems of linear parabolic systems with Neumann boundary conditions involving constant time delays were considered. Using the results of (Wang, 1975), the existence of a unique solution of such parabolic systems was discussed. A characterization of optimal control in terms of the adjoint system is given. This characterization was used to derive specific properties of optimal control (bang-bangness, uniqueness, etc.). These results were also extended to certain cases of nonlinear control without convexity and to certain fixed-time problems.

Consequently, in (Kowalewski 1988; 1990a; 1990b; 1993; 1998; 1999; 2001; Kowalewski and Duda, 1992), linear quadratic problems for parabolic systems with time delays given in various forms (constant time delays, time-varying delays, time delays given in the integral form, etc.) were presented.

In particular, in (Kowalewski and Krakowiak, 2000), time-optimal distributed control problems for parabolic systems with deviating arguments appearing in the integral form both in state equations and in Neumann boundary conditions were considered.

In this paper, we consider the time-optimal boundary control problem for a linear parabolic system in which time lags appear in the integral form both in the state equation and in Neumann boundary condition.

The existence and uniqueness of solutions of such parabolic equations are proved. Optimal control is characterized by the adjoint equation. Using this characterization, particular properties of time-optimal boundary control are proved, i.e. bang-bangness, uniqueness, etc.

2. Existence and Uniqueness of Solutions

Consider now the distributed-parameter system described by the following parabolic delay equation:

$$\frac{\partial y}{\partial t} + A(t)y + \int_a^b b(x,t)y(x,t-h)\,\mathrm{d}h = u,$$
$$x \in \Omega,\ t \in (0,T),\ h \in (a,b), \quad (1)$$

$$y(x,t') = \Phi_o(x,t'), \quad x \in \Omega,\ t' \in [-b,0), \quad (2)$$
$$y(x,0) = y_0(x), \quad x \in \Omega, \quad (3)$$

$$\frac{\partial y}{\partial \eta_A} = \int_a^b c(x,t)y(x,t-h)\,\mathrm{d}h + v,$$
$$x \in \Gamma,\ t \in (0,T),\ h \in (a,b), \quad (4)$$

$$y(x,t') = \Psi_o(x,t'), \quad x \in \Gamma, \quad t' \in [-b,0), \quad (5)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded, open set with boundary Γ, which is a C^∞-manifold of dimension $(n-1)$. Locally, Ω is totally on one side of Γ.

$$y \equiv y(x,t;u), \quad u \equiv u(x,t), \quad v \equiv v(x,t),$$
$$Q \equiv \Omega \times (0,T), \quad \bar{Q} = \bar{\Omega} \times [0,T], \quad Q_0 = \Omega \times [-b,0),$$
$$\Sigma = \Gamma \times (0,T), \quad \Sigma_0 = \Gamma \times [-b,0).$$

Furthermore, T is a specified positive number representing a time horizon, b is a given real C^∞ function defined on $\bar{Q}$, c is a given real C^∞ function defined on $\sum$, h is a time lag such that $h \in (a,b)$, Φ_0 and Ψ_0 are initial functions defined on Q_o and Σ_o, respectively.

The parabolic operator $\frac{\partial}{\partial t} + A(t)$ in the state equation (1) satisfies the hypothesis of Section 1, Chapter 4 of (Lions and Magenes, 1972, Vol. 2, p. 2), and $A(t)$ is given by

$$A(t)y = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial y(x,t)}{\partial x_j}\right), \quad (6)$$

and the functions $a_{ij}(x,t)$ are real C^∞ functions defined on $\bar{Q}$(the closure of Q) satisfying the ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\varphi_i\varphi_j \ge \alpha \sum_{i=1}^{n} \varphi_i^2, \alpha > 0,$$
$$\forall (x,t) \in \bar{Q}, \forall \varphi_i \in \mathbb{R}. \quad (7)$$

Equations (1)–(5) constitute a Neumann problem. The left-hand side of (4) is written in the following form:

$$\frac{\partial y}{\partial \eta_A} = \sum_{i,j=1}^{n} a_{ij}(x,t)\cos(n,x_i)\frac{\partial y(x,t)}{\partial x_j} = q(x,t),$$
$$x \in \Gamma,\ t \in (0,T), \quad (8)$$

where $\partial y/\partial \eta_A$ is the normal derivative at Γ, directed towards the exterior of Ω, $\cos(n,x_i)$ is the i-th direction cosine of n, with n being the normal at Γ exterior to Ω, and

$$q(x,t) = \int_a^b c(x,t)y(x,t-h)\,\mathrm{d}h + v(x,t)$$
$$x \in \Gamma,\ t \in (0,T),\ h \in (a,b). \quad (9)$$

First, we shall prove sufficient conditions for the existence of a unique solution of the mixed initial-boundary value problem (1)–(5) for the case where the distributed control $v \in L^2(Q)$. For this purpose, for any pair of real numbers $r, s \ge 0$, we introduce the Sobolev space $H^{r,s}(Q)$ (Lions and Magenes 1972, Vol. 2, p. 6) defined by

$$H^{r,s}(Q) = H^0\big(0,T;H^r(\Omega)\big) \cap H^s\big(0,T;H^0(\Omega)\big), \quad (10)$$

which is a Hilbert space normed by

$$\left(\int_0^T \|y(t)\|^2_{H^r(\Omega)}\,\mathrm{d}t + \|y\|^2_{H^s\left(0,T;H^0(\Omega)\right)}\right)^{1/2}, \quad (11)$$

where the spaces $H^r(\Omega)$ and $H^s(0,T;H^0(\Omega))$ are defined in Chapter 1 of (Lions and Magenes 1972, Vol. 1).

The existence of a unique solution for the mixed initial-boundary value problem (1)–(5) on the cylinder Q can be proved using a constructive method, i.e., first, solving (1)–(5) on the subcylinder Q_1 and then on Q_2, etc., until the procedure covers the whole cylinder Q. In this way the solution in the previous step determines the next one.

For simplicity, we introduce the following notation:

$$E_j \triangleq ((j-1)a, ja), \quad Q_j = \Omega \times E_j, \quad Q_0 = \Omega \times [-b,0)$$

$$\Sigma_j = \Gamma \times E_j, \quad \Sigma_0 = \Gamma \times [-b, 0) \quad \text{for } j = 1, \dots, K.$$

Using Theorem 15.2 of (Lions and Magenes 1972, Vol. 2, p. 81), we can prove the following lemma:

Lemma 1. *Let*

$$u \in H^{1/2,-1/4}(Q),\ v \in L^2(\Sigma), \tag{12}$$

$$f_j \in H^{-1/2,-1/4}(Q_j), \tag{13}$$

where

$$f_j(x,t) = u(x,t) - \int_a^b b(x,t) y_{j-1}(x, t-h)\,\mathrm{d}h$$

$$y_{j-1}\big(\cdot, (j-1)a\big) \in H^{1/2}(\Omega), \tag{14}$$

$$q_j \in L^2(\Sigma_j), \tag{15}$$

where

$$q_j(x,t) = \int_a^b c(x,t) y_{j-1}(x, t-h)\,\mathrm{d}h + v(x,t).$$

Then there exists a unique solution $y_j \in H^{3/2,3/4}(Q_j)$ for the mixed initial-boundary value problem (1), (4), (14).

Proof. We observe that for $j = 1$, $y_{j-1}|_{Q_0}(x, t-h) = \Phi_0(x, t-h)$ and $y_{j-1}|_{\Sigma_0}(x, t-h) = \Psi_0(x, t-h)$. Then the assumptions (13)–(15) are fulfilled if we assume that $\Phi_0 \in H^{3/2,3/4}(Q_0)$, $y_0 \in H^{1/2}(\Omega)$, and $\Psi_0 \in L^2(\Sigma_0)$. These assumptions are sufficient to ensure the existence of a unique solution $y_1 \in H^{3/2,3/4}(Q_1)$. In order to extend the result to Q_2, we have to prove that $y_1(\cdot, a) \in H^{1/2}(\Omega)$, $y_1|_{\Sigma_1} \in L^2(\Sigma_1)$ and $f_2 \in H^{-1/2,-1/4}(Q_2)$. Indeed, from Theorems 2.1 and 2.2 of (Kowalewski, 1998) $y_1 \in H^{3/2,3/4}(Q_1)$ implies that the mapping $t \to y_1(\cdot, t)$ is continuous from $[0, a]$ into $H^{3/4}(\Omega) \subset H^{1/2}(\Omega)$. Thus, $y_1(\cdot, a) \in H^{1/2}(\Omega)$. Then using the trace theorem (Theorem 2.3 of (Kowalewski, 1998)) we can verify that $y_1 \in H^{3/2,3/4}(Q_1)$ implies that $y_1 \to y_1|_{\Sigma_1}$ is a linear, continuous mapping of $H^{3/23/4}(Q_1)$ into $H^{1,1/2}(\Sigma_1)$. Thus, $y_1|_{\Sigma_1} \in L^2(\Sigma_1)$. Also, it is easy to notice that the assumption (13) follows from the fact that $y_1 \in H^{3/2,3/4}(Q_1)$ and $u \in H^{-1/2,-1/4}(Q)$. Then, there exists a unique solution $y_2 \in H^{3/2,3/4}(Q_2)$. The foregoing result is now summarized for $j = 3, \dots, K$. ■

Theorem 1. *Let y_0, Φ_0, Ψ_0, v and u be given with $y_0 \in H^{1/2}(\Omega)$, $\Phi_0 \in H^{3/2,3/4}(Q_0)$, $\Psi_0 \in L^2(\Sigma_0)$, $v \in L^2(\Sigma)$ and $u \in H^{-1/2,-1/4}(Q)$. Then there exists a unique solution $y \in H^{3/2,3/4}(Q)$ for the mixed initial-boundary value problem (1)–(5). Moreover, $y(\cdot, ja) \in H^{1/2}(\Omega)$ for $j = 1, \dots, K$.*

3. Problem Formulation. Optimization Theorems

Now, we shall formulate the time-optimal problem for (1)–(5) in the context of Theorem 1, that is,

$$v \in U = \left\{ v \in L^2(\Sigma) \ : | \ v(x,t) \ | \le 1 \right\}. \tag{16}$$

We shall define the reachable set Y such that

$$Y = \left\{ y \in L^2(\Omega) \ : \ \|y - z_d\|_{L^2(\Omega)} \le \epsilon \right\}, \tag{17}$$

where z_d, ϵ are given with $z_d \in L^2(\Omega)$ and $\epsilon > 0$.

The solving of the stated time-optimal problem is equivalent to hitting the target set Y in minimum time, that is, minimizing the time t, for which $y(t; v) \in Y$ and $v \in U$.

Moreover, we assume that

$$\text{there exists a } T > 0 \text{ and } v \in U \text{ with } y(T; v) \in Y. \tag{18}$$

Theorem 2. *If the assumption (18) holds, then the set Y is reached in minimum time t^* by admissible control $v^* \in U$. Moreover,*

$$\int_\Omega \big(z_d - y(t^*; v^*)\big)\big(y(t^*; v) - y(t^*; v^*)\big)\,\mathrm{d}x \le 0, \quad \forall v \in U. \tag{19}$$

Outline of the proof. Let us define the following set:

$$t^* = \inf\{t : y(t; v) \in Y \text{ for some } v \in U\}. \tag{20}$$

The minimum is well defined, as (18) guarantees that this set is nonempty. By definition, we can choose $t_n \downarrow t^*$ and admissible control $\{v_n\}$ such that

$$y(t_n; v_n) \in Y, \ \ n = 1, 2, 3, \dots. \tag{21}$$

Each v_n is defined on $\Gamma \times (0, t_n) \supset \Gamma \times (0, t^*)$. To simplify the notation, we denote the restriction of v_n to $\Gamma \times (0, t^*)$ again by v_n. The admissible control set then forms a weakly compact, convex set in $L^2(\Gamma \times (0, t^*)$, and so we can extract a weakly convergent subset $\{v_m\}$ which converges weakly to some admissible control v^*.

Consequently, Theorem 1 implies that $y(t; v) \in H^{1/2}(\Omega) \subset L^2(\Omega)$ for each $v \in L^2(\Sigma)$ and $t > 0$. Then using Theorem 1.2 of (Lions 1971, p. 102) and Theorem 1 it is easy to verify that the mapping $v \to y(t^*; v)$, from $L^2(\Gamma \times (0, t^*))$ into $L^2(\Omega)$, is continuous. Since any continuous linear mapping between Banach spaces is also weakly continuous (Dunford and Schwartz 1958, Thm. V. 3.15), the affine mapping $v \to y(t^*; v)$ must also be weakly continuous. Hence

$$y(t^*; v_m) \to y(t^*; v^*) \ \text{ weakly in } \ L^2(\Omega). \tag{22}$$

Moreover, $\mathrm{d}y(v)/\mathrm{d}t \in L^2([0,t^*], H^0(\Omega))$, for each $v \in U$, by the definition of $H^{3/2,3/4}(\Omega \times (0,t^*))$, and

$$\begin{aligned}
&\|y(t_m;v_m) - y(t^*;v_m)\|_{L^2(\Omega)} \\
&\quad = \Big\| \int_{t^*}^{t_m} \dot{y}(\sigma;v_m)\,\mathrm{d}\sigma \Big\|_{L^2(\Omega)} \\
&\quad \le \sqrt{t_m - t^*} \left(\int_{t^*}^{t_m} \|\dot{y}(\sigma;v_m)\|^2_{L^2(\Omega)}\,\mathrm{d}\sigma \right)^{1/2}. \qquad (23)
\end{aligned}$$

Applying Theorem 1.2 of (Lions, 1971) and Theorem 1 again, the set $\{\dot{y}(v_m)\}$ must be bounded in $L^2(0,t^*;H^0(\Omega))$, and hence

$$\|y(t_m;v_m) - y(t^*;v_m)\|_{L^2(\Omega)} \le M\sqrt{t_m - t^*}. \qquad (24)$$

Combining (22) and (24) shows that

$$\begin{aligned}
&y(t_m;v_m) - y(t^*;v^*) \\
&\quad = \Big(y(t_m;v_m) - y(t^*;v_m)\Big) \\
&\qquad + \Big(y(t^*;v_m) - y(t^*;v^*)\Big) \qquad (25)
\end{aligned}$$

converges weakly to zero in $L^2(\Omega)$, and so $y(t^*;v^*) \in Y$ as Y is closed and convex, and hence weakly closed. This shows that Y is reached in time t^* by admissible control; accordingly, t^* must be the minimum time and v^* optimal control.

We shall now prove the second part of our theorem. From Theorem 3.1 of (Lions and Magenes, 1972, Vol. 1, p.19), $y(v) \in H^{3/2,3/4}(Q)$ implies that the mapping $t \to y(t;v)$, from $[0,T]$ into $H^{3/4}(\Omega) \subset H^{1/2}(\Omega) \subset L^2(\Omega)$, is continuous for each fixed v, and so $y(t^*;v) \notin \operatorname{int} Y$, for any $v \in U$, by the minimality of t^*.

From our earlier remarks, the set

$$\mathcal{A}(t^*) = \{y(t^*;v_x) : v_x \in U\} \qquad (26)$$

is weakly compact and convex in $L^2(\Omega)$. Applying Theorem 21.11 of (Choquet, 1969) to the sets $\mathcal{A}(t^*)$ and Y shows that there exists a nontrivial hyperplane $z' \in L^2(\Omega)$ separating these sets, that is,

$$\int_\Omega z' y(t^*;v)\,\mathrm{d}x \le \int_\Omega z' y(t^*;v^*)\,\mathrm{d}x \le \int_\Omega z' y\,\mathrm{d}x, \qquad (27)$$

for all $v \in U$ and $y \in L^2(\Omega)$ with

$$\|y - z_d\|_{L^2(\Omega)} \le \varepsilon. \qquad (28)$$

From the second inequality in (27), z' must support the set Y at $y(t^*;v^*)$. Moreover, since $L^2(\Omega)$ is a Hilbert space, z' must be of the form

$$z' = \lambda\big(z_d - y(t^*;v^*)\big) \quad \text{for some} \quad \lambda > 0. \qquad (29)$$

Subsequently, dividing (27) by λ gives the desired result (19). ■

4. Optimization Theorems

We shall apply Theorem 2 to the control problem of (1)–(5). To simplify (19), we introduce the adjoint equation and for, every $v \in U$, we define the adjoint variable $p = p(v) = p(x,t;v)$ as the solution of the equation

$$\begin{aligned}
&-\frac{\partial p(v)}{\partial t} + A^*(t)p(v) + \int_a^b b(x,t+h) \\
&\qquad \times p(x,t+h;v)\,\mathrm{d}h = 0, \\
&\qquad x \in \Omega, \quad t \in (0,t^*-b), \quad h \in (a,b), \qquad (30)
\end{aligned}$$

$$\begin{aligned}
&-\frac{\partial p(v)}{\partial t} + A^*(t)p(v) = 0, \\
&\qquad x \in \Omega, \quad t \in (t^*-b, t^*-b+a) \qquad (31)
\end{aligned}$$

$$p(x,t^*;v) = z_d(x) - y(x,t^*;v), \quad x \in \Omega, \qquad (32)$$

$$p(x,t;v) = 0, x \in \Omega, \quad t \in [t^*-b+a, t^*), \qquad (33)$$

$$\begin{aligned}
&\frac{\partial p(v)}{\partial \eta_{A^*}}(x,t) = \int_a^b c(x,t+h)p(x,t+h;v)\,\mathrm{d}h, \\
&\qquad x \in \Gamma, \quad t \in (t^*-b),\ h \in (a,b), \qquad (34)
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial p(v)}{\partial \eta_{A^*}}(x,t) = 0, \\
&\qquad x \in \Gamma, \quad t \in (t^*-b, t^*-b+a), \qquad (35)
\end{aligned}$$

where

$$\frac{\partial p(v)}{\partial \eta_{A^*}}(x,t) = \sum_{i,j=1}^n a_{ji}(x,t)\cos(n,x_i)\frac{\partial p(v)}{\partial x_j}(x,t), \qquad (36)$$

and

$$A^*(t)p = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial p}{\partial x_i}\right). \qquad (37)$$

The existence of a unique solution for the problem (30)–(35) on the cylinder $\Omega \times (0,t^*)$ can be proved using a constructive method. It is easy to notice that for given z_d and u, the problem (30)–(35) can be solved backwards in time starting from $t = t^*$, i.e., first, solving (30)–(35) on the subcylinder Q_K and then on Q_{K-1}, etc., until the procedure covers the whole cylinder $\Omega \times (0,t^*)$. For this purpose, we may apply Theorem 1 (with an obvious change of variables). Hence, using Theorem 1, the following result can be proved:

Theorem 3. *Let the hypothesis of Theorem 1 be satisfied. Then for given $z_d \in L^2(\Omega)$ and any $v \in L^2(\Sigma)$, there exists a unique solution $p(v) \in H^{3/2,3/4}(\Omega \times (0,t^*))$ for the adjoint problem (30)–(35).*

We simplify (19) using the adjoint equation (30)–(35). For this purpose, setting $v = v^*$ in (30)–(35), multiplying both sides of (30), (31) by $y(v) - y(v^*)$, then integrating over $\Omega \times (0, t^* - b)$ and $\Omega \times (t^* - b, t^*)$, respectively, and then adding both sides of (30), (31), we get

$$
\begin{aligned}
&\int_0^{t^*}\int_\Omega \left(-\frac{\partial p(v^*)}{\partial t} + A^*(t)p(v^*)\right)(y(v) - y(v^*))\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^{t^*-b}\int_\Omega \Big(\int_a^b b(x,t+h)p(x,t+h;v^*)\,\mathrm{d}h\Big) \\
&\quad \times \Big[y(x,t;v) - y(x,t;v^*)\Big]\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_\Omega p(x,t^*;v^*)\big(y(x,t^*;v) - y(x,t^*;v^*)\big)\,\mathrm{d}x \\
&\quad + \int_0^{t^*}\int_\Omega p(v^*)\frac{\partial}{\partial t}\big(y(v) - y(v^*)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^{t^*}\int_\Omega A^*(t)p(v^*)\big(y(v) - y(v^*)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^{t^*-b}\int_\Omega\int_a^b b(x,t+h)p(x,t+h;v^*) \\
&\quad - \Big(y(x,t;v) - y(x,t;v^*)\Big)\,\mathrm{d}h\,\mathrm{d}x\,\mathrm{d}t = 0.
\end{aligned}
\tag{38}
$$

Then, applying (32), the formula (38) can be expressed as

$$
\begin{aligned}
&\int_\Omega \big(z_d - y(t^*;v^*)\big)\big(y(t^*;v) - y(t^*;v^*)\big)\,\mathrm{d}x \\
&= \int_0^{t^*}\int_\Omega p(v^*)\frac{\partial}{\partial t}\big(y(v) - y(v^*)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^{t^*}\int_\Omega A^*(t)p(v^*)\big(y(v) - y(v^*)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_a^b\int_\Omega\int_0^{t^*-b} b\,(x,t+h)\,p\,(x,t+h;v^*) \\
&\quad \times \Big(y(x,t;v) - y(x,t;v^*)\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}h.
\end{aligned}
\tag{39}
$$

Using Eqn. (1), the first integral on the right-hand side of (39) can be rewritten as

$$
\begin{aligned}
&\int_0^{t^*}\int_\Omega p(v^*)\frac{\partial}{\partial t}\big(y(v) - y(v^*)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^{t^*}\int_\Omega p(v^*)A(t)\big(y(v) - y(v^*)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^{t^*}\int_\Omega p(x,t;v^*) \\
&\quad \times \Big(\int_a^b b(x,t)(y(x,t-h;v) - y\big(x,t-h;v^*)\big)\,\mathrm{d}h\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^{t^*}\int_\Omega p(v^*)A(t)\big(y(v) - y(v^*)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^{t^*}\int_\Omega\int_a^b p(x,t;v^*)b(x,t) \\
&\quad \times \big(y(x,t-h;v) - y(x,t-h;v^*)\big)\,\mathrm{d}h\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^{t^*}\int_\Omega p(v^*)A(t)\big(y(v) - y(v^*)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_a^b\int_\Omega\int_0^{t^*} p(x,t;v^*)b(x,t) \\
&\quad \times \big(y(x,t-h;v) - y(x,t-h;v^*)\big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}h \\
&= -\int_0^{t^*}\int_\Omega p(v^*)A(t)\big(y(v) - y(v^*)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_a^b\int_\Omega\int_{-h}^{t^*-h} p(x,t'+h;v^*)b(x,t'+h) \\
&\quad \times \big(y(x,t';v) - y(x,t';v^*)\big)\,\mathrm{d}t'\,\mathrm{d}x\,\mathrm{d}h \\
&= -\int_0^{t^*}\int_Q p(v^*)A(t)\big(y(v) - y(v^*)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_a^b\int_\Omega\int_{-h}^{0} p(x,t'+h;v^*)b(x,t'+h) \\
&\quad \times \big(y(x,t';v) - y(x,t';v^*)\big)\,\mathrm{d}t'\,\mathrm{d}x\,\mathrm{d}h
\end{aligned}
$$

$$- \int_a^b \int_\Omega \int_0^{t^*-b} p(x,t'+h;v^*)b(x,t'+h)$$
$$\times \big(y(x,t';v) - y(x,t';v^*)\big)\,\mathrm{d}t'\,\mathrm{d}x\,\mathrm{d}h$$
$$- \int_a^b \int_\Omega \int_{t^*-b}^{t^*-h} p(x,t'+h;v^*)b(x,t'+h)$$
$$\times \big(y(x,t';v) - y(x,t';v^*)\big)\,\mathrm{d}t'\,\mathrm{d}x\,\mathrm{d}h$$
$$= \int_0^{t^*} \int_\Omega p(v^*)(u - v^*)\mathrm{d}x\,\mathrm{d}t$$
$$- \int_0^{t^*} \int_\Omega p(v^*)A(t)\big(y(v) - y(v^*)\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$- \int_a^b \int_\Omega \int_{-h}^{0} p(x,t'+h;v^*)b(x,t'+h)$$
$$\times \big(y(x,t';v) - y(x,t';v^*)\big)\,\mathrm{d}t'\,\mathrm{d}x\,\mathrm{d}h$$
$$- \int_a^b \int_\Omega \int_0^{t^*-b} p(x,t'+h;v^*)b(x,t'+h)$$
$$\times \big(y(x,t';v) - y(x,t';v^*)\big)\,\mathrm{d}t'\,\mathrm{d}x\,\mathrm{d}h$$
$$- \int_a^b \int_\Omega \int_{t^*-b+h}^{t^*} p(x,t;v^*)b(x,t)$$
$$\times \big(y(x,t-h;v) - y(x,t-h;v^*)\big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}h. \quad (40)$$

The second integral on the right-hand side of (39), in view of Green's formula, can be expressed as

$$\int_0^{t^*} \int_\Omega A^*(t)p(v^*)\big(y(v) - y(v^*)\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^{t^*} \int_\Omega p(v^*)A(t)\big(y(v) - y(v^*)\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^{t^*} \int_\Gamma p(v^*)\left(\frac{\partial y(v)}{\partial \eta_A} - \frac{\partial y(v^*)}{\partial \eta_A}\right)\,\mathrm{d}\Gamma\,\mathrm{d}t$$
$$- \int_0^{t^*} \int_\Gamma \frac{\partial p(v^*)}{\partial \eta_{A^*}}\big(y(v) - y(v^*)\big)\,\mathrm{d}\Gamma\,\mathrm{d}t. \quad (41)$$

Using the boundary condition (4), the second component on the right-hand side of (41) can be written as

$$\int_0^{t^*} \int_\Gamma p(v^*)\Big[\frac{\partial y(v)}{\partial \eta_A} - \frac{\partial y(v^*)}{\partial \eta_A}\Big]\,\mathrm{d}\Gamma\,\mathrm{d}t$$
$$= \int_0^{t^*} \int_\Gamma p(x,t;v^*)$$
$$\times \left(\int_a^b c(x,t)\big(y(x,t-h;v) - y(x,t-h;v^*)\big)\,\mathrm{d}h\right)\mathrm{d}\Gamma\,\mathrm{d}t$$
$$+ \int_0^{t^*} \int_\Gamma p(x,t;v^*)(v - v^*)\,\mathrm{d}\Gamma\,\mathrm{d}t$$
$$= \int_0^{t^*} \int_\Gamma \int_a^b p(x,t;v^*)c(x,t)$$
$$\times \big(y(x,t-h;v) - y(x,t-h;v^*)\big)\,\mathrm{d}h\,\mathrm{d}\Gamma\,\mathrm{d}t$$
$$+ \int_0^{t^*} \int_\Gamma p(x,t;v^*)(v - v^*)\,\mathrm{d}\Gamma\,\mathrm{d}t$$
$$= \int_a^b \int_\Gamma \int_0^{t^*} p(x,t;v^*)c(x,t)$$
$$\times \big(y(x,t-h;v) - y(x,t-h;v^*)\big)\,\mathrm{d}t\,\mathrm{d}\Gamma\,\mathrm{d}h$$
$$+ \int_0^{t^*} \int_\Gamma p(x,t;v^*)(v - v^*)\,\mathrm{d}\Gamma\,\mathrm{d}t$$
$$= \int_a^b \int_\Gamma \int_{-h}^{t^*-h} p(x,t'+h;v^*)c(x,t'+h)$$
$$\times \big(y(x,t';v) - y(x,t';v^*)\big)\,\mathrm{d}t'\,\mathrm{d}\Gamma\,\mathrm{d}h$$
$$+ \int_0^{t^*} \int_\Gamma p(x,t;v^*)(v - v^*)\,\mathrm{d}\Gamma\,\mathrm{d}t$$
$$= \int_a^b \int_\Gamma \int_{-h}^{0} p(x,t'+h;v^*)c(x,t'+h)$$
$$\times \big(y(x,t';v) - y(x,t';v^*)\big)\,\mathrm{d}t'\,\mathrm{d}\Gamma\,\mathrm{d}h$$
$$+ \int_a^b \int_\Gamma \int_0^{t^*-b} p(x,t'+h;v^*)c(x,t'+h)$$
$$\times \big(y(x,t';v) - y(x,t';v^*)\big)\,\mathrm{d}t'\,\mathrm{d}\Gamma\,\mathrm{d}h$$

$$+\int_a^b\int_\Gamma\int_{t^*-b}^{t^*-h} p(x,t'+h;v^*)c(x,t'+h)$$
$$\times\big(y(x,t';v)-y(x,t';v^*)\big)\,\mathrm{d}t'\ \mathrm{d}\Gamma\,\mathrm{d}h$$
$$+\int_0^{t^*}\int_\Gamma p(x,t;v^*)(v-v^*)\,\mathrm{d}\Gamma\,\mathrm{d}t$$
$$=\int_a^b\int_\Gamma\int_{-h}^{0} p(x,t'+h;v^*)c(x,t'+h)$$
$$\times\big(y(x,t';v)-y(x,t';v^*)\big)\,\mathrm{d}t'\,\mathrm{d}\Gamma\,\mathrm{d}h$$
$$+\int_a^b\int_\Gamma\int_{0}^{t^*-b} p(x,t'+h;v^*)c(x,t'+h)$$
$$\times\big(y(x,t';v)-y(x,t';v^*)\big)\,\mathrm{d}t'\,\mathrm{d}\Gamma\,\mathrm{d}h$$
$$+\int_a^b\int_\Gamma\int_{t^*-b+h}^{t^*} p(x,t+h;v^*)c(x,t)$$
$$\times\big(y(x,t-h;v)-y(x,t-h;v^*)\big)\,\mathrm{d}t\,\mathrm{d}\Gamma\,\mathrm{d}h$$
$$+\int_0^{t^*}\int_\Gamma p(x,t;v^*)(v-v^*)\,\mathrm{d}\Gamma\,\mathrm{d}t. \tag{42}$$

The last component in (41) can be rewritten as

$$\int_0^{t^*}\int_\Gamma \frac{\partial p(v^*)}{\partial\eta_{A^*}}\big(y(v)-y(v^*)\big)\,\mathrm{d}\Gamma\,\mathrm{d}t$$
$$=\int_0^{t^*-b}\int_\Gamma \frac{\partial p(v^*)}{\partial\eta_{A^*}}\big(y(v)-y(v^*)\big)\,\mathrm{d}\Gamma\,\mathrm{d}t$$
$$+\int_{t^*-b}^{t^*}\int_\Gamma \frac{\partial p(v^*)}{\partial\eta_{A^*}}\big(y(v)-y(v^*)\big)\,\mathrm{d}\Gamma\,\mathrm{d}t. \tag{43}$$

Substituting (42), (43) into (41) and then (40), (41) into (39), we obtain

$$\int_\Omega \big(z_d-y(t^*;v^*)\big)\big(y(t^*;v)-y(t^*;v^*)\big)\,\mathrm{d}x$$
$$=-\int_0^{t^*}\int_\Omega p(v^*)A(t)\big(y(v)-y(v^*)\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_a^b\int_\Omega\int_{-h}^{0} b(x,t+h)p(x,t+h;v^*)$$
$$\times\big(y(x,t;v)-y(x,t;v^*)\big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}h$$
$$-\int_a^b\int_\Omega\int_{0}^{t^*-b} b(x,t+h)p(x,t+h;v^*)$$
$$\times\big(y(x,t;v)-y(x,t;v^*)\big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}h$$
$$+\int_0^{t^*}\int_\Omega p(v^*)A(t)\big(y(v)-y(v^*)\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_a^b\int_\Omega\int_{t^*-b+h}^{t^*} p(x,t;v^*)b(x,t)$$
$$\times\big(y(x,t-h;v)-y(x,t-h;v^*)\big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}h$$
$$+\int_a^b\int_\Gamma\int_{-h}^{0} c(x,t+h)p(x,t+h;v^*)$$
$$\times\big(y(x,t;v)-y(x,t;v^*)\big)\,\mathrm{d}t\,\mathrm{d}\Gamma\,\mathrm{d}h$$
$$+\int_a^b\int_\Gamma\int_{0}^{t^*-b} c(x,t+h)p(x,t+h;v^*)$$
$$\times\big(y(x,t;v)-y(x,t;v^*)\big)\,\mathrm{d}t\,\mathrm{d}\Gamma\,\mathrm{d}h$$
$$+\int_0^{t^*}\int_\Gamma p(x,t;v^*)(v-v^*)\,\mathrm{d}\Gamma\,\mathrm{d}t$$
$$=\int_a^b\int_\Gamma\int_{t^*-b+h}^{t^*} p(x,t;v^*)c(x,t)$$
$$\times\big(y(x,t-h;v)-y(x,t-h;v^*)\big)\,\mathrm{d}t\,\mathrm{d}\Gamma\,\mathrm{d}h$$
$$+\int_0^{t^*}\int_\Gamma p(x,t;v^*)(v-v^*)\,\mathrm{d}\Gamma\,\mathrm{d}t$$
$$-\int_0^{t^*-b}\int_\Gamma \frac{\partial p(v^*)}{\partial\eta_{A^*}}\big(y(x,t;v)-y(x,t;v^*)\big)\,\mathrm{d}t\,\mathrm{d}\Gamma\,\mathrm{d}h$$
$$-\int_{t^*-b}^{t^*}\int_\Gamma \frac{\partial p(v^*)}{\partial\eta_{A^*}}\big(y(x,t;v)-y(x,t;v^*)\big)\,\mathrm{d}\Gamma\,\mathrm{d}t$$
$$+\int_a^b\int_\Omega\int_{0}^{t^*-b} b(x,t+h)p(x,t+h;v^*)$$
$$\times\big(y(x,t;v)-y(x,t;v^*)\big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}h. \tag{44}$$

Then, using the fact that $y(x,t;v)=y(x,t;v^*)=\Phi_0(x,t)$ for $x\in\Omega$ and $t\in[-b,0)$, and $y(x,t;v)=y(x,t;v^*)=\Psi_0(x,t)$ for $x\in\Gamma$ and $t\in[-b,0)$, we obtain

$$\int_{\Omega} \big(z_d - y(t^*; v^*)\big)\big(y(t^*; v) - y(t^*; v^*)\big)\,\mathrm{d}x = \int_0^{t^*}\int_{\Gamma} p(v^*)(v - v^*)\,\mathrm{d}\Gamma\,\mathrm{d}t. \quad (45)$$

Substituting (45) into (19) gives

$$\int_0^{t^*}\int_{\Gamma} p(v^*)(v - v^*)\,\mathrm{d}\Gamma\,\mathrm{d}t \le 0, \quad \forall v \in U. \quad (46)$$

The foregoing result is now summarized.

Theorem 4. *The optimal control v^* is characterized by the condition (46). Moreover, in particular,*

$$v^*(x,t) = \mathrm{sign}\big(p(x,t;v^*)\big), \quad x \in \Gamma, \quad t \in (0,t^*), \quad (47)$$

whenever $p(v^*)$ is nonzero.

This property leads to the following result:

Theorem 5. *If the coefficients of the operator $A(t)$ and the functions $b(x,t)$ and $c(x,t)$ are analytic, and Ω has analytic boundary Γ, then there exists unique optimal control for the mixed initial-boundary value problem (1)–(5). Moreover, the optimal control is bang-bang, that is, $|v^*(x,t)| \equiv 1$, almost everywhere and the unique solution of (1)–(5), (30)–(35), (47).*

The idea of the proof of Theorem 5 is the same as in the case of Theorem 3.4 in (Kowalewski and Krakowiak, 2000).

5. Conclusions and Perspectives

The results presented in the paper can be treated as a generalization of the results obtained by Knowles (1978), and Kowalewski and Krakowiak (2000) onto the case of time-optimal boundary control of parabolic systems with deviating arguments appearing in the integral form both in state equations and in boundary conditions.

We considered a different type of control, namely, the control function defined at the boundary of the spatial domain. Sufficient conditions for the existence of a unique solution of such parabolic equations with Neumann boundary conditions are proved (Lemma 1 and Theorem 1). The optimal control is characterized by using the adjoint equation (Theorems 2 and 3). The uniqueness and bang-bang properties of the optimal control are proved (Theorems 4 and 5).

The condition (18) plays a fundamental role in controllability problems for time-delay parabolic systems. With regard to the controllability assumption (18), we can investigate the exact controllability problem for the parabolic system (1)–(5).

In this paper, we considered the time-optimal boundary control problem for parabolic systems with nonhomogeneous Neumann boundary conditions. We can also consider an analogous minimum time problem for systems with nonhomogeneous Dirichlet boundary conditions. Finally, we can consider the time-optimal control problem for discrete time delay distributed parameter systems. The ideas mentioned above will be developed in forthcoming papers.

Acknowledgments

The first author would like to thank Professor A.J. Pritchard, the Director of the Control Theory Centre, for his invitation and hospitality at the Department of Mathematics of the University of Warwick. The author would like to express his gratitude to Professor A.J. Pritchard for many discussions and valuable suggestions provided during the author's stay at the Control Theory Centre of the University of Warwick.

The research presented here was carried out within the research programme of the AGH University of Science and Technology, no. 10.10.120.31.

References

Choquet G. (1969): *Lectures on Analysis*. — Vol. 2, New York: W.A. Benjamin.

Dunford N. and Schwartz J. (1958): *Linear Operators, Vol. 1*. — New York: Wiley.

Knowles G. (1978): *Time optimal control of parabolic systems with boundary conditions involving time delays*. — J. Optim. Theory Applics., Vol. 25, No. 4, pp. 563–574.

Kowalewski A. (1988): *Boundary control of distributed parabolic system with boundary condition involving a time-varying lag*. — Int. J. Contr., Vol. 48, No. 6, pp. 2233–2248.

Kowalewski A. (1990a): *Feedback control for a distributed parabolic system with boundary condition involving a time-varying lag*. — IMA J. Math. Contr. Inf., Vol. 7, No. 2, pp. 143–157.

Kowalewski A. (1990b): *Optimal control of distributed parabolic systems involving time lags*. — IMA J. Math. Contr. Inf., Vol. 7, No. 4, pp. 375–393.

Kowalewski A. (1993): *Optimal control of parabolic systems with time-varying lags*. — IMA J. Math. Contr. Inf., Vol. 10, No. 2, pp. 113–129.

Kowalewski A. (1998): *Optimal control of distributed parabolic systems with multiple time-varying lags*. — Int. J. Contr., Vol. 69, No. 3, pp. 361–381.

Kowalewski A. (1999): *Optimization of parabolic systems with deviating arguments*. — Int. J. Contr., Vol. 72, No. 11, pp. 947–959.

Kowalewski A. (2001): *Optimal Control of Infinite Dimensional Distributed Parameter Systems with Delays*. — Cracow: University of Mining and Metallurgy Press.

Kowalewski A. and Duda J. (1992): *On some optimal control problem for a parabolic system with boundary condition involving a time-varying lag*. — IMA J. Math. Contr. Inf., Vol. 9, No. 2, pp. 131–146.

Kowalewski A. and Krakowiak A.(2000): *Time-optimal control of a parabolic system with time lags given in the integral form*. — IMA J. Math. Contr. Inf., Vol. 17, No. 3, pp. 209–225.

Lions J.L. (1971): *Optimal Control Systems Governed by Partial Differential Equations*. — Berlin: Springer.

Lions J.L. and Magenes E. (1972): *Non-Homogeneous Boundary Value Problems and Applications, Vols. 1 and 2*. — Berlin: Springer.

Wang P.K.C. (1975): *Optimal control of parabolic systems with boundary conditions involving time delays*. — SIAM J. Contr., Vol. 13, No. 2, pp. 274–293.

Received: 6 March 2006