ANALYSIS OF SOME DUAL PROPERTIES IN DISCRETE DYNAMIC SYSTEMS

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The problem of duality in nonlinear and linear systems is considered. In addition to the known duality between controllability and observability, new dual notions and their properties are investigated. A way to refine these properties through an isomorphic transformation of the original systems is suggested.

Keywords: nonlinear and linear systems, algebraic approach, error correction, reachability, duality.

1. Introduction

The idea of duality is a powerful tool to investigate some problems in the theory of dynamic systems. A well-known fact is the duality between controllability and observability in linear dynamic systems as well as the one between optimal regulator and observer design (Kwakernaak and Sivan, 1972). The duality in this case is established by means of matrix analysis.

The problem of duality between controllability and observability in continuous-time nonlinear systems was considered in (Hermann and Krener, 1977) by differential-geometric methods and it was pointed out that it is, mathematically, just the duality between vector fields and differential forms. Decomposable systems in category with products and coproducts were considered in (Arbib and Manes, 1974), and it was shown that the duality between controllability and observability is of the form of dual commutative diagrams. Discrete-time nonlinear dynamic systems were investigated by methods of the so-called algebra of functions developed by Zhirabok and Shumsky (1993). It was shown that the duality between controllability and observability is of the form of dual expressions based on function algebra tools (operations and operators) and dual commutative diagrams describing the main definitions as properties of controllability and observability (Zhirabok, 1998). It is natural that duality is expressed by means of the same mathematical technique with which the problems of controllability and observability are studied.

In this paper, we investigate two new dual problems and their properties. The first one is connected with error correction. It is known that the ability of digital systems to correct errors caused by malfunctions in their elements can be obtained via error-correcting codes (Peterson and Weldon, 1972), i.e., by using certain redundancy. However, in some cases, the system may have the error correction property due to its operation features that can be considered as a natural redundancy. The problem of analyzing this property will be called the error correction degree problem.

The next problem is associated with finding an accuracy degree of the final state of a given system under some known control and an unknown (or known with a limited accuracy degree) initial state. This problem will be termed the reachability degree problem.

At first glance, these problems are not dual. The goal of this paper is to show that they are dual mathematically and this duality is established by means of the algebra of functions. Besides, a way to improve the error correction property and increase the reachability degree is suggested. A conference version of this paper was given in (Michtchenko and Zhirabok, 2001).

The paper is organized as follows: Section 2 describes the problem in detail. It starts with the specification of nonlinear and linear dynamic systems under consideration. Then, definitions of self-correction errors and the reachability degree are introduced. Section 3 is devoted to the approach based an the algebra of functions. Brief descriptions of algebraic tools and their properties in use are given. In Section 4, a solution to the error correction problem is given and an illustrative example is considered. In Section 5, a way to find a reachability degree of the final state is given and an illustrative example is considered. Section 6 analyses the duality between the error correction and reachability degree problems. In Section 7, a way to improve the error correction and reachability degrees based on an isomorphic transformation of a given system is suggested. The problem of inverse function design and the properties of isomorphic systems are investigated. In Section 7, the problem under consideration is studied for linear dynamic systems. The Jordan canonical form to improve the error correction and reachability degrees is suggested. Section 9 concludes the paper.

2. Problem Description

The problems under consideration are initially solved for nonlinear discrete dynamic systems described by difference equations:

$$x(t+1) = f\big(x(t), u(t)\big), \tag{1}$$

where $x \in \mathrm{X} \subset \mathbb{R}^n$ and $u \in \mathrm{U} \subset \mathbb{R}^s$ are the state and control vectors, respectively, and f is a nonlinear vector function. The obtained results are then applied to linear dynamic systems:

$$x(t+1) = Fx(t) + Gu(t), \tag{2}$$

where F and G are known matrices. Denote the model (1) by $\Sigma = (X, U, f)$.

Let $x^1(t_0)$ and $x^2(t_0)$ be expected and real values of a state vector, respectively, at the moment $t = t_0$ and define $\varepsilon = x^1(t_0) - x^2(t_0)$ as an error. The term "expected" means that $x^1(t_0) = f(x(t_0-1), u(t_0-1))$; the real value $x^2(t_0)$ differs from the expected one due to a malfunction in the system (e.g., in delayers) at $t = t_0$. We shall further assume that $t_0 = 0$.

The error ε is said to be self-corrected if $x^1(k) = x^2(k)$ for some $t = k$ where $x^1(k)$ (resp. $x^2(k)$) is a state to which the system transfers from the initial state $x^1(0)$ (resp. $x^2(0)$) under the control $U(k) = \{u(0), u(1), \dots, u(k-1)\}$. The error and self-correction processes are shown in Fig. 1.

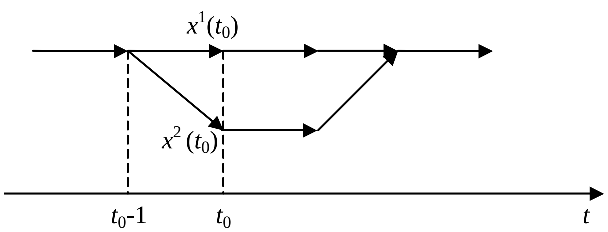

Fig. 1. Illustration of the error correction property for $k = 2$.

The problem is to describe the class of all self-correction errors and find a way to improve the self-correction property.

The accuracy of the state vector is described by a vector of functions φ defined on the set X. For example, if $\varphi(x) = x_1$, then the value of the first state component is known. If $\varphi(x) = [x_1 x_2 + x_4]^{\mathrm{T}}$, then the value of the first component and the sum of the second and the fourth one are known. In this case it can be said that the state x is known with an accuracy of φ.

The problem is formulated as follows: For a given system with the initial state known with an accuracy of φ and the control $U(k)$, find the accuracy ψ of the final state. This accuracy will be called the reachability degree. Besides, by analogy with self-correction analysis, find a way to increase the accuracy of the final state.

To solve these problems, the so-called algebra of functions developed for nonlinear systems in (Shumsky and Zhirabok, 2005; Zhirabok and Shumsky, 1993) and used for solving various problems in (Zhirabok, 1998; 2000) will be used. Main tools of the algebra of functions will be presented in the next section.

3. Algebra of Functions

Vector functions are elements of this algebra, which includes some binary relations, operations and operators.

1. Partial preordering relation $\le$: for any functions $\alpha : X \to S$ and $\beta : X \to W$ write $\alpha \le \beta$ if $\gamma\alpha = \beta$ for some function $\gamma : S \to W$, i.e., $\gamma(\alpha(x)) = \beta(x)$ for all $x \in X$ where S and W are some sets. If $\alpha \le \beta$ and $\beta \le \alpha$, then write $\alpha \approx \beta$.

2. Operation $\times$: the Cartesian product $\alpha \times \beta$ of the functions α and β is a function γ such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[d, "\gamma"] \arrow[dr, "\beta"] & \\ S & S \times W \arrow[l, "\pi_S"'] \arrow[r, "\pi_W"] & W \end{tikzcd} \tag{3}$$

is commutative, i.e., $\alpha(x) = \pi_S(\gamma(x))$ and $\beta(x) = \pi_W(\gamma(x))$ for all $x \in X$ where $\times$ is the Cartesian product of the sets S and W, π_S and π_W are projections: $\pi_S(s, w) = s$ and $\pi_W(s, w) = w$ for all $(s, w) \in S \times W$. From the definition of the Cartesian product of the sets it follows that γ is unique (Goldblad, 1979). There is an equivalent definition of the operation $\times$:

$$\alpha \times \beta = \max(\gamma \mid \gamma \le \alpha, \gamma \le \beta).$$

Diagram (3) results in the equalities $\alpha = \pi_S(\alpha \times \beta)$ and $\beta = \pi_W(\alpha \times \beta)$. It can be shown that $\alpha \times \beta = \left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right]$.

3. Binary relation Δ: $(\alpha, \beta) \in \Delta$, if $\beta f \ge \alpha\pi_X \times \pi_U$ or for some function $\gamma : S \times U \to W$ and all $(x, u) \in X \times U$ the equality $\beta(f(x,\, u)) = \gamma(\alpha(x), u)$ holds.

4. Operators $\boldsymbol{M}$ and $\boldsymbol{m}$: $\boldsymbol{M}(\beta)$ is a function satisfying the conditions

$$\big(\boldsymbol{M}(\beta), \beta\big) \in \Delta, \quad (\alpha, \beta) \in \Delta, \quad \alpha \le \boldsymbol{M}(\beta), \tag{4}$$

$\boldsymbol{m}(\alpha)$ is a function satisfying the conditions

$$\big(\alpha, \boldsymbol{m}(\alpha)\big) \in \Delta, \quad (\alpha, \beta) \in \Delta, \quad \boldsymbol{m}(\alpha) \le \beta. \tag{5}$$

The operator $\boldsymbol{M}$ can be calculated as follows: If β is a scalar function and $\beta(f(x,\, u)) = \sum_{i=1}^{d} a_i(x) b_i(u)$ where the functions $b_1, b_2, \ldots, b_d$ are linearly independent, then $\boldsymbol{M}(\beta) = a_1 \times a_2 \times \cdots \times a_d$. If $\beta = \beta_1 \times \beta_2 \times \cdots \times \beta_l$, then $\boldsymbol{M}(\beta) \approx \boldsymbol{M}(\beta_1) \times \boldsymbol{M}(\beta_2) \times \cdots \times \boldsymbol{M}(\beta_l)$. In the linear case, when $\beta(x) = Bx$ for some matrix B and the system is described by the model (2), we have $\boldsymbol{M}(B) = BF$, since the composition $\beta(f(x, u))$ is of the form $BFx + BGu$.

From the definition of the relation Δ and (5) it follows that $\boldsymbol{m}(\alpha)$ is a vector function with a maximal number of functionally independent components. Therefore, each of these components is a composition of variables from the left-hand side of Eqn. (1), and the corresponding composition on the right-hand side of this equation depends on the components of the function α. The term "functionally independent" is a generalization of the term "linearly independent": the functions $\gamma_1, \gamma_2, \ldots, \gamma_k$ are functionally independent if no nontrivial function φ exists such that $\varphi(\gamma_1(x), \gamma_2(x), \ldots, \gamma_k(x)) = 0$ for all $x \in X$ (Korn and Korn, 1961, pp. 4,5–6).

A formal procedure of evaluating the operator $\boldsymbol{m}$ demands a special operation (in addition to $\times$) and is rather difficult. It can be found in (Zhirabok and Shumsky, 1993). In simple cases, one can use the following rule explained on the basis of a dynamic system described by the following difference equations:

$$
\begin{aligned}
x_1(t+1) &= u_1(t)x_4(t),\\
x_2(t+1) &= u_1(t)x_3(t) + u_2(t),\\
x_3(t+1) &= u_2(t)\big(x_3(t) + x_4(t)\big)\big(x_1(t) + x_2(t)\big)\\
&\qquad + u_1(t)u_2(t),\\
x_4(t+1) &= u_1(t)\big(x_3(t) + x_4(t)\big) - u_2(t)\\
&\qquad \big(x_3(t) + x_4(t)\big)\big(x_1(t) + x_2(t)\big). \qquad (6)
\end{aligned}
$$

Consider the function $\alpha(x) = x_3 + x_4$. Find a vector function in the form of compositions of the variables x_i at the moment $t+1$ containing a maximal number of functionally independent components. Thus, the corresponding compositions of the variables x_i at the moment t depend on the sum $x_3(t) + x_4(t)$ and the control $u(t)$. Clearly, it is the function

$$\boldsymbol{m}\big(\alpha(x)\big) = \alpha_1(x) = (x_1 + x_2) \times (x_3 + x_4)$$

because

$$x_1(t+1) + x_2(t+1) = u_1(t)\big(x_3(t) + x_4(t)\big) + u_2(t)$$

and

$$x_3(t+1) + x_4(t+1) = u_1(t)\big(x_3(t) + x_4(t)\big) + u_1(t)u_2(t).$$

By analogy, one obtains

$$\boldsymbol{m}\big(\alpha_1(x)\big) = \boldsymbol{m}^2\big(\alpha(x)\big) = (x_1 + x_2) \times x_3 \times x_4.$$

In the linear case, when $\alpha(x) = Ax$ for some matrix A and the system is described by the model (2), the operator **m** can be implemented as follows: if $[\, Q \ \ N \,]$ is a matrix of a maximal rank such that

$$\begin{bmatrix} Q & N \end{bmatrix} \begin{bmatrix} F \\ A \end{bmatrix} = 0,$$

then $\boldsymbol{m}(A) = Q$.

The relations $\leq$ and Δ, the operation and operators have the following properties:

1. $\alpha \leq \beta \Rightarrow \alpha\delta \leq \beta\delta$;
2. $(\alpha \times \beta)\delta = \alpha\delta \times \beta\delta$;
3. if $(\alpha,\, \beta) \in \Delta$ and $\gamma \leq \alpha$, then $(\gamma,\, \beta) \in \Delta$;
4. $(\alpha,\, \beta) \in \Delta \Leftrightarrow \boldsymbol{m}(\alpha) \leq \beta \Leftrightarrow \alpha \leq \boldsymbol{M}(\beta)$;
5. if $\alpha \leq \beta$, then $\boldsymbol{m}(\alpha) \leq \boldsymbol{m}(\beta)$ and $\boldsymbol{M}(\alpha) \leq \boldsymbol{M}(\beta)$;
6. $\boldsymbol{M}(\boldsymbol{m}(\alpha)) \geq \alpha$, $\boldsymbol{m}(\boldsymbol{M}(\beta)) \leq \beta$.

4. Self-Correction Property Analysis

4.1. Theoretical Results. The main tools of the algebra of functions are operators $\boldsymbol{M}$ and $\boldsymbol{m}$, which are dual to each other by their definitions and properties. This duality allows one to use some property obtained with the help of the operator $\boldsymbol{M}$ in order to obtain a dual property based on the operator $\boldsymbol{m}$, and vice versa.

To solve the problems under consideration, we need an auxiliary result. The states x and x^0 are said to be φ-equivalent if $\varphi(x) = \varphi(x^0)$.

Lemma 1. *The φ-equivalence of states at the moment t implies the ψ–equivalence of states at $t+1$ under the arbitrary control $u(t)$ if and only if $(\varphi,\psi) \in \Delta$ that is equivalent to $\boldsymbol{m}(\varphi) \leq \psi$ or $\varphi \leq \boldsymbol{M}(\psi)$.*

Proof. (Necessity): Assume that $x(t)$ and $x^0(t)$ are states such that $\varphi(x(t)) = \varphi(x^0(t))$. Define the function γ for the state $x(t)$ and the control $u(t)$ as follows:

$$\psi\left(f\big(x(t),\, u(t)\big)\right) = \gamma\left(\varphi\big(x(t)\big), u(t)\right). \qquad (7)$$

Since $x(t+1) = f(x(t),\, u(t))$ and $x^0(t+1) = f(x^0(t),\, u(t))$, we have that $\varphi(x(t)) = \varphi(x^0(t))$ implies $\psi(f(x(t),\, u(t))) = \psi(f(x^0(t),\, u(t)))$ by assumption. This means that if the state $x(t)$ on the right-hand side of (7) is replaced by $x^0(t)$, then this equality is true. Therefore, the function γ is defined correctly. Then, by the definition of the relation Δ, the inclusion $(\varphi,\psi) \in \Delta$ holds, which, by the properties of the operators $\boldsymbol{m}$ and $\boldsymbol{M}$, is equivalent to the inequalities $\boldsymbol{m}(\varphi) \leq \psi$ and $\varphi \leq \boldsymbol{M}(\psi)$.

(Sufficiency): Assume that the inclusion $(\varphi,\psi) \in \Delta$ is true for the functions φ and ψ, i.e., for some function γ under the arbitrary control $u(t)$ the equality (7) holds. Let also the states $x(t)$ and $x^0(t)$ be φ-equivalent. Then (7) yields $\psi(f(x(t),\, u(t))) = \psi(f(x^0(t),\, u(t)))$, i.e., the states $x(t+1) = f(x(t),\, u(t))$ and $x^0(t+1) = f(x^0(t),\, u(t))$ are ψ-equivalent. This completes the proof. ■

Introduce the minimal (with respect to the relation $\leq$) function φ as follows: $\varphi(x^1(0)) = \varphi(x^2(0))$ where $x^1(0)$ and $x^2(0)$ are expected and real (due to a malfunction at the moment $t = 0$) values of the state vector at $t = 0$. In our case, this means that for the state $x(0)$ different from $x^1(0)$ and $x^2(0)$ the inequality $\varphi(x(0)) \neq \varphi(x^1(0))$ holds. For example, if ε is an error in the first component of the vector x, then $\varphi(x) = x_2 \times x_3 \times \cdots \times x_n$.

Using Lemma 1, we obtain the main result of this section.

Theorem 1. *The error ε is self-corrected by the time $t = k$ if and only if $\varphi \leq \boldsymbol{M}^k(\boldsymbol{e})$. Here $\boldsymbol{M}^{i+1} = \boldsymbol{M}(\boldsymbol{M}^i)$ and $\boldsymbol{e}$ is the identity function: $\boldsymbol{e}(x) = x, \forall x \in X$.*

Proof. (Necessity): From Lemma 1 it follows that if the state is known with an accuracy of φ at the moment $t = 0$, then it will be known with an accuracy of φ_1 or better at $t = 1$ if and only if the inequality $\varphi \leq \boldsymbol{M}(\varphi_1)$ holds. By analogy, a similar result is true for all i, $i = 0, 1, \dots, k-1$: $\varphi_i \leq \boldsymbol{M}(\varphi_{\mathrm{i}+1})$ with $\varphi_0 = \varphi$. Because, by definition, the error ε is self-corrected by the time $t = k$, we have $\varphi_k = \boldsymbol{e}$ and $\varphi_{k-1} \leq \boldsymbol{M}(\boldsymbol{e})$. Since $\varphi_1 \leq \boldsymbol{M}(\varphi_2)$, we get $\varphi \leq \boldsymbol{M}(\varphi_1) \leq \boldsymbol{M}^2(\varphi_2)$ by the definition of the operator $\boldsymbol{M}$. By analogy, one obtains the chain of inequalities $\varphi \leq \boldsymbol{M}(\varphi_1) \leq \cdots \leq \boldsymbol{M}^i(\varphi_i) \leq \cdots \leq \boldsymbol{M}^k(\varphi_k) = \boldsymbol{M}^k(\boldsymbol{e})$.

(Sufficiency): Let $\varphi \leq \boldsymbol{M}^k(\boldsymbol{e})$. Define the function φ_i as follows: $\varphi_i = \boldsymbol{M}^{k-i}(\boldsymbol{e}), i = 1, 2, \dots, k$, $\varphi_k = \boldsymbol{e}$. Consider the functions φ and $\varphi_1 \leq \boldsymbol{M}^{k-1}(\boldsymbol{e})$. From the properties of the operators $\boldsymbol{M}$ and $\boldsymbol{m}$ it follows that $\boldsymbol{m}(\varphi) \leq \boldsymbol{m}(\boldsymbol{M}^k(\boldsymbol{e})) \leq \boldsymbol{M}^{k-1}(\boldsymbol{e}) = \varphi_1$, which gives $(\varphi,\varphi_1) \in \Delta$. By the definition of the relation Δ this means that a function γ_0 exits such that $\varphi_1(f(x,u)) = \gamma_0(\varphi(x), u)$ for all $(x, u) \in X \times U$. If the states $x(0)$ and $x^0(0)$ satisfy the condition $\varphi(x(0)) = \varphi(x^0(0))$, then the equality $\varphi_1(f(x(0),(u(0))) = \varphi_1(f(x^0(0), u(0)))$, or $\varphi_1(x(1)) = \varphi_1(x^0(1))$, holds for some arbitrary control $u(0)$. Then it can be shown that $\boldsymbol{m}(\varphi_1) \leq \boldsymbol{m}(\boldsymbol{M}^{k-1}(\boldsymbol{e})) \leq \boldsymbol{M}^{k-2}(\boldsymbol{e}) = \varphi_2$, $(\varphi_1,\varphi_2) \in \Delta$, and $\varphi_2(f(x, u)) = \gamma_1(\varphi_1(x), u)$ for some function γ_1 and for all $(x, \mathrm{u}) \in X \times U$. Therefore, $\varphi_2(f(x(1),(u(1))) = \varphi_2(f(x^0(1), u(1)))$, or $\varphi_2(x(2)) = \varphi_2(x^0(2))$, holds for the arbitrary control $u(1)$. By analogy, one obtains $(\varphi_{k-1},\varphi_{\mathrm{k}}) \in \Delta$ and $\varphi_k(x(k)) = \varphi_k(x^0(k))$. Since $\varphi_k = \boldsymbol{e}$, we get $x(k) = x^0(k)$, i.e., the error described by the function φ will be corrected by the time $t = k$ under the arbitrary control $U(k)$. The proof is complete. ■

The function φ is said to describe the error correction degree if $\varphi = \boldsymbol{M}^k(\boldsymbol{e}) \approx \boldsymbol{M}^{k+1}(\boldsymbol{e})$ for some k.

By the definition of the function e, the inequality $\boldsymbol{e} \leq \boldsymbol{M}(\boldsymbol{e})$ holds. Since the operator $\boldsymbol{M}$ is monotonic, we have $\boldsymbol{M}(\boldsymbol{e}) \leq \boldsymbol{M}^2(\boldsymbol{e})$ and $\boldsymbol{e} \leq \boldsymbol{M}(\boldsymbol{e}) \leq \boldsymbol{M}^2(\boldsymbol{e}) \leq \dots$ The equivalence $\boldsymbol{M}^k(\boldsymbol{e}) \approx \boldsymbol{M}^{k+1}(\boldsymbol{e})$ for some k implies $\boldsymbol{M}^k(\boldsymbol{e}) \approx \boldsymbol{M}^{k+v}(\boldsymbol{e})$ for all $v = 1, 2, \dots$ This means that if an error is not corrected at the k-th step, it is never correct.

Clearly, if $\boldsymbol{M}(\boldsymbol{e}) \approx \boldsymbol{e}$, the system does not have the self-correction property.

4.2. Illustrative Example.

For the system described by the model (6), we wish to find the error correction degree. Suppose that a malfunction may occur in each component of the state vector. Hence

$$\varphi_1(x) = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \varphi_2(x) = \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix},$$

$$\varphi_3(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix}, \quad \varphi_4(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

To obtain the function $\mathbf{M}(\mathbf{e})$, write the composition $\boldsymbol{e}(f(x, u))$:

$$\boldsymbol{e}\big(f(x,\, u)\big) = f(x,\, u) = \begin{bmatrix} u_1 x_4 \\ u_1 x_3 + u_2 \\ u_2(x_3 + x_4)(x_1 + x_2) + u_1 u_2 \\ u_1(x_3 + x_4) - u_2(x_3 + x_4)(x_1 + x_2) \end{bmatrix}.$$

Thus $\boldsymbol{M}(\boldsymbol{e})(x) \approx (x_1 + x_2) \times x_3 \times x_4$, and the condition of Theorem 1 is not fulfilled for all functions φ_i.

To obtain the function $\boldsymbol{M}^2(\boldsymbol{e})$, write down the composition

$$\boldsymbol{M}(\boldsymbol{e})\big(f(x,\, u)\big) = \begin{bmatrix} u_1(x_4 + x_3) + u_2 \\ u_2(x_3 + x_4)(x_1 + x_2) + u_1 u_2 \\ u_1(x_3 + x_4) - u_2(x_3 + x_4)(x_1 + x_2) \end{bmatrix}.$$

Therefore $\boldsymbol{M}^2(\boldsymbol{e})(x) \approx (x_1 + x_2) \times (x_3 + x_4)$, and the condition $\varphi_i \leq \boldsymbol{M}^2(\boldsymbol{e})$ is not fulfilled for $i = 1, 2, 3, 4$.

Since

$$M^2(e)\big(f(x,\,u)\big) = \begin{bmatrix} u_1(x_4+x_3)+u_2 \\ u_1(x_3+x_4)+u_1u_2 \end{bmatrix},$$

we get $M^3(e)(x) = x_3 + x_4$. Since $\varphi_1 \le M^3(e)$ and $\varphi_2 \le M^3(e)$, the errors in the first and second components will be corrected in the third step.

The analysis shows that $M^4(e)(x) = M^3(e)(x) = x_3 + x_4$, and hence the errors in the third and fourth components are not corrected. However, the sum $x_3 + x_4$ has the self-correction property. Transform the initial system into the system with the state vector x^* determined as follows: $x_1^* = x_1,\ x_2^* = x_2,\ x_3^* = x_3,\ x_4^* = x_3 + x_4$. This gives the following description:

$$\begin{aligned} x_1^*(t+1) &= u_1(t)\big(x_4^*(t) - x_3^*(t)\big), \\ x_2^*(t+1) &= u_1(t)x_3^*(t) + u_2(t), \\ x_3^*(t+1) &= u_2(t)x_4^*(t)\big(x_1^*(t) + x_2^*(t)\big) + u_1(t)u_2(t), \\ x_4^*(t+1) &= u_1(t)x_4^*(t). \end{aligned} \tag{8}$$

Calculations yield

$$\begin{aligned} M^*(e)(x^*) &\approx (x_1^*{+}x_2^*) \times x_3^* \times x_4^*, \\ M^{*2}(e)(x^*) &\approx (x_1^* + x_2^*) \times x_4^*, \\ M^{*3}(e)(x^*) &\approx x_4^*, \end{aligned}$$

where M^* is an operator of the transformed system. More clearly, the errors in the first three components will be corrected in the third step. Indeed, if a malfunction occurs at $t = 0$ in the first or second components of the state vector x^*, then at $t = 1$ the error ε occurs in the third component, at $t = 2$ in the first and second components, and at $t = 3$ it disappears (see Table 1). It can be shown that the error in the third component disappears at $t = 2$. This corresponds to the form of the function $M^{*2}(e)$.

Table 1. Error propagation.

Components	$t=1$	$t=2$	$t=3$
x_1^*	—	$-\varepsilon$	—
x_2^*	—	ε	—
x_3^*	ε	—	—
x_4^*	—	—	—

5. Reachability Degree Analysis

5.1. Theoretical Results. The notion of φ-equivalence is connected with an accuracy of φ in the following way: Let the state $x(t)$ at the moment t be known with an accuracy of φ, i.e., the value of the function $\varphi(x(t))$ is known. Assume also that the state $x(t+1)$ calculated on the basis of the state $x(t)$ and the control $u(t)$ is known with an accuracy of ψ. Clearly, if the state $x^0(t)$ is φ-equivalent to $x(t)$, i.e., $\varphi(x(t)) = \varphi(x^0(t))$, then the state $x^0(t+1)$ calculated on the basis of the state $x^0(t)$ and the control $u(t)$ is known also with an accuracy of ψ. Consequently, the equality $\psi(f(x(t),\,u(t))) = \psi(f(x^0(t),\,u(t)))$ holds. Thus, with an accuracy of φ (or ψ) in mind, one has to take into consideration the class of φ-equivalent (ψ-equivalent) states and the binary relation Δ.

Theorem 2. *If the initial state $x(0)$ is known with an accuracy of φ, then under the control $U(k)$ the state $x(k)$ is known with an accuracy of ψ if and only if $m^k(\varphi) \le \psi$ where $m^{i+1} = m(m^i)$.*

Proof. (Necessity): From Lemma 1 and the compatibility between the accuracy φ and φ-equivalence it follows that the accuracy ψ_1 with which the state $x(1)$ at the moment $t = 1$ can be obtained under the accuracy φ of the state $x(0)$ at $t = 0$ and the arbitrary control $u(0)$ can be specified by the inequality $m(\varphi) \le \psi_1$. By analogy, the accuracy ψ_2 with which the state $x(2)$ at the moment $t = 2$ can be obtained under the accuracy ψ_1 of the state $x(1)$ at $t = 1$ and the arbitrary control $u(1)$ can be specified by the inequality $m(\psi_1) \le \psi_2$. At the moment $t = k$, the corresponding inequality is $m(\psi_{k-1}) \le \psi_k = \psi$. By the properties of the operator m and the transitivity of the relation $\le$, the inequality $m(\varphi) \le \psi_1$ implies $m^2(\varphi) \le m(\psi_1) \le \psi_2$. By analogy, $m^3(\varphi) \le m(\psi_2) \le \psi_3$ and, eventually, $m^k(\varphi) \le m(\psi_{k-1}) \le \psi_k = \psi$.

(Sufficiency): Let $m^k(\varphi) \le \psi$. Then from the properties of the operators M and m it follows that $M(m^k(\varphi)) \le M(\psi)$ and $m^{k-1}(\varphi) \le M(m^k(\varphi)) \le M(\psi)$. Writing $\psi_{k-1} = M(\psi)$, we get $m(\psi_{k-1}) = m(M(\psi)) \le \psi$. This inequality means that the functions ψ_{k-1} and ψ specify the ψ_{k-1}-and ψ-equivalent states at the moments $t = k{-}1$ and $t = k$, respectively. In other words, the accuracy ψ at $t = k$ can be obtained under the accuracy ψ_{k-1} at $t = k-1$. By analogy, it can be shown that the inequality $m^{k-1}(\varphi) \le \psi_{k-1}$ results in $m^{k-2}(\varphi) \le M(\psi_{k-1})$, and the functions $\psi_{k-2} = M(\psi_{k-1}) = M^2(\psi)$ and ψ_{k-1} specify the ψ_{k-2}- and ψ_{k-1}– equivalent states at $t = k - 2$ and $t = k - 1$, respectively. In other words, the accuracy ψ_{k-1} at $t = k - 1$ can be obtained under the accuracy ψ_{k-2} at $t = k - 2$. By analogy, one concludes that $m(\varphi) \le \psi_1 = M^{k-1}(\psi)$ and the functions φ and ψ_1 specify the φ – and ψ_1 -equivalent states at $t = 0$ and $t = 1$, respectively. Thus, in the i-th step, the system transfers from a state known with an accuracy of ψ_{i-1} under the control $u(i)$ into a state known with an accuracy of ψ_i, $i = 1, 2, \ldots, k$, $\psi_0 = \varphi$, $\psi_k = \psi$. It follows that the system transfers from the initial state known with an accuracy of φ under the control $U(k)$ into the final state known with an accuracy of ψ. The proof is complete. ■

Consider the important specific case when the initial state in unknown. In this case we have $\varphi = \mathbf{1}$, and the condition of Theorem 2 takes the form $\boldsymbol{m}^k(\mathbf{1}) \leq \psi$. Here $\mathbf{1}$ is the constant function: $\mathbf{1}(x) = c = \text{const}, \forall x \in X$. This case will be considered later.

Consider some properties of functions in the form $\boldsymbol{m}^i(1)$. From the definitions of the function $\mathbf{1}$ and the relation $\leq$ it follows that $\mathbf{1} \geq m(1)$. This results in $\boldsymbol{m}(1) \geq m^2(1)$. By analogy, one obtains the chain of inequalities $\mathbf{1} \geq m(1) \geq \dots \geq m^i(1) \geq \dots$.

Assume that the relation $\boldsymbol{m}^p(\mathbf{1}) \approx \boldsymbol{m}^{p+1}(\mathbf{1})$ holds for some p. From the properties of the operator $\boldsymbol{m}$ it follows that $\boldsymbol{m}^p(\mathbf{1}) \approx \boldsymbol{m}^{p+v}(\mathbf{1})$ for all $v = 1, 2, \dots$ For the problem under consideration this means that the reachability degree obtained by the time $t = p$ from an unknown initial state cannot be improved. If $\boldsymbol{m}(1) \approx \mathbf{1}$, one can say that the system reachability degree from an unknown initial state is equal to zero.

5.2. Illustrative Example. For the system described by (6), we wish to find the reachability degree from an unknown initial state. Because

$$x_1(t+1) + x_2(t+1) = u_1(t)\big(x_3(t) + x_4(t)\big) + u_2(t)$$

and

$$x_3(t+1)+x_4(t+1) = u_1(t)\big(x_3(t)+x_4(t)\big)+u_1(t)u_2(t),$$

we have

$$x_1(t+1) + x_2(t+1) - \big(x_3(t+1) + x_4(t+1)\big) = u_2(t) - u_1(t)u_2(t),$$

which implies

$$\boldsymbol{m}(\mathbf{1}) = x_1 + x_2 - (x_3 + x_4).$$

By analogy,

$$\boldsymbol{m}^2(\mathbf{1}) = x_1 + x_2 - (x_3 + x_4).$$

Accordingly, the reachability degree from the unknown initial state can be estimated by the function $\psi(x) = x_1 + x_2 - (x_3 + x_4)$.

Analogously with Section 4, transform the initial system into the system Σ^* with the state vector x^* determined as follows: $x_1^* = x_1$, $x_2^* = x_2$, $x_3^* = x_3$, $x_4^* = x_1 + x_2 - (x_3 + x_4)$. This gives the following description of the system Σ^*:

$$\begin{aligned} x_1^*(t+1) &= u_1(t)\Big(x_1^*(t) + x_2^*(t) - \big(x_3^*(t) + x_4^*(t)\big)\Big), \\ x_2^*(t+1) &= u_1(t)x_3^*(t) + u_2(t), \\ x_3^*(t+1) &= u_2(t)\big(x_1^*(t) + x_2^*(t) - x_4^*(t)\big) \\ &\quad \big(x_1^*(t) + x_2^*(t)\big) + u_1(t)u_2(t), \\ x_4^*(t+1) &= u_2(t) - u_1(t)u_2(t). \end{aligned} \tag{9}$$

It follows that $\psi^*(x^*) = \boldsymbol{m}^*(\mathbf{1}^*) = x_4^*$ where the asterisk denotes the elements of the system Σ^* corresponding to the ones of the system Σ. In terms of the relation $\leq$, the functions ψ and ψ^* are not comparable but it seems that information about a single component of the state vector is more preferable than that about some linear combination of these components.

6. Duality

First of all, the functions $\mathbf{1}$ and $\boldsymbol{e}$ are dual because they are unity and zero in the lattice of equivalent functions classes, respectively, due to the property $\boldsymbol{e} \leq \psi \leq \mathbf{1}$ for any arbitrary function ψ defined on the set X. The operators $\boldsymbol{M}$ and $\boldsymbol{m}$ are dual due to their definitions (see the relations (3) and (4)) and properties. Finally, the inequalities $\varphi \leq \boldsymbol{M}^k(\boldsymbol{e})$ and $\boldsymbol{m}^k(\mathbf{1}) \leq \psi$, as well as the chains of inequalities $\boldsymbol{e} \leq \boldsymbol{M}(\boldsymbol{e}) \leq \dots \leq \boldsymbol{M}^i(\boldsymbol{e}) \leq$ and $\mathbf{1} \geq \boldsymbol{m}(1) \geq \dots \geq \boldsymbol{m}^i(1) \geq \cdots$, are dual.

Results obtained via self-correction analysis can be used to solve some problems of reachability analysis as follows: Let $\alpha = \boldsymbol{M}^k(\boldsymbol{e})$. Then $\boldsymbol{m}(\alpha) = \boldsymbol{m}(\boldsymbol{M}^k(\boldsymbol{e})) \leq \boldsymbol{M}^{k-1}(\boldsymbol{e})$. By analogy, $\boldsymbol{m}^2(\alpha) = \boldsymbol{m}(\boldsymbol{M}^{k-1}(\boldsymbol{e})) \leq \boldsymbol{M}^{k-2}(\boldsymbol{e})$ and, eventually, $\boldsymbol{m}^k(\alpha) \leq \boldsymbol{e}$. As $\boldsymbol{e}$ is the least element with respect to the relation $\leq$, we get $\boldsymbol{m}^k(\alpha) \approx \boldsymbol{e}$. Since the inequality $\gamma \leq \alpha$ implies $\boldsymbol{m}^k(\gamma) \leq \boldsymbol{m}^k(\alpha)$, we obtain $\boldsymbol{m}^k(\gamma) \approx \boldsymbol{e} \approx \boldsymbol{m}^k(\alpha)$.

This result can be interpreted as follows: The inequality $\gamma \leq \alpha$ means that the function γ assures an accuracy degree which is not worse than that of the function α. This, however, is unnecessary because $\boldsymbol{m}^k(\gamma) \approx \boldsymbol{m}^k(\alpha)$ holds. Consequently, the function α specifies the least (in terms of the relation $\leq$) accuracy degree of the system initial state with which the greatest accuracy degree of the final state will be obtained under the control $U(k)$, i.e., the final state will be exactly known.

As has been shown in Section 4.2, $\alpha = \boldsymbol{M}^3(\boldsymbol{e}) = x_3 + x_4$, and then $\boldsymbol{m}^3(\alpha) \approx \boldsymbol{e}$. This can be confirmed by the calculations which were partially performed in Section 3, where it was shown that

$$\boldsymbol{m}(\alpha) = \boldsymbol{m}(x_3 + x_4) = (x_1 + x_2) \times (x_3 + x_4)$$

and

$$\boldsymbol{m}^2(\alpha) = (x_1 + x_2) \times x_3 \times x_4.$$

The next step gives

$$\boldsymbol{m}^3(\alpha) = x_1 \times x_2 \times x_3 \times x_4 = \boldsymbol{e}.$$

In much the same way, the following problem can be solved: Find an accuracy of the system initial state such that the accuracy of the final state will be no less than ψ under the control $U(k)$. Clearly, this accuracy is specified by the function $\varphi = \boldsymbol{M}^k(\psi)$ since $\boldsymbol{m}^k(\varphi) \leq \psi$ in this case.

Dually, assume that $\beta = \boldsymbol{m}^k(\mathbf{1})$. Then, by analogy with the previous case, it can be shown that $\boldsymbol{M}^k(\beta) \geq \mathbf{1}$ or (due to the definition of the function **1**) $\boldsymbol{M}^k(\beta) \approx \mathbf{1}$. Because the inequality $\beta \leq \gamma$ implies $\boldsymbol{M}^k(\beta) \leq \boldsymbol{M}^k(\gamma)$, we have $\boldsymbol{M}^k(\gamma) \approx \mathbf{1} \approx \boldsymbol{M}^k(\beta)$. These results can be interpreted as follows: According to Section 4, a malfunction in the system resulting in an arbitrary error can be corrected to a state known with an accuracy which is no less than β.

If the desired accuracy is given by the function δ, then the errors which can be corrected by the time $t = k$ with an accuracy of δ are specified by the function $\varphi = \boldsymbol{m}^k(\delta)$.

7. Increase in the error correction and reachability degrees

7.1. General Relationships. The examples in Sections 4.2 and 5.2 show the idea of increasing the error correction and reachability degrees through an isomorphic transformation of the initial system. Consider this idea in the general case.

Recall that a function $\Phi : X \to X^*$ is an isomorphism $\Sigma \to \Sigma^* = (X^*, U^*, f^*)$ with $U = U^*$ if the following diagram is commutative:

$$\begin{tikzcd} X \times U \arrow[r, "f"] \arrow[d, "\Phi\pi_X \times \pi_U"'] & X \arrow[d, "\Phi"] \\ X^* \times U \arrow[r, "f^*"] & X^* \end{tikzcd}$$

i.e., $\Phi f = f^*(\Phi\pi_X \times \pi_U)$, or $\Phi f(x,u) = f^*(\Phi(x), u)$ for all $(x,u) \in X \times U$ where π_X and π_U are projections: $\pi_X(x,u) = x$ and $\pi_U(x,u) = u$ for all $(x,u) \in X \times U$. In this case, a function $\Phi^{-1} : X^* \to X$ must exist such that $\Phi^{-1}\Phi \approx \boldsymbol{e}$, $\Phi\Phi^{-1} \approx \boldsymbol{e}^*$, $\Phi^{-1}(f^*(x^*,u)) = f^*(\Phi^{-1}(x^*), u)$ for all $(x^*, u) \in X^* \times U$.

Assume that the relationships

$$\boldsymbol{M}^{k+1}(\boldsymbol{e}) \approx \boldsymbol{M}^k(\boldsymbol{e}) \approx \rho_1 \times \rho_2 \times \cdots \times \rho_m \tag{10}$$

for the self-correction degree analysis and

$$\boldsymbol{m}^{k+1}(\mathbf{1}) \approx \boldsymbol{m}^k(\mathbf{1}) \approx \rho_1 \times \rho_2 \times \cdots \times \rho_m \tag{11}$$

for the reachability degree analysis hold for some k and $n-m$ components of the state vector exist (with no loss of generality, we assume that these are $x_1, x_2, \ldots, x_{n-m}$) such that

$$\begin{aligned}\Phi(x) = x^* = x_1 \times x_2 \times \cdots \times x_{n-m} \times \rho_1(x) \times \rho_2(x) \\ \times \cdots \times \rho_m(x) \approx \boldsymbol{e}^*(x^*).\end{aligned} \tag{12}$$

The last assumption is a basis for an isomorphic transformation of the system to increase the error correction and reachability degrees. Our goal is to show that the components $x_1^* = x_1, x_2^* = x_2, \ldots, x_{n-m}^* = x_{n-m}$ of the system Σ^* will be corrected by the time $t = k$. Dually, the last m components of the system Σ^* specified by the functions $\rho_1, \rho_2, \ldots, \rho_m$ will be exactly known at the time $t = k$ under an unknown initial state and the control $U(k)$.

Rewrite (12) as

$$\Phi = \pi_1 \times \pi_2 \times \cdots \times \pi_{n-m} \times \rho_1 \times \rho_2 \times \cdots \times \rho_m, \tag{13}$$

where π_j is the projection: $\pi_j(x) = x_j$, $j = 1, 2, \ldots, n$. The notation π_j will be useful for formal transformations.

The expression (13) for the function ϕ means that the first $n-m$ components of the initial basis are retained, i.e., $x_1^* = x_1,\ x_2^* = x_2, \ldots, x_{n-m}^* = x_{n-m}$. The last components are exposed to nontrivial transformations.

7.2. Inverse Function Design. Consider the case when each function ρ_j contains only one variable different from $x_1, x_2, \ldots, x_{n-m}$. Assume that this variable is x_{n-m+j}. This can be achieved by changing the indices of the functions $\rho_1, \rho_2, \ldots, \rho_m$. For simplicity, consider the function ρ_1 only.

Let $x_{i_1}, x_{i_2}, \ldots, x_{i_p}$ be the arguments of this function with $x_{i_p} = x_{n-m+1}$. Assume that

$$\begin{aligned}\pi_{i_1} \times \pi_{i_2} \times \cdots \times \pi_{i_{p-1}} \times \rho_1(\pi_{i_1} \times \pi_{i_2} \times \cdots \times \pi_{i_p}) \\ \approx \pi_{i_1} \times \pi_{i_2} \times \cdots \times \pi_{n-m+1}.\end{aligned}$$

Also suppose that the functions $\rho_2, \ldots, \rho_{\mathrm{m}}$ have similar structures and properties.

The example of Section 4.2 with the system (8) gives $m = 1$. The variables $x_{i_1}, x_{i_2}, x_{i_3}$ are x_1, x_2, x_3 and $\rho_1(x) = x_3 + x_4$. The isomorphism Φ, denoted by Φ_e, is of the form $\Phi_e = \pi_1 \times \pi_2 \times \pi_3 \times \rho_1(\pi_3 \times \pi_4)$ and can be represented by the matrix

$$\Phi_e = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The example of Section 5.2 with the system (9) gives $m = 1$. The variables $x_{i_1}, x_{i_2}, x_{i_3}$ are x_1, x_2, x_3 and $\rho_1(x) = x_1 + x_2 - x_3 - x_4$. The isomorphism Φ, denoted by Φ_r, is of the form $\Phi_r = \pi_1 \times \pi_2 \times \pi_3 \times \rho_1(\pi_1 \times \pi_2 \times \pi_3 \times \pi_4)$ and can be represented by the matrix

$$\Phi_r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & -1 \end{bmatrix}.$$

The function Φ in (13) is the required isomorphic transformation of the initial system. Assume that the inverse function Φ^{-1} exists and has the following form:

$$\begin{aligned}\Phi^{-1} &= \pi_1^* \times \pi_2^* \times \cdots \times \pi_{n-m}^* \\ &\quad \times \mu_1(\pi_{i_1}^* \times \pi_{i_2}^* \times \cdots \times \pi_{i_p}^*) \\ &\quad \times \mu_2(\dots) \times \cdots \times \mu_m(\dots),\end{aligned} \tag{14}$$

where the function μ_1 is the inverse one of ρ_1 in the following sense:

$$\begin{aligned}&\rho_1\Big(\pi_{i_1}^* \times \pi_{i_2}^* \times \cdots \times \pi_{i_{p-1}}^* \times \mu_1(\pi_{i_1}^* \times \pi_{i_2}^* \times \cdots \times \pi_{i_p}^*)\Big) \\ &\qquad \approx \pi_{i_p}^* = \pi_{n-m+1}^*.\end{aligned} \tag{15}$$

The last relationship is a result of transforming the following expression:

$$\begin{aligned}&\rho_1(\pi_{i_1} \times \pi_{i_2} \times \cdots \times \pi_{i_p})\big(\pi_1^* \times \pi_2^* \times \cdots \times \pi_{n-m}^* \\ &\times \mu_1(\pi_{i_1}^* \times \pi_{i_2}^* \times \cdots \times \pi_{i_p}^*) \times \mu_2(\dots) \times \cdots \times \mu_m(\dots)\big),\end{aligned}$$

which is a part of the composition $\Phi\Phi^{-1}$ taking account of the fact that $x_{i_p} = x_{n-m+1}$ and the function μ_1 is located in the $(n-m+1)$-th position in (14).

It is supposed that the functions $\mu_2, \dots, \mu_{\mathrm{m}}$ have similar properties. The system (8) gives the following results: $\mu_1(x^*) = x_4^* - x_3^*$; the isomorphism Φ_e^{-1} is of the form $\Phi_e^{-1} = \pi_1^* \times \pi_2^* \times \pi_3^* \times \mu_1(\pi_3^* \times \pi_4^*)$ and can be represented by the matrix

$$\Phi_e^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix},$$

(15) takes the form $\rho_1(\pi_3^* \times \mu_1(\pi_3^* \times \pi_4^*)) = \pi_4^*$, because the function ρ_1 sums up their arguments and the function μ_1 is subtraction.

The system (9) yields the following results: $\mu_1(x^*) = x_1^* + x_2^* - (x_3^* + x_4^*)$; the isomorphism Φ_r^{-1} is of the form $\Phi_r^{-1} = \pi_1^* \times \pi_2^* \times \pi_3^* \times \mu_1(\pi_1^* \times \pi_2^* \times \pi_3^* \times \pi_4^*)$ and can be represented by the matrix

$$\Phi_r^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & -1 \end{bmatrix},$$

(15) takes the form

$$\rho_1\big(\pi_1^* \times \pi_2^* \times \pi_3^* \times \mu_1(\pi_1^* \times \pi_2^* \times \pi_3^* \times \pi_4^*)\big) = \pi_4^*,$$

because the function ρ_1 performs the operation $x_1 + x_2 - x_3 - x_4$, and the function μ_1 the operation $x_1^* + x_2^* - x_3^* - x_4^*$.

In this case, the properties of the projections and the operation give the following result for the composition $\Phi\Phi^{-1}$:

$$\begin{aligned}\Phi\Phi^{-1} &= \Big(\pi_1 \times \pi_2 \times \cdots \times \pi_{n-m} \times \rho_1(\pi_{i_1} \times \pi_{i_2} \\ &\qquad \times \cdots \times \pi_{i_p}) \times \rho_2(\dots) \times \cdots \times \rho_m(\dots)\Big) \\ &\qquad \Big(\pi_1^* \times \pi_2^* \times \cdots \times \pi_{n-m}^* \times \mu_1(\pi_{i_1}^* \times \pi_{i_2}^* \\ &\qquad \times \cdots \times \pi_{i_p}^*) \times \mu_2(\dots) \times \cdots \times \mu_m(\dots)\Big) \\ &= \pi_1^* \times \pi_2^* \times \cdots \times \pi_{n-m}^* \times \Big(\rho_1(\pi_{i_1} \times \pi_{i_2} \\ &\qquad \times \cdots \times \pi_{i_p}) \times \rho_2(\dots) \times \cdots \times \rho_m(\dots)\Big) \\ &\qquad (\pi_1^* \times \pi_2^* \times \cdots \times \pi_{n-m}^* \times \mu_1(\pi_{i_1}^* \times \pi_{i_2}^* \\ &\qquad \times \cdots \times \pi_{i_p}^*) \times \mu_2(\dots) \times \cdots \times \mu_m(\dots)) \\ &= \pi_1^* \times \pi_2^* \times \cdots \times \pi_{n-m}^* \times \Big(\rho_1\Big(\pi_{i_1}^* \times \pi_{i_2}^* \\ &\qquad \times \cdots \times \pi_{i_{p-1}}^* \times \mu_1(\pi_{i_1}^* \times \pi_{i_2}^* \times \cdots \times \pi_{i_p}^*)\Big) \\ &\qquad \times \rho_2\big(\cdots \times \mu_2(\dots)\big) \times \dots\Big) \\ &\approx \pi_1^* \times \pi_2^* \times \cdots \times \pi_{n-m}^* \times \pi_{n-m+1}^* \\ &\qquad \times \pi_{n-m+2}^* \times \cdots \times \pi_n^* \approx \boldsymbol{e}.\end{aligned}$$

Accordingly, the function Φ^{-1} from (14) is indeed the inverse of Φ. To clarify the role of this function in the problems under consideration, we shall indicate some properties of the operators $\boldsymbol{M}$ and $\boldsymbol{m}$ for isomorphic systems.

7.3. Properties of the Operators M and m for Isomorphic Systems.

Let Φ be an isomorphism $\Sigma \to \Sigma^*$, i.e., $\Phi f = f^*(\Phi\pi_X \times \pi_U)$, and $\beta = \beta^*\Phi$ for some functions $\beta : X \to W$ and $\beta^* : X^* \to W$. Then $\beta f = \beta^*\Phi f = \beta^* f^*(\Phi\pi_X \times \pi_U)$.

By the definition of the operator $M*$ for the system Σ^*, the inclusion $(\boldsymbol{M}^*(\beta^*), \beta^*) \in \Delta$ holds. It gives the inequalities

$$\beta^* f^* \geq \boldsymbol{M}^*(\beta^*)\pi_{X^*} \times \pi_U$$

and, with the function $\Phi\pi_X \times \pi_U$, the inequality

$$\begin{aligned}&\beta^* f^*(\Phi\pi_X \times \pi_U) = \beta f \\ &\qquad \geq \big(\boldsymbol{M}^*(\beta^*)\pi_{X^*} \times \pi_U\big)(\Phi\pi_X \times \pi_U).\end{aligned}$$

Since $\pi_{X^*}(\Phi\pi_X \times \pi_U) = \Phi\pi_X$ and $\pi_U(\Phi\pi_X \times \pi_U) = \pi_U$ due to the properties of the projections π_{X^*} and π_U, we obtain $\beta f \geq \boldsymbol{M}^*(\beta^*)\Phi\pi_X \times \pi_U$. By the definition of Δ, this inequality is equivalent to the inclusion $(\boldsymbol{M}^*(\beta^*)\Phi, \beta) \in \Delta$, which implies, by the definition of the operator $\boldsymbol{M}$,

$$\boldsymbol{M}(\beta) = \boldsymbol{M}(\beta^*\Phi) \geq \boldsymbol{M}^*(\beta^*)\Phi. \tag{16}$$

Since $\beta = \beta^*\Phi$ implies $\beta^* = \beta\Phi^{-1}$, (16) allows us to write down the inequality $\boldsymbol{M}^*(\beta^*) \geq \boldsymbol{M}(\beta)\Phi^{-1}$, or $\boldsymbol{M}^*(\beta^*)\Phi \geq \boldsymbol{M}(\beta)$, which gives the desired relationship $\boldsymbol{M}(\beta) = \boldsymbol{M}(\beta^*\Phi) \approx \boldsymbol{M}^*(\beta^*)\Phi$, or $\boldsymbol{M}^*(\beta^*) \approx \boldsymbol{M}(\beta)\Phi^{-1}$. It can be shown that this one is true for the k-th degree of the operator $\boldsymbol{M}$:

$$\boldsymbol{M}^{*k}(\beta^*) \approx \boldsymbol{M}^{*k}(\beta\Phi^{-1}) \approx \boldsymbol{M}^k(\beta)\Phi^{-1}. \tag{17}$$

The definitions of the identity functions $\boldsymbol{e}{:}X \to X$ and $\boldsymbol{e}^*{:}X^* \to X^*$ and the equivalence $\approx$ imply $\boldsymbol{e} \approx \Phi\boldsymbol{e} = \boldsymbol{e}^*\Phi \approx \boldsymbol{e}^*$ if Φ is an isomorphism. Then (10) and (17) with $\beta^* = \boldsymbol{e}^*$, $\beta = \boldsymbol{e}$, and $\boldsymbol{e}^* = \boldsymbol{e}\Phi^{-1}$ can be transformed as follows:

$$\begin{aligned}
&\boldsymbol{M}^{*k}(\boldsymbol{e}^*) \approx \boldsymbol{M}^{*k}(\boldsymbol{e}\Phi^{-1}) \approx \boldsymbol{M}^k(\boldsymbol{e})\Phi^{-1}\\
&\approx \Big(\rho_1(\dots) \times \rho_2(\dots) \times \dots \times \rho_m(\dots)\Big)\Phi^{-1}\\
&\approx \Big(\rho_1(\dots) \times \rho_2(\dots) \times \dots \times \rho_m(\dots)\Big)\\
&\quad \Big(\pi_1^* \times \pi_2^* \times \dots \times \pi_{n-m}^* \times \mu_1(\pi_{i_1}^* \times \pi_{i_2}^*\\
&\quad \times \dots \times \pi_{i_p}^*) \times \mu_2(\dots) \times \dots \times \mu_m(\dots)\Big)\\
&= \rho_1\Big(\pi_{i_1}^* \times \pi_{i_2}^* \times \dots \times \pi_{i_{p-1}}^* \times \mu_1\\
&\quad (\pi_{i_1}^* \times \pi_{i_2}^* \times \dots \times \pi_{i_p}^*)\Big) \times \rho_2\Big(\dots \times \mu_2(\dots)\Big)\\
&\quad \times \dots \times \rho_m\Big(\dots \times \mu_m(\dots)\Big).
\end{aligned}$$

The property (15) for the function ρ_1 and a similar one for the functions $\rho_1, \dots, \rho_m$ yield with the above expression the relationship $\boldsymbol{M}^{*k}(\boldsymbol{e}^*) \approx \pi_{n-m+1}^* \times \dots \times \pi_n^*$. By Theorem 1, this gives the following result: an error in the components $x_1^*, x_2^*, \dots, x_{n-m}^*$ of the system Σ^* state vector will be corrected by the time $t = k$.

As has been shown in Section 4.2,

$$\begin{aligned}
&\boldsymbol{M}^{*3}(\boldsymbol{e}^*) \approx \boldsymbol{M}^3(\boldsymbol{e})\Phi^{-1}\\
&\qquad \approx \rho_1\big(\pi_3^* \times \mu_1(\pi_3^* \times \pi_4^*)\big) = \pi_4^*,
\end{aligned}$$

and therefore the errors in the first three components will be corrected in the third step.

Consider briefly similar properties for the operator $\boldsymbol{m}$. From the definition of $\boldsymbol{m}^*$ for the system Σ^*, it follows that the inclusion $(\beta^*, \boldsymbol{m}^*(\beta^*)) \in \Delta^*$ holds. It is equivalent to the inequality $\boldsymbol{m}^*(\beta^*)f^* \geq \beta^*\pi_{X^*} \times \pi_U$, which, with the function $\Phi\pi_X \times \pi_U$, gives

$$\boldsymbol{m}^*(\beta^*)\Phi f \geq (\beta^*\pi_{X^*} \times \pi_U)(\Phi\pi_X \times \pi_U).$$

Because

$$\pi_{X^*}(\Phi\pi_X \times \pi_U) = \Phi\pi_X$$

and

$$\pi_U(\Phi\pi_X \times \pi_U) = \pi_U,$$

we get

$$\boldsymbol{m}^*(\beta^*)\Phi f \geq \beta^*\Phi\pi_X \times \pi_U = \beta\pi_X \times \pi_U.$$

By the definition of Δ this means that $(\beta, \boldsymbol{m}^*(\beta^*)\Phi) \in \Delta$ and, by the definition of the operator $\boldsymbol{m}$,

$$\boldsymbol{m}(\beta) = \boldsymbol{m}(\beta^*\Phi) \leq \boldsymbol{m}^*(\beta^*)\Phi. \tag{18}$$

Analogously to the operator $\boldsymbol{M}$, we write down $\boldsymbol{m}^*(\beta^*) \leq \boldsymbol{m}^*(\beta^*)\Phi^{-1}$ and $\boldsymbol{m}^*(\beta^*)\Phi \leq \boldsymbol{m}(\beta)$ which, with (18), gives the relations $\boldsymbol{m}^*(\beta^*) \approx \boldsymbol{m}^*(\beta^*)\Phi^{-1}$ and

$$\boldsymbol{m}^{*k}(\beta^*) \approx \boldsymbol{m}^{*k}(\beta\Phi^{-1}) \approx \boldsymbol{m}^k(\beta)\Phi^{-1}. \tag{19}$$

The functions $\mathbf{1}{:}\ X \to \{c\}$ and $\mathbf{1}^*{:}\ X^* \to \{c\}$ are such that $\mathbf{1} = \mathbf{1}^*\Phi$. Then (11) and (19) with $\beta^* = \mathbf{1}^*$, $\beta = \mathbf{1}$, and $\mathbf{1}^* = \mathbf{1}\Phi^{-1}$ can be written down as follows:

$$\begin{aligned}
\boldsymbol{m}^{*k}(\mathbf{1}^*) &\approx \boldsymbol{m}^{*k}(\mathbf{1}\Phi^{-1}) \approx \boldsymbol{m}^k(\mathbf{1})\Phi^{-1}\\
&\approx \big(\rho_1(\dots) \times \rho_2(\dots) \times \dots \times \rho_m(\dots)\big)\Phi^{-1}\\
&\approx \pi_{n-m+1}^* \times \dots \times \pi_n^*.
\end{aligned}$$

In accordance with Theorem 2, this means that the components $x_{n-m+1}^*, \dots, x_n^*$ will be exactly known by the time $t = k$. This result is dual to the one obtained above: an error in the components $x_1^*, x_2^*, \dots, x_{n-m}^*$ will be corrected by the time $t = k$.

As has been shown in Section 5.2,

$$\begin{aligned}
\boldsymbol{m}^*(\mathbf{1}^*) &\approx \boldsymbol{m}(\mathbf{1})\Phi^{-1}\\
&\approx \rho_1\big(\pi_1^* \times \pi_2^* \times \pi_3^* \times \mu_1(\pi_1^* \times \pi_2^* \times \pi_3^* \times \pi_4^*)\big)\\
&= \pi_4^*,
\end{aligned}$$

and therefore the forth component of the state vector x^* will be exactly known at $t = 1$.

8. Linear Systems

8.1. Theoretical Results. In the linear case, when our system is described by (2), the problems under consideration can be solved more simply.

As has been shown in Section 3, if $\beta(x) = Bx$ for some matrix B, then $\boldsymbol{M}(B) = BF$ for the system (2). In the linear case, $\boldsymbol{e}(x) = Ex$ (E is the identity matrix) and $\boldsymbol{M}(E) = F$, $\boldsymbol{M}^k(E) = F^k$.

Recall that the operator $\boldsymbol{m}$ can be calculated as follows: If $\begin{bmatrix} Q & N \end{bmatrix}$ is a matrix of a maximal rank such that

$$\begin{bmatrix} Q & N \end{bmatrix} \begin{bmatrix} F \\ B \end{bmatrix} = 0,$$

then $\boldsymbol{m}(B) = Q$. The function **1** in the linear case is the n-dimensional row vector $0 = [00 \dots 0]$ filled with zeros.

The equality

$$\begin{bmatrix} Q & N \end{bmatrix} \begin{bmatrix} F \\ \mathbf{0} \end{bmatrix} = 0$$

yields $QF = 0$, and therefore $\boldsymbol{m}(0) = Q$ is a matrix with maximal numbers of rows orthogonal to the columns of the matrix F. To obtain the matrix $\boldsymbol{m}^2(0)$, consider the equality

$$\begin{bmatrix} Q_1 & N_1 \end{bmatrix} \begin{bmatrix} F \\ \boldsymbol{m}(\mathbf{0}) \end{bmatrix} = 0,$$

which gives $Q_1 F = -N_1 \boldsymbol{m}(\mathbf{0}) = -N_1 Q$ and $Q_1 F^2 = -N_1 QF = 0$. Therefore, $\boldsymbol{m}^2(\mathbf{0}) = Q_1$ is a matrix with maximal numbers of rows orthogonal to the columns of the matrix F^2. By analogy, the matrix $\boldsymbol{m}^k(\mathbf{0})$ is defined on the basis of the matrix F^k.

Clearly, if $\det(F) \neq 0$, then $\boldsymbol{M}(E) = F \approx E$, and the system does not have the self-correction property. Dually, $QF = 0$ implies $\mathbf{m}(\mathbf{0}) = \mathbf{0}$, and the reachability degree of the system from unknown initial state is zero.

Let $\det(F) = 0$ and $F^k \approx F^{k+1}$ for some k, i.e., $AF^k = F^{k+1}$ and $BF^{k+1} = F^k$ for some matrices A and B. Clearly, these equalities are equivalent to the conditions

$$\operatorname{rank}(F^k) = \operatorname{rank} \begin{bmatrix} F^k \\ F^{k+1} \end{bmatrix}$$

and

$$\operatorname{rank}(F^{k+1}) = \operatorname{rank} \begin{bmatrix} F^{k+1} \\ F^k \end{bmatrix},$$

respectively. That is, $F^k \approx F^{k+1}$ if and only if

$$\operatorname{rank}(F^k) = \operatorname{rank}(F^{k+1}).$$

In the linear case, the self-correction condition for the error ε, i.e., the inequality $\varphi \leq \boldsymbol{M}^k(\boldsymbol{e}) = \boldsymbol{M}^k(E) = F^k$ takes the following form: if the matrix F^k contains zero columns with numbers $i_1, i_2, \dots, i_m$, then the errors in the components of the state vector with these numbers will be corrected. Dually, the condition $\boldsymbol{m}^k(\mathbf{1}) \leq \psi$ takes the following form: If the matrix F^k contains zero rows with numbers $i_1, i_2, \dots, i_m$, then the vector state components with these numbers will be exactly known at the time $t = k$.

To improve the error correction and reachability degrees, we have to use an isomorphic transformation of the system based on the Jordan canonical form of the matrix F. It is known (Bellman, 1960; Lankaster, 1969) that for a square $n \times n$ matrix F there exists a nonsingular matrix Φ such that

$$F^* = \Phi F \Phi^{-1} = \begin{bmatrix} L_{k_1}(\lambda_1) & & & 0 \\ & L_{k_2}(\lambda_2) & & \\ 0 & & \dots & \\ & & & L_{k_r}(\lambda_r) \end{bmatrix}, \quad (20)$$

where $k_1 + k_2 + \dots + k_r = n$ and $\lambda_1, \lambda_2, \dots, \lambda_r$ are the eigenvalues of F. Furthermore, $L_k(\lambda)$ is the Jordan block (i.e., a $k \times k$ matrix) which has the form

$$L_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & \dots & 0 & \lambda \end{bmatrix}, \quad (21)$$

where k is the multiplicity of the eigenvalue λ.

Let $\operatorname{rank}(F) = m < n$. Then zero is an eigenvalue of the matrix F with the multiplicity $l \geq n - m$ (Bellman, 1960). From (20) and (21) it follows that if all eigenvalues are nonzero, then all rows of the matrix $F^* = \Phi F \Phi^{-1}$ are linearly independent. Because Φ is a nonsingular matrix and $\operatorname{rank}(F) = m < n$, we obtain $\operatorname{rank}(F^*) = m - n$, and hence the matrix F^* contains $n - m$ Jordan blocks corresponding to the zero eigenvalue of the form

$$L_k(0) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & \\ & & & & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}. \quad (22)$$

If $k = 1$, then $L_1(0) = 0$. Therefore, errors which may occur in the components of the vector $x^* = \Phi x$ corresponding to the zero columns in matrices of the form (22) are self-corrected. Dually, the components of the state vector $x^* = \Phi x$, corresponding to zero rows in matrices of the form (22), will be exactly known at the time $t = 1$ under the control $u(0)$ from an unknown initial state at $t = 0$.

It is known that the square of the Jordan canonical form (20) gives a matrix of a block-diagonal structure. In this case, the Jordan blocks in the matrix $(F^*)^2$ are equal to the squares of the corresponding Jordan blocks in the matrix F^*. This statement is true for an arbitrary degree

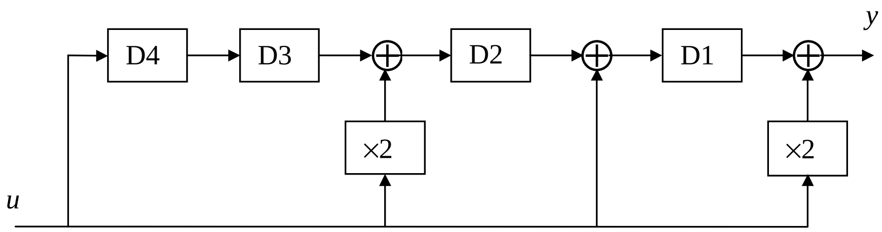

Fig. 2. Scheme of a multiplier; $D1$–$D4$ are delayers.

of this matrix. For example, if a Jordan block has the form

$$L_3(0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

then

$$L_3^2(0) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_3^3(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Consequently, if the third row of the block $L_3(0)$ corresponds to the p-th component of the state vector x^*, then the component $p-1$ at $t=2$ and the component $p-2$ at $t=3$ will be exactly known. Dually, if the first column of the block $L_3(0)$ corresponds to the q-th component of the vector x^*, then the errors in the components $q+1$ and $q+2$ will be corrected at $t=2$ and $t=3$, respectively.

The following general conclusion can be formulated: A quantity of the transformed system components with exactly known values at the moment $t=1$ from an unknown initial state is equal to the defect of the matrix F. The number of the transformed system components with exactly known values after several steps is equal to l zero eigenvalues of this matrix, and the number of these steps is equal to the maximal dimension of the block (22) in the Jordan canonical form (20). A dual statement for self-corrected errors is evident.

Notice that the structure of the matrix (20) and the block (22) yield the following result: Let $\mathrm{rank}(F) = m < n$. Then a decomposition of the initial system into two subsystems exists such that

$$x^0(t+1) = G^0 u(t),$$

$$x^{00}(t+1) = F^0 x^0(t) + F^{00} x^{00}(t) + G^{00} u(t) \quad (23)$$

with some matrices G^0, F^0, F^{00}, and G^{00} where $\dim(x^0) = n-m$ and $\dim(x^{00}) = m$. The components of the vector x^0 are formed out of the ones of the vector $x^* = \Phi x$, which correspond to the zero row of the matrix (20); the vector x^{00} is formed from other components. From the model (23) it follows that the components of the vector x^0 will be corrected (exactly known) at the moment $t=1$ if an error occurs at $t=0$ (the system starts from an unknown initial state).

8.2. Real Example. Consider a device for multiplying the polynomials by $1+2x^2+x^3+2x^4$, both defined on the Galois field GF(3), cf. Fig. 2 (Gill, 1971). The description of this multiplier is the following:

$$x(t+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(t) + 2u(t),$$

where y is the output. Clearly, for this system, $F = L_k(0)$ with $k=4$, and therefore an error in the i-th delayer will be corrected at $t=i$, $i=1,2,3,4$, which follows from Fig. 2.

9. Conclusions

In this paper, the former results obtained by the author with the use of an algebraic approach were developed to study the problem of duality in linear and nonlinear dynamic systems. In addition to the known form of duality connected with controllability and observability, this paper indicated a new manifestation of duality in linear and nonlinear dynamic systems. It is based on two main mathematical inequalities, $\varphi \le \boldsymbol{M}^k(\boldsymbol{e})$ and $\boldsymbol{m}^k(\boldsymbol{1}) \le \psi$, and their properties that made it possible to introduce two notions and investigate their properties. The first is the error correction degree and the second is the reachability degree. A duality between these parameters based on their mathematical properties was shown. A way to improve the obtained properties through an isomorphic transformation of the original system was suggested. In the linear case, this technique reduces to the Jordan canonical form.

It is hoped that these new notions of duality will be useful in solving some problems connected with the controllability and observability analysis.

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References

Arbib M. and Manes E. (1974): *Foundation of system theory: Decomposable systems.* — Automatica, Vol. 10, No. 2, pp. 285–302.

Bellman R. (1960): *Introduction to Matrix Analysis.* — New York: McGraw-Hill Company, Inc.

Gill A. (1971): *Linear Sequential Circuits. Analysis, Synthesis, and Application.* — New York: McGraw-Hill.

Goldblad R. (1979): *Topoi. The Categorical Analysis of Logic.* — Amsterdam: North-Holland.

Hermann R. and Krener A. (1977): *Nonlinear controllability and observability.* — IEEE Trans. Automat. Contr., Vol. AC-22, No. 5, pp. 728–740.

Korn G.A. and Korn T.M. (1961): *Mathematical Handbook.* — New York: McGraw-Hill.

Lankaster P. (1969): *Theory of Matrices.* — London: Academic Press.

Kwakernaak H. and Sivan R. (1972): *Linear Optimal Control Systems.* — New York: Wiley.

Michtchenko A. and Zhirabok A. (2001): *Application of invariance for error correction in nonlinear systems.* — Proc. 12-th Int. Symp. *Danube Adria Association for Automation and Manufacturing*, Vienna, Austria, pp. 297–298.

Peterson W. and Weldon E. (1972): *Error-Correcting Codes.* — London: Mita Press.

Shumsky A. and Zhirabok A. (2005): *Nonlinear diagnostic filter design: Algebraic and geometric points of view*. — Proc. 16-th IFAC World Congress, Prague, Czech Republic, (on CD-ROM).

Zhirabok A. and Shumsky A. (1993): *A new mathematical technique for nonlinear systems research.* — Proc. 12-th IFAC World Congress, Sydney, Australia, Vol. 3, pp. 485–488.

Zhirabok A. (1998): *Observability and controllability properties of nonlinear dynamic systems.* — Int. J. Comput. Syst. Sci., Vol. 37, No. 1, pp. 1–4.

Zhirabok A. (2000): *On canonical decomposition of nonlinear dynamic systems,* In: Nonlinear Control in the Year 2000, Vol. 2, (F. Lamnabhi-Lagarrigue, A. Isidori, W. Respondek, Eds.). — London: Springer, pp. 555–564.

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