Int. J. Appl. Math. Comput. Sci., 2016, Vol. 26, No. 2, 495–516
DOI: 10.1515/amcs-2016-0035

A COMPREHENSIVE SURVEY ON FORMAL CONCEPT ANALYSIS, ITS RESEARCH TRENDS AND APPLICATIONS

PREM KUMAR SINGH [a], CHERUKURI ASWANI KUMAR [b,*], ABDULLAH GANI [a]

[a]Centre for Mobile Cloud Computing Research, Faculty of Computer Science and Information Technology
University of Malaya, Kuala Lumpur 50603, Malaysia
e-mail: premsingh.csjm@gmail.com

[b]School of Information Technology and Engineering
VIT University, Vellore 632014, India
e-mail: cherukuri@acm.org

In recent years, FCA has received significant attention from research communities of various fields. Further, the theory of FCA is being extended into different frontiers and augmented with other knowledge representation frameworks. In this backdrop, this paper aims to provide an understanding of the necessary mathematical background for each extension of FCA like FCA with granular computing, a fuzzy setting, interval-valued, possibility theory, triadic, factor concepts and handling incomplete data. Subsequently, the paper illustrates emerging trends for each extension with applications. To this end, we summarize more than 350 recent (published after 2011) research papers indexed in Google Scholar, IEEE Xplore, ScienceDirect, Scopus, SpringerLink, and a few authoritative fundamental papers.

Keywords: concept lattice, formal concept analysis, formal concept, formal context, Galois connection.

1. Introduction

Formal concept analysis (FCA) is a mathematical framework based on lattice theory (Wille, 1982). FCA starts data analysis from a given incidence matrix in which each row corresponds to objects, each column corresponds to attributes, and the matrix field value denotes the relationship between them. One of the major outputs of this model is the concept lattice, reflecting generalization and specialization between the derived formal concepts from the incidence matrix (Davey and Priestley, 2002). Formal concepts are a basic unit of thought and play-major role in knowledge processing tasks containing distinct extents (sets of objects) and intents (corresponding common attributes) (Ganter and Wille, 1999). To handle the uncertainty and vagueness in data, FCA has been successfully extended with a fuzzy setting, an interval-valued fuzzy setting, possibility theory, a rough setting and triadic concept analysis. These extensions have independent background mathematics, algorithms, and outputs. Several algorithms are available in the literature on FCA (Doerfel *et al.*, 2012; Poelmans *et al.*, 2014), its notions (Kuznetsov and Obiedkov, 2002; Poelmans *et al.*, 2013b), theoretical analysis (Aswani Kumar and Singh, 2014; Sarmah *et al.*, 2015), algorithms (Dias and Vieira, 2015; Kuznetsov and Obiedkov, 2002; Kuznetsov and Poelmans, 2013) and applications (Poelmans *et al.*, 2013b; Yan *et al.*, 2015). The current paper is unique and different from the above cited works mainly due to two aspects: first, it provides the necessary mathematical background for each of the new extensions of FCA that is discussed, and second, it discusses applications for each extension. This paper provides a summary of the trends and applications of FCA after 2011. Further, the paper also provides pointers to most authoritative literature on FCA. To achieve this, we have collected 544 articles from prominent indexing systems.

2. Survey methodology

This systematic study has been conducted with the help of research papers published after 2011. The rationale

*Corresponding author

behind is that the summary of FCA findings till 2011 was analyzed in a series of papers by Poelmans *et al.* (2013a; 2013b; 2014). A total of 544 research papers have been collected from the prominent indexing systems such as Scopus, Google Scholar, leading scientific data bases such as the ACM Digital Library, IEEE Xplore, ScienceDirect, SpringerLink, etc. Also, we have referred to the proceedings of prominent FCA conferences like *ICFCA, ICCS, CLA*, etc.

The methodology we have used to extract the articles is based on the following keywords: formal concept analysis (FCA), formal concept, fuzzy formal concept, concept lattice, fuzzy concept lattice and Galois connection. From the 544 collected papers we have shortlisted 352 works based on their innovative content. From these papers the following research trends identified: (a) FCA with granular computing, (b) FCA with a fuzzy setting, (c) FCA with an interval-valued fuzzy setting, (d) FCA with possibility theory, (e) FCA with rough set theory, (f) triadic concept analysis in a fuzzy setting, (g) factor concepts, and (h) concept lattices of incomplete data.

3. Formal concept analysis

In this section we provide a brief background of FCA, its tools and current research issues.

3.1. Background. FCA is a mathematical model for knowledge processing tasks. It receives data, structured in the form of objects, attributes and the relation between them. This relation is represented as in the form of a formal context $-\mathbf{F} = (X, Y, R)$ where X is a set of objects, Y is a set of attributes and R is a binary relation between them, of Table 1 (where $a, b, c, \ldots, o$ represent the attributes $y_1, y_2, \ldots, y_{15}$, respectively). From the given context, FCA derives a set of objects (A) and the set of all attributes (B) that are in common for these objects. Similarly, the dual operation on the set of attributes (B) identifies the common objects objects (A) using the concept forming operator.

Definition 1. (*Concept forming operators*) The operators $\uparrow$: $2^X \to 2^Y$ and $\downarrow$: $2^Y \to 2^X$ are defined for every $A \subseteq X$ and $B \subseteq Y$ by:

$$A^{\uparrow} = \{y \in Y \mid \forall x \in A : (x, y) \in R\},$$
$$B^{\downarrow} = \{x \in X \mid \forall y \in B : (x, y) \in R\},$$

where $A^{\uparrow}$ is the set of all attributes shared by all objects from A. Similarly, $B^{\downarrow}$ is the set of all objects sharing all attributes from B. The formal concept is a pair (A, B) of $A \subseteq X$ and $B \subseteq Y$ such that $A^{\uparrow} = B$ and $B^{\downarrow} = A$. The collection of all such pairs of concepts forms a concept lattice under the closure operation.

Definition 2. (*Concept lattice*) The concept lattice structure determines the hierarchy of formal concepts which follows the partial ordering principle: $(A_1, B_1) \leq (A_2, B_2)$ iff $A_1 \leq A_2$ ($B_2 \leq B_1$) and provides generalization and specialization between the concepts, i.e., (A_1, B_1) is more specific than (A_2, B_2) ((A_2, B_2) is more general). The attributes of each formal concept are inherited from the most general maximum node, while the objects are inherited from the most specific minimum node. Several algorithms have been proposed for generating the concept lattice (Bartl *et al.*, 2011; Codocedo *et al.*, 2011; Kuznetsov and Obiedkov, 2002; Outrata and Vychodil, 2012) including parallel and recursive algorithms (Fu and Mephu Nguifo, 2004; Krajca *et al.*, 2008; Langdon *et al.*, 2011). The attribute implications are represented in the form of $A \to B$ over the set Y (Ganter and Wille, 1999). There are several patents granted for the inventions that are based on FCA. Table 2 summarizes some of such patents.

3.2. Tools and ssoftware in FCA. Several tools and packages are developed to handle the FCA tasks such as generating concepts, attribute implications, etc. (`http://www.upriss.org.uk/fca/fca.html`). Following is a summary of some of the available tools:

1. ToscanaJ: Provides a view for conceptual schemas and optimized for a non-technical audience, `http://toscanaj.sourceforge.net/`.

2. ConExp: Implements the basic functionality of FCA with a crisp setting, `http://conexp.sourceforge.net/`.

3. ConExp-NG: Is an extension of ConExp with the focus on usability and maintainability, `https://github.com/fcatools/conexp-ng`.

4. Conexp-clj: Allows us to handle the formal context, relational algebra with formal contexts, many-valued contexts, attribute exploration, lattice layouts by NextClosure or Iceberg Concepts and fuzzy FCA, `https://github.com/exot/conexp-clj/`.

5. Galicia: Is an open environment and handles binary and relational contexts, `http://www.iro.umontreal.ca/~galicia/`.

6. FcaStone: Is a command-line utility that converts between the file formats of commonly used FCA tools (such as ToscanaJ, ConExp and Galicia) or FCA formats to other graphics formats (dot, fig, svg, ...), `http://fcastone.sourceforge.net/`.

Table 1. Binary formal context.

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
x_1	x		x			x		x		x					x
x_2	x			x		x		x		x					x
x_3	x		x			x		x			x				x
x_4	x		x		x			x		x		x	x		
x_5	x		x		x			x			x	x	x		
x_6	x			x		x		x						x	x
x_7	x			x	x		x		x					x	
x_8		x		x				x						x	

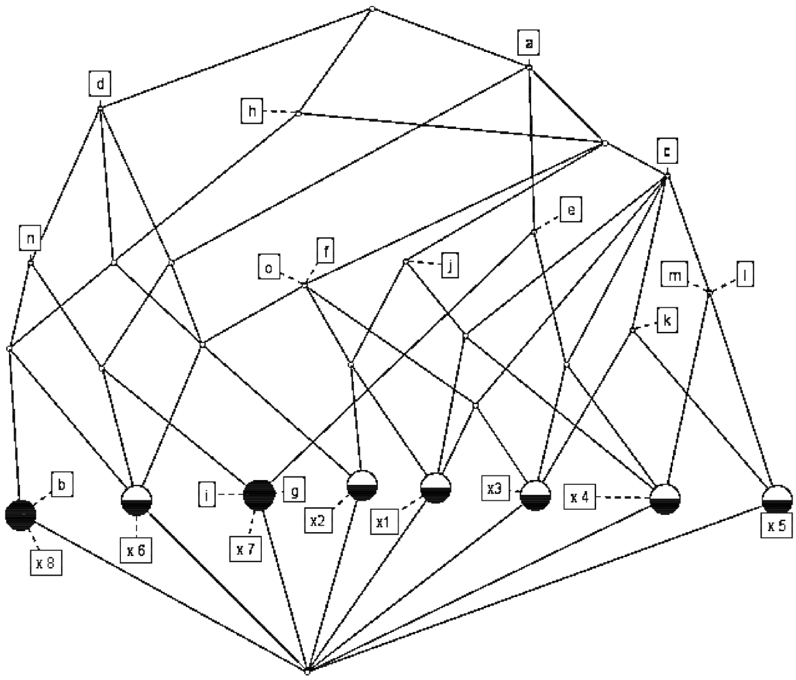


Fig. 1. Formal concept lattice for the context shown in Table 1.

7. Lattice Navigator: Provides three applications of FCA using a single setup file: Lattice Navigator, Context Editor, Lattice Visualizer, `http://www.fca.radvansky.net/news.php`.

8. Colibri-concepts: Permits to explore only part of a concept lattice which is most useful when working with huge lattices, `http://code.google.com/p/colibri-concepts/`.

3.3. Issues in FCA. Hierarchical order visualization of formal concepts in the concept lattice structure is an important concern for practical applications of FCA (Aswani Kumar, 2011a). In this process, one of the major issues is the size of the concept lattice constructed from "a large formal context" (Codocedo *et al.*, 2011; Aswani Kumar *et al.*, 2015a; Aswani Kumar and Srinivas, 2010; Singh and Gani, 2015). The concept lattice constructed from the large context becomes complex and impractical. Hence, handling a large formal context and reducing the size of the concept lattice are addressed as real issues in practical applications of FCA (Dias and Vieira, 2015; Singh *et al.*, 2015a; 2015b).

The issue includes a number of formal concepts, and implications generated from a large context can be exponential while counting them is P-complete and P-hard (Babin and Kuznetsov, 2013; Bartl *et al.*, 2011; Bazhanov and Obiedkov, 2014; Obiedkov, 2012; Slezak, 2012). This problem also merges with a fuzzy formal context (Denniston *et al.*, 2013; Ma and Zhang, 2013), a decision formal context (Li *et al.*, 2012a; 2012b) multi adjoint concept lattices (Medina and Ojeda-Aciego, 2012; Medina, 2012a; 2012b), and granular computing (Tadrat *et al.*, 2012; Yang *et al.*, 2011a). Subsequently, some metrics are proposed to measure the stability and importance of obtained concepts (Kuznetsov, 2013; Martin *et al*., 2013; Pei *et al*., 2013). In the next section we illustrate the mathematics behind each of the above categorized research trends in FCA with an illustrative example.

4. Current research trends in FCA

In this section we describe trends of FCA such as granular computing, FCA with a fuzzy setting, an interval-valued fuzzy setting, possibility theory, a rough setting, a triadic setting, factor concepts and the incomplete context.

4.1. FCA with granular computing. In this section we discuss a method for reducing the concepts at a chosen granulation of their (computed) weight (Butka *et al*., 2012; Lei and Tian, 2012; Ma and Zhang, 2013).

Table 2. Some important patents on FCA and their inventions.

Patent information	Invention
US patent (May 19,2005) US2005/0108252A1	Attribute implications
US patent (May 25,2006) US2006/0112108A1	Information retrieval
US patent (Sep 21,2006) US2006/0212470A1	Organizing the information
International patent(April 5,2007) WO2007/038375A2	Processing patient records
US patent (Jul 7,2007) US2005/0149510A1	Mapping of context
US patent (Jun 10,2010) US2010/0153092A1	Identifying similar word
China patent (April 6,2011) CN2017885100	Dynamic mining system
China patent (Aug 24,2011) CN101699444B	Remote sensing
US patent (Jan 5,2012) US2012/0005210A1	To structure a database
International patent (Feb 2,2012) WO2012014938A1	Electronic repository
China patent (Jun 20,2012) CN102508767A	FCA based software maintenance
US patent(Feb 26,2013) US8386489B2	Conceptual similarity
China patent (May 29,2013) CN103123607A	Software maintenance
China patent (Jun 26,2013) CN103176902A	Software error locations
US patent (Jul 25, 2013) US20130191735A1	Sentiments analysis
US patent(Aug 1,2013) US2013/0198195A1	Resume classification
European patent (Oct 2,2013) EP2645274A1	Reducing the lattice
International patent WO 2014 013327A1 (Jan23,2014)	Traffic measurement

The reason is that the number of concepts increases exponentially in the worst case. In this case, granular computing provides a path to process the large context into less time based on the requirement when dealing with numeric processing (Pedrycz, 2013). An information granule is the basic notion of granular computing, which can be defined broadly as a collection of information. This notion has been recently introduced into the concept lattice as an attempt to decrease the computation time (Belohlavek *et al.*, 2013; Wu *et al.*, 2009; 2012; Li *et al.*, 2015; Xu and Li, 2015). In general, the information granule regarded as a collection of elements drawn together by their closeness (resemblance, proximity, functionality, etc.) articulated in terms of some useful spatial (Ciobanu and Vaideanu, 2014; Singh and Gani, 2015; Singh and Aswani Kumar, 2015a; Aswani Kumar *et al*., 2015a), bidrectional (Aswani Kumar *et al*., 2015b), temporal (Belohlavek and Trnecka, 2013; Dias *et al*., 2013; Dias and Vieira, 2013), or functional relationships (Singh and Aswani Kumar, 2012b; Vityaev *et al*., 2012; Zhang *et al*., 2012). Selecting the level to find some important concepts in the large context is based on user requirements.

Definition 3. (*Granular concept*) Information granularity has been engaged in one way or another in quantifying the lack of numeric precision computed by different methods. The computed weight (w) of any given concepts indicates the importance of attributes (Y) where $0 \leq w \leq 1$. This process gives the priority to the concepts whose weight is more than the chosen threshold θ ($0 \leq \theta \leq 1$) (Belohlavek and Macko, 2011; Babin and Kuznetsov, 2012).

Example 1. For illustration of the granular based concept lattice, a context shown in Table 1 has been considered (Junli *et al*., 2013). Let us analyse any object $x_j \in X$ of a given context and compute its probability $P(y_j/x_i)$ for possessing the corresponding attribute y_i. Then the average information weight $E(y_i)$, of x_i to provide the attribute $y_i \in Y$ can be computed as follows (and shown in Tables 3 and 4) (Junli *et al*., 2013):

$$E(y_i) = -\sum_{i=1}^{m} P(y_i/x_j)\log_2(P(y_i/x_j)), \qquad (1)$$

where m represents the total number of attributes

$$w_i = \frac{E(y_i)}{\sum\limits_{i=1}^{m} E(y_i)} \qquad (2)$$

$$Weight(B) = \frac{\sum(w_i)}{m}, \qquad (3)$$

where B is the intent.

The removal of formal concepts at a chosen granulation is shown in Table 5. Subsequently, it can be applied to FCA with fuzzy attributes as well (Singh *et al*., 2015a; Xu and Li, 2015). ♦

4.2. FCA with a fuzzy setting. FCA has been extended with a fuzzy setting for handling vagueness and uncertainty in data using the following definitions.

Definition 4. (*Fuzzy formal context*) It is a triplet $\mathbf{K} = (X, Y, \tilde{R})$, where X is a set of objects, Y is a set of attributes and $\tilde{R}$ is an L-relation: $X \times Y \to L$.

Definition 5. (*Residuated lattice*) A residuated lattice $\mathbf{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ is the basic structure of truth degrees, and it is complete iff (i) $(L, \wedge, \vee, 0, 1)$ is a complete lattice, (ii) $(L, \otimes, 1)$ is commutative monoid, (iii) $\otimes$ and $\to$ are adjoint operators, i.e., $a \otimes b \leq c$ iff $a \leq b \to c, \forall a, b, c \in L$ and defined distinctly (Davey and Priestley, 2002; Macko, 2013).

Definition 6. (*Fuzzy Galois connection*) For any **L**-set $A \in L^X$ of objects, and $B \in L^Y$ of attributes we can define an **L**-set of $A^{\uparrow} \in L^Y$ attributes and **L**-set $B^{\downarrow} \in L^X$ of objects as follows (Belohlavek and Vychodil, 2012; Pocs, 2012):

1. $A^{\uparrow}(y) = \bigwedge_{x \in X} (A(x) \to \tilde{R}(x, y))$,

2. $B^{\downarrow}(x) = \bigwedge_{y \in Y} (B(y) \to \tilde{R}(x, y))$.

Definition 7. (*Fuzzy formal concept*) It is a pair of $(A, B) \in L^X \times L^Y$ satisfying $A^{\uparrow} = B$ and $B^{\downarrow} = A$, where A is called the (fuzzy) extent and B is called the (fuzzy) intent.

Example 2. For illustration, we have considered a fuzzy context shown in Table 6. For concept generation and lattice structure, the interested readers can refer to the works of Belohlavek and Vychodil (2005), Kaiser and Schmidt (2013), Kang *et al.* (2012a), Martin and Majidian (2013) or Martin *et al.* (2013). ♦

Definition 8. (*Implication*) Implication over a attribute set Y is an expression $A \Rightarrow B$, where $A, B \subseteq L^Y$. It represents "if it is (very) true that an object has all attributes from A, then it has also all attributes from B (Massanet *et al.*, 2013; Glodeanu, 2012). The notions

Table 3. Computed weight for each attributes of Table 1.

y_i	P(y_i)	E(y_i)	w_i
a	0.875	0.169	0.026
b	0.125	0.375	0.057
c	0.500	0.500	0.076
d	0.500	0.500	0.076
e	0.375	0.531	0.081
f	0.500	0.500	0.076
g	0.125	0.375	0.057
h	0.875	0.169	0.026
i	0.125	0.375	0.057
j	0.375	0.531	0.081
k	0.250	0.500	0.076
l	0.250	0.500	0.076
m	0.250	0.500	0.076
n	0.375	0.531	0.081
o	0.500	0.500	0.076

Table 4. Computed weight and deviation for each concepts of Fig. 1.

Node	Intent	Average	$W(B)$	$D(y)$
c_0	$\oslash$	1	1	0
c_1	a	0.026	0.026	0
c_2	d	0.76	0.076	0
c_3	h	0.026	0.026	0
c_4	ae	0.054	0.054	0.055
c_5	ah	0.026	0.026	0
c_6	ad	0.051	0.051	0.0326
c_7	ach	0.043	0.043	0.029
c_8	adn	0.061	0.061	0.031
c_9	dhn	0.061	0.061	0.031
c_{10}	$acfho$	0.056	0.056	0.028
c_{11}	$adfho$	0.056	0.056	0.028
c_{12}	$adein$	0.064	0.064	0.024
c_{13}	$bdghn$	0.059	0.059	0.022
c_{14}	$acfhjo$	0.060	0.060	0.027
c_{15}	$adfhjo$	0.060	0.060	0.027
c_{16}	$acfhko$	0.059	0.059	0.026
c_{17}	$adfhno$	0.060	0.060	0.027
c_{18}	$acehjlm$	0.063	0.063	0.026
c_{19}	$acehklm$	0.063	0.063	0.025
c_{20}	$abcdefghijklmno$	1	1	0

Table 5. Removed concepts at chosen granulation.

$W(B)$	θ	Removed concepts
1	$0.076 < \theta \leq 1$	c_1,c_2,c_3,c_4,c_5,c_6, c_7 c_8,c_9,c_{10}, c_{11}, c_{12}, c_{13}, c_{14},c_{15},c_{17},c_{18},c_{19}
0.076	$0.064 < \theta \leq 0.076$	c_1,c_3,c_4,c_5,c_6,c_7, c_8,c_9,c_{10},c_{11}, c_{12},c_{13}, c_{14}, c_{15},c_{17},c_{18},c_{19}
0.64	$0.063 < \theta \leq 0.064$	c_1,c_3,c_4,c_5,c_6,c_7, c_8,c_9,c_{10}, c_{11},c_{13},c_{14}, c_{15},c_{17},c_{18},c_{19}
0.063	$0.061 < \theta \leq 0.063$	c_1,c_3,c_4,c_5,c_6, c_7,c_8,c_9, c_{10},c_{11}, c_{13},c_{14},c_{15},c_{17}
0.061	$0.060 < \theta \leq 0.061$	c_1,c_3,c_4,c_5,c_6,c_7,c_{10}, c_{11},c_{13},c_{14},c_{15},c_{17}
0.06	$0.059 < \theta \leq 0.06$	c_1,c_3,c_4,c_5,c_6, c_7,c_{10},c_{11},c_{13}
0.059	$0.056 < \theta \leq 0.059$	c_1,c_3,c_4,c_5,c_6, c_7,c_{10},c_{11}
0.056	$0.054 < \theta \leq 0.056$	c_1,c_3,c_4,c_5,c_6,c_7
0.054	$0.051 < \theta \leq 0.054$	c_1,c_3,c_5,c_6,c_7
0.051	$0.043 < \theta \leq 0.051$	c_1,c_3,c_5,c_7
0.043	$0.026 < \theta \leq 0.043$	c_1,c_3,c_5
0.026	$0 < \theta \leq 0.026$	$\oslash$

"being very true", "to have an attribute" and logical connective "if-then" are determined by the chosen L (Belohlavek *et al.*, 2013b; Zhai *et al.*, 2012; 2013; Massanet, 2013).

Example 3. Table 6 generate following implications (i) $(s, 0.5/l, f) \to (s, l, f, n)$, (ii) $(0.5/s, 0.5/n) \to (s, n)$, (iii) $(l, f) \to (l, f, 0.5/n)$, (iv) $(0.5/l) \to (0.5/l, f)$, (v) $(f, 0.5/n) \to (l, f, 0.5/n)$ (vi) $(n) \to (s, n)$. These six attribute implications are sufficient to determine all the fuzzy formal concepts generated from Table 6. ♦

Recently many researchers focused on the analysis of a fuzzy context having similar attributes set (Alcalde *et al.*, 2012a; 2012b; 2015; Li and Mi, 2013).

Example 4. For illustration, two fuzzy contexts having a similar attribute set are shown in Tables 7 and 8, with CS: *Computer science*, AC: *Accounting*, ME: *Mechanical*, CK: *Cooking*, and $C_1, \ldots, C_5$ representing candidates. The context shown in Table 7 and 8 can be connected using the composition $\tilde{R}_1 * \tilde{R}_2 = \tilde{R}_3$ as shown in Table 9. For the employment of $Waiter$ most of the candidates are eligible, where C_2 is more suitable having membership value 1 (Singh and Aswani Kumar, 2015b; Tho *et al.*, 2006; Wang and Xu, 2011). ♦

Table 6. Fuzzy formal context.

	Size		Distance	
	small (s)	large (l)	far (f)	near(n)
$Mercury(Me)$	1	0	0	1
$Venus(Ve)$	1	0	0	1
$Earth(Ea)$	1	0	0	1
$Mars(Ma)$	1	0	0.5	1
$Jupiter(Ju)$	0	1	1	0.5
$Saturn(Sa)$	0	1	1	0.5
$Uranus(Ur)$	0.5	0.5	1	0
$Neptune(Ne)$	0.5	0.5	1	0
$Pluto(Pl)$	1	0	1	0

Table 7. Requirements of knowledge for employment in a company: $\tilde{R}_1$.

	CS	AC	ME	CK
$Domestichelper$	0.1	0.3	0.1	1.0
$Waiter$	0.0	0.4	0.0	0.7
$Accountant$	0.9	1.0	0.0	0.0
$Carsalesman$	0.5	0.7	0.9	0.0

Table 8. Knowledge of candidate for employment: $\tilde{R}_2$.

	CS	AC	ME	CK
C_1	0.5	0.8	0.3	0.6
C_2	0.2	0.5	0.1	1.0
C_3	0.0	0.2	0.0	0.3
C_4	0.9	0.4	0.1	0.5
C_5	0.7	0.5	0.2	0.1

4.3. FCA with an interval valued fuzzy setting.

For adequate analysis of fuzzy attributes, FCA has been extended to an interval-valued fuzzy setting as described below (Singh and Aswani Kumar, 2012a).

Definition 9. (*Interval number*) It is an $D - [a^-, b^+]$ with $0 \le a^- \le b^+ \le 1$. For interval numbers $D_1 = [a_1^-, b_1^+]$ and $D_2 = [a_2^-, b_2^+]$, we can define $(D[0,1], \le, \vee, \wedge)$ is a complete lattice with $[0,0]$ as the least element and $[1,1]$ as the greatest element.

Definition 10. (*Interval-valued fuzzy set*) An interval-valued fuzzy set I in V is defined as

$$I = \left\{(v, [\mu_I^-(v), \mu_I^+(v)]) : v \in V\right\},$$

where $\mu_I^-(v)$ and $\mu_I^+(v)$ are fuzzy subsets of V such that $\mu_I^-(v) \le \mu_I^+(v)$ for all $v \in V$. For interval-valued fuzzy sets $I = [\mu_I^-(v), \mu_I^+(v)]$ and $J = [\mu_J^-(v), \mu_J^+(v)]$ in V we can define

- $I \cup J = (v, \max(\mu_I^-(v)), \ \mu_J^-(v)), \max(\mu_I^+(v), \mu_J^+(v)))$, where, $v \in V$;
- $I \cap J = (v, \min(\mu_I^-(v)), \ \mu_J^-(v)), \ \min(\mu_I^+(v), \mu_J^+(v)))$, where $v \in V$.

Definition 11. (*Fuzzy graph*) A fuzzy graph $G = (V, \mu, \rho)$ is a non-empty set V together with a pair of functions $\mu : V \to [0,1]$ and $\rho : V \times V \to [0,1]$, such that, for all v_1, v_2 in V, $\rho(v_1, v_2) \le \mu(v_1) \wedge \mu(v_2)$, where μ is said to be the fuzzy vertex set and ρ is the fuzzy edges set of G.

Definition 12. (*Interval-valued fuzzy graph*) An interval-valued fuzzy graph of a graph G, is a pair (I, J) where $I = [\mu_I^-, \mu_I^+]$ is an interval-valued fuzzy set on V and $J = [\mu_J^-, \mu_J^+]$ is an interval valued fuzzy relation on the set E such that

$$\mu_J^-(pq) \le \min(\mu_I^-(p), \mu_I^-(q)),$$

$$\mu_J^+(pq) \le \min(\mu_I^+(p), \mu_I^+(q))$$

for all $pq \in E$.

Example 5. Suppose that $V = \{p, q, r\}$ and $E = \{pq, qr, rp\}$. Let I be an interval-valued fuzzy set of V and J be an interval-valued fuzzy set of $E \subseteq V \times V$ defined by

$$I = \{(p/0.2, q/0.3, r/0.4), (p/0.4, q/0.5, r/0.6)\},$$

Table 9. Composition of fuzzy contexts: $\tilde{R}_3 = \tilde{R}_1 * \tilde{R}_2$.

	C_1	C_2	C_3	C_4	C_5
$Domestichelper$	0.6	1.0	0.3	0.5	0.1
$Waiter$	0.9	1.0	0.6	0.8	0.4
$Accountant$	0.6	0.3	0.1	0.4	0.5
$Carsalesman$	0.4	0.2	0.5	0.2	0.3

$$J = \{(pq/0.1, qr/0.2, rp/0.1), (pq/0.3, qr/0.4, rp/0.4)\}.$$

Then it can be presented in an interval-valued fuzzy graph as shown in Fig. 2 (Akram and Dudek, 2011). ♦

Definition 13. (*Complete graph*) An interval-valued fuzzy graph G is complete if

$$\mu_J^-(pq) = \min(\mu_I^-(p), \mu_I^-(q))$$

and

$$\mu_J^+(pq) = \min(\mu_I^+(p), \mu_I^+(q)),$$

for all $pq \in E$.

Example 6. Consider graph $G = (V, E)$ such that $V = (p, q, r)$, $E = (pq, qr, rp)$ and I, J are defined as follows:

$$I = ((p/0.2, q/0.3, r/0.4), (p/0.4, q/0.5, r/0.5));$$

$$J = ((pq/0.2, qr/0.3, rp/0.2), (pq/0.4, qr/0.5, rp/0.4)).$$

Then $G = (I, J)$ is an interval-valued fuzzy complete graph. ♦

Definition 14. The composition, join, and product of two interval-valued fuzzy graphs G1 and G2 are again an interval-valued fuzzy graph.

Example 7. (*Interval-valued fuzzy context*) It is a triplet $(X, Y, \mathbf{I})$ where X represents objects, Y represents attributes and $\mathbf{I}$ represents interval-valued fuzzy relation:

$$\mathbf{I} = \Big\{((x, y), [\mu_{\tilde{I}}^-(x, y), \mu_{\tilde{I}}^+(x, y)]) : (x, y) \in X \times Y\Big\}$$

(cf. Alcalde *et al.*, 2011). As an example we have considered a context shown in Table 10 (Djouadi and Prade, 2009). ♦

Definition 15. (*Interval-valued fuzzy concept*) (*Singh* et al., *2015b*) It is a pair $((x_i, [\mu_{\tilde{R}}^-(x), \mu_{\tilde{R}}^+(x)]), (y_j, [\mu_{\tilde{R}}^-(y), \mu_{\tilde{R}}^+(y)]))$, which satisfies $(x_i, [\mu_{\tilde{R}}^-(x), \mu_{\tilde{R}}^+(x)]) = (subb(y))^{\downarrow}$ and $(y_j, [\mu_{\tilde{R}}^-(y), \mu_{\tilde{R}}^+(y)]) = (subb(x))^{\uparrow}$, where $subb$ is used for subset (Djouadi, 2011). For example, the following interval-valued fuzzy formal concepts can be generated from Table 10 (Singh and Aswani Kumar, 2014):

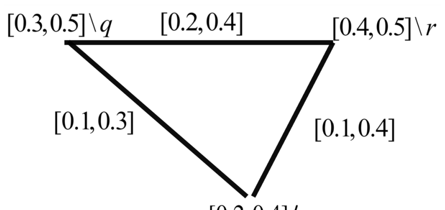


Fig. 2. Interval-valued fuzzy graph for Example 5.

1. $\{\emptyset, [1.0, 1.0]/y_1 + [1.0, 1.0]/y_2 + [1.0, 1.0]/y_3\}$,
2. $\{[0.9, 1.0]/x_1 + [0.8, 1.0]/x_2 + [0.3, 0.6]/x_3 + [0.2, 0.4]/x_4, [1.0, 1.0]/y_1\}$,
3. $\{[0.5, 0.7]/x_1 + [0.0, 1.0]/x_2 + [1.0, 1.0]/x_3 + [0.6, 1.0]/x_4, [1.0, 1.0]/y_2\}$,
4. $\{[0.0, 0.2]/x_1 + [0.5, 0.5]/x_2 + [0.8, 0.8]/x_3 + [0.0, 0.1]/x_4, [1.0, 1.0]/y_3\}$,
5. $\{[0.5, 1.0]/x_1 + [0.0, 1.0]/x_2 + [0.3, 1.0]/x_3 + [0.2, 1.0]/x_4, [1.0, 1.0]/y_1 + [1.0, 1.0]/y_2\}$,
6. $\{[0.0, 1.0]/x_1 + [0.5, 1.0]/x_2 + [0.3, 0.8]/x_3 + [0.0, 0.4]/x_4, [1.0, 1.0]/y_1 + [1.0, 1.0]/y_3\}$,
7. $\{[0.0, 0.2]/x_1 + [0.0, 1.0]/x_2 + [0.8, 1.0]/x_3 + [0.0, 1.0]/x_4, [1.0, 1.0]/y_2 + [1.0, 1.0]/y_3\}$,
8. $\{[1.0, 1.0]/x_1 + [1.0, 1.0]/x_2 + [1.0, 1.0]/x_3 + [1.0, 1.0]/x_4, \emptyset\}$.

The interval-valued fuzzy concept lattice for the above generated concepts is shown in Fig. 3. This extension has been successfully applied in information retrieval and the rule mining tasks (Zerarga and Djouadi, 2013; Zhai *et al.*, 2012).

4.4. FCA with possibility theory.

FCA is augmented with possibility theory for handling uncertainty in data. In this section, we provide a summary of the four basic set-functions of possibility theory in terms of FCA. The possibility distribution π, defined on a universe U, is equated to the characteristic (membership) function of a fuzzy set H in U and the two set-functions (S, T) are associated with π as follows (Dubois and Prade, 2012).

Definition 16. (*Potential possibility*) A possibility measure is $\pi : \pi(S) = \max_{s \in S} \pi(s)$. It estimates to what extent event S is consistent with the information represented by π and characterized by $\pi(S \cup T) =$

Table 10. Interval-valued fuzzy formal context.

	y_1	y_2	y_3
x_1	[0.9, 1.0]	[0.5, 0.1]	[0.0, 0.2]
x_2	[0.8, 1.0]	[0.0, 1.0]	[0.5, 0.5]
x_3	[0.3, 0.6]	[1.0, 1.0]	[0.8, 0.8]
x_4	[0.2, 0.4]	[0.6, 1.0]	[0.0, 0.1]

$\max(\pi(S), \pi(T))$, where as $\pi(\oslash) = 0$ if $\pi(s)$ is normalized (i.e., there exists $\pi(U) = 1$). However, in the Boolean case (the set H is non empty and crisp), $\pi(S) = 1$ iff $S \cap H \neq \oslash$, otherwise 0.

Definition 17. (*Actual necessity*) It expresses the necessity (certainty) an event is true as the opposite event is more impossible as follows: $N(S) = 1 - \pi(S^-) = 1 - \max_{s \notin S} \pi(s)$, where $\notin S = U/S$. $N(S)$ estimates to what extent event S is implied by the information H represented by π and characterized by decomposition property $N(S \cap T) = \min(N(S), N(T))$ whereas $N(\oslash) = 0$ if N is normalized (i.e., there exists $N(U) = 1$). However, in the Boolean case $N(A) = 1$ iff $\oslash \neq H \subseteq A$, otherwise 0.

Definition 18. (*Actual possibility*) A measure of "actual (or guaranteed) possibility" $\Delta(S) = \max_{s \in S} \pi(s)$. It estimates to what extent all elements in S are possible and characterized by $\Delta(S \cup T)$ =min$(\Delta(S), \Delta(T))$, whereas $\Delta(\oslash) = 1$ by convention (hence $\Delta \leq \pi$ and $\Delta(U) = 0$ if π is anti-normalized (i.e., there exists u such that $\pi(u) = 0$). However, in the Boolean case, $\Delta(A)$=1 iff $S \subset H$ (if $H \neq U$), otherwise 0.

Definition 19. (*Potential necessity*) A dual measure of "potential necessity or certainty" $\nabla(S) = 1 - \nabla(S^-) = 1 - \max_{s \notin S} \pi(s)$, which estimates to what extent there exists at least one value in the complement of S that has a zero (or more generally a low) degree of possibility and is characterized by $\nabla(S \cup T) = \max(\nabla(S), \nabla(T))$ whereas $\nabla(\oslash) = 1$ if π is anti-normalized and $\nabla(U) = 0$. However, in the Boolean case, $\nabla(S) = 1$ iff $S \cap H \neq U$, otherwise 0.

The above operators can be combined with each other in a meaningful way in a formal context $K = (X, Y, R)$ as follows:

1. X^π is the set of objects that satisfy at least one attributes in Y.

$$X^\pi = \{x \in X | Y \cap R(x) \neq \oslash\} \\ = \{x \in X | \exists y \in Y : xRy \neq \oslash\}.$$

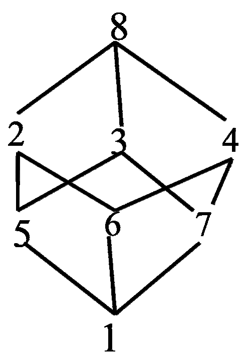


Fig. 3. Interval-valued fuzzy concept lattice of Table 10.

2. X^N is the set of objects such that any objects satisfied by one of them is necessarily in Y:

$$X^N = \{x \in X | R(x) \subset Y\} \\ = \{x \in X | \forall y \in Y : (xRy \Rightarrow y \neq Y)\}.$$

3. X^Δ is the set of objects that satisfy all attributes in Y:

$$X^\Delta = \{x \in X | \forall y \in Y (y \in y \Rightarrow xRy\} \\ = \{x \in X | Y \subset R(x)\}.$$

4. X^∇ is the set of objects that do not satisfy at least one attributes in Y^-.

$$X^\nabla = \{x \in X | Y \cup R(x) \neq X\} \\ = \left\{x \in X | \exists y \in Y^- : xR^- y\right\}.$$

Definition 20. (*Derivational operator*) The derivational operators are defined in an L-context for the fuzzy set $\tilde{Y} \in L^Y$ ($\tilde{X} \in L^X$):

(i) $\tilde{X}^\delta(x)$=$\wedge_{x \in X}$ $(\tilde{X}(x) \rightarrow R(x, y))$,

(ii) $\tilde{X}^\pi(x)$=$\vee_{x \in X}$ $(\tilde{X}(x) * R(x, y))$,

(iii) $\tilde{X}^N(x)$=$\wedge_{x \in X}$ $(R(x, y) \rightarrow \tilde{(}X)(x))$,

(iv) $\tilde{X}^\nabla(x)$=$\vee_{x \in X}$ $(-\tilde{X}(x) * -R(x, y))$,

where $\rightarrow$ denotes a fuzzy implication and $*$ denotes a fuzzy conjunction.

Definition 21. (*Formal concept with possibility theory*) It is a pair $(\tilde{X}, \tilde{Y})$ such that $\tilde{X}^\Delta = \tilde{Y}$ and $\tilde{Y}^\Delta = \tilde{X}$ (similarly for other operators), and it follows the infimum and supremum property given by

$$\bigwedge_{j \in J} (X_j, Y_j) = (\bigcap_{j \in J} X_j, (\bigcup_{j \in J} Y_j)^{\Delta\Delta}),$$

$$\bigwedge_{j \in J} (X_j, Y_j) = ((\bigcup_{j \in J} X_j)^{\Delta\Delta}, \bigcap_{j \in J} Y_j).$$

4.5. FCA with rough set theory.

Rough set theory (RST) deals with uncertainty and imperfect knowledge. It was introduced in FCA by Yao (2004) and Yao *et al.* (2012).

Definition 22. (*Approximation operator*) The dual approximation operators $^\circ$ and $^\Delta$:$2^X \rightarrow 2^Y$ can be defined as below:

$$X^\circ = \{y \in Y \mid \forall x \in X (xIY \Rightarrow x \in X)\} \\ = \{y \in Y \mid I_y \subseteq X)\}.$$

$$X^{\Delta} = \{y \in Y \mid \exists x \in X(xIY \wedge x \in X)\}$$
$$= \{y \in Y \mid I_y \cap X \neq \oslash)\} = \bigcup_{x \in X} xI.$$

Similarly, other pairs of approximation operators $^{\circ}$ and $^{\Delta}$:$2^Y \to 2^X$ can be defined as below:

$$Y^{\circ} = \{x \in X \mid \forall y \in Y(xIY \Rightarrow y \in Y)\}$$
$$= \{y \in Y \mid xI \subseteq Y)\}.$$
$$Y^{\Delta} = \{x \in X \mid \exists y \in Y(xIY \wedge y \in Y)\}$$
$$= \{x \in X \mid xI \cap Y \neq \oslash)\} = \bigcup_{y \in Y} Iy.$$

Based on the above notions, two new concept lattices in rough set theory can be introduced as follows.

Definition 23. (Object and attribute oriented concept) A pair $(A, B), A \subseteq X, B \subseteq Y$ is called an object oriented concept if $X = Y^{\Delta}$ and $Y = X^{\circ}$. The set of all object oriented formal concepts forms a lattice. Specifically, the meet $\wedge$ and join $\vee$ are defined by

$$(x_1, y_1) \wedge (x_2, y_2) = ((y_1 \cap y_2)^{\Delta}, y_1 \cap y_2),$$
$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \cup x_2, (x_1 \cup x_2)^{\circ}).$$

Similarly, a pair (A, B), $A \subseteq X$, $B \subseteq Y$ is called an attribute oriented concept if $X = Y^{\circ}$ and $Y = X^{\Delta}$. All the generated property oriented formal concepts form a lattice. Specifically, the meet $\wedge$ and join $\vee$ are defined by

$$(x_1, y_1) \wedge (x_2, y_2) = ((x_1 \cap x_2, x_1 \cap x_2)^{\Delta}),$$
$$(x_1, y_1) \vee (x_2, y_2) = ((y_1 \cup y_2)^{\circ}, (y_1 \cup y_2)).$$

Example 8. For illustration, we have considered a formal context shown in Table 11. The object oriented concepts

Table 11. Formal context.

	y_1	y_2	y_3	y_4	y_5
x_1	×		×	×	×
x_2	×		×		
x_3		×			×
x_4		×			×
x_5	×				
x_6	×	×			×

generated from Table 11 are

1. $\{(x_1, x_2, x_3, x_4, x_5, x_6), (y_1, y_2, y_3, y_4, y_5)\}$,
2. $\{(x_1, x_2, x_5, x_6), (y_1, y_3, y_4)\}$,
3. $\{(x_1, x_2, x_3, x_6), (y_2, y_3, y_4, y_5)\}$,
4. $\{(x_1, x_2), (y_3, y_4)\}$,
5. $\{(x_1, x_3, x_4, x_6), (y_2, y_4, y_5)\}$,
6. $\{(x_1), (y_4)\}$,
7. $\{(x_3, x_4, x_5), (y_2)\}$,
8. $\{\oslash, \oslash\}$.

Similarly, the attribute oriented formal concepts generated from Table 11 are

1. $\{(x_1, x_2, x_3, x_4, x_5, x_6), (y_1, y_2, y_3, y_4, y_5)\}$,
2. $\{(x_2, x_3, x_4, x_5, x_6), (y_1, y_2, y_3, y_5)\}$,
3. $\{(x_1, x_2, x_5), (y_1, y_3, y_4, y_5)\}$,
4. $\{(x_3, x_4, x_5, x_6), (y_1, y_2, y_5)\}$,
5. $\{(x_2, x_5), (y_1, y_3)\}$,
6. $\{(x_3, x_4), (y_2, y_5)\}$,
7. $\{x_5, y_1\}$,
8. $\{\oslash, \oslash\}$,

where $\oslash$ represents the null set.

The object and attribute oriented concept lattices are shown in Figs. 4 and 5, respectively. These two concept lattices differ in representations of the involved subsets of objects and their attributes. Recently, this extension has been applied in several research domains (Ganter and Meschke, 2011; Yang *et al.*, 2011b; Kang *et al.*, 2012b; Slezak, 2012; Wang and Li, 2012; Yang, 2011; Zhao and Liu, 2011) as well as for concept approximation (Chen *et al.*, 2015; Saquer and Deogun, 2001). ♦

4.6. Triadic concept analysis in a fuzzy setting.

Extension to a triadic context handles more attributes or conditional attributes in a crisp as well as a fuzzy setting.

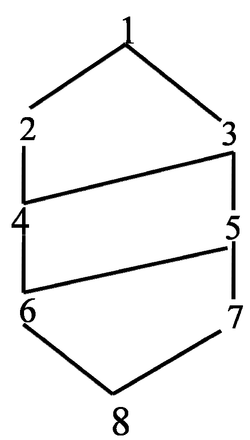


Fig. 4. Object oriented concept lattice of Table 11.

Definition 24. (*Triadic context*) It is defined as a quadruple $K = (X, Y, Z, \tilde{I})$, where X represents a set of objects, Y represents a set of attributes and Z represents a set of conditions, i.e, if $(x, y, z) \in \tilde{I}$ means object x has attribute y under condition z, whereas in the case of fuzzy attributes I represents the relationship among them using fuzzy membership-value. From a triadic context, the number of dyadic contexts can be derived as follows (Ignatov *et al.*, 2015): Given a fuzzy set $C_k \in L^{X_k}$, K induces a dyadic fuzzy context $K^{ij}_{C_k} = (X_i, Y_j, \tilde{I}^{ij}_{C_k})$, where $I^{ij}_{C_k}$ is defined by (Belohlavek and Osicka, 2012a).

$$\tilde{I}^{ij}_{C_k}(x_i, y_j) = \bigwedge_{z_k \in Z_k} C_k(x_k) \to \tilde{I}(x_i, y_j, z_k).$$

The pair $(x_i, y_j) \in \tilde{I}^{ij}_{C_k}$ iff for each $x_k \in X_k$ implies $(x_i, y_j, z_k) \in \tilde{I}$. The concept forming operator can be induced by a dyadic context $K^{ij}_{C_k}$, i.e., for a fuzzy set $C_i \in L^{X_i}$ we can define a fuzzy set $C^{i,j}_{C_k} \in L^{Y_j} = \wedge_{x_i \in X_i} C_i(x_i) \to \tilde{I}^{ij}_{C_k}(x_i, y_j)$.

Definition 25. (*Triadic fuzzy concepts*) It is a triplet (A_1, A_2, A_3) consisting of fuzzy sets $A_1 \in L^X$, $A_2 \in L^Y$, $A_3 \in L^Z$ such that $A_i = A_j^{i,j,A_k}$, $A_j = A_k^{j,k,A_i}$, and $A_k = A_i^{k,i,A_j}$ and can be shown in the concept trilattice.

Example 9. For illustration, a triadic context shown in Table 12 is considered, where objects (Beef Steak, Cheese Salad, Vegetable Plate and Fried Chicken Wings) represent dishes; attributes (Taste: T, Aroma: A, Look: L, and Price: P) represent features of the dishes; customers (Fry, Bender, Leela, Zoidberg) represent evaluation of the dishes (Belohlavek and Osicka, 2012b). The degree 0 stands for *bad*, 1/2 for *neutral* and 1 for *excellent*. Table 13 depicts five triadic fuzzy concepts generated from the context shown in Table 12, which provide the following information: Concept No. 1 represents customers who evaluate taste and aroma of beaf steak and fried chicken wings as excellent whereas their look is evaluated as neutral. Concept No. 2 represents that customers who like salad for its excellent taste, aroma and look, whereas its price evaluates as neutral. Concept No. 3 represents customers having no preferences in food. Concept No. 4 represents customers who like beef steak and partly fried chicken wings for their excellent taste and look and at least neutral aroma. Concept No. 5 shows that there is no customer who finds all properties of given dishes excellent.

♦

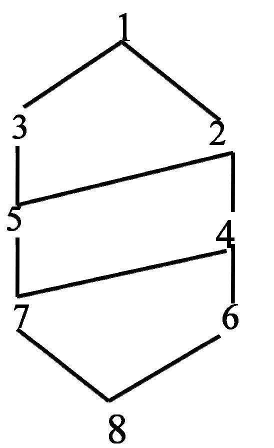


Fig. 5. Property oriented concept lattice of Table 11.

4.7. Factor concepts.

In this section, data analysis using factor concepts is described (Belohlavek, 2012; Ganter and Glodeanu, 2012; Glodeanu and Ganter, 2012; Glodeanu, 2011).

Definition 26. (*Factor concepts*) A subset of formal concepts F generated from the given formal context **F** such that $\bigcup_{(A,B) \in F(A \times B)} = R$ is called factorization. If F is minimal with respect to its cardinality, then it is called

Table 12. Triadic fuzzy formal context.

		Steak	Salad	Veg	Wings
Fry	Taste	1	0.5	0	1
	Aroma	1	0	0	1
	Look	1	0.5	0.5	0.5
	Price	0	0.5	1	0.5
Bender	Taste	1	0	0	1
	Aroma	1	0	0	1
	Look	1	0.5	0	0.5
	Price	0.5	0	0	1
Leela	Taste	0.5	1	0.5	0
	Aroma	0	1	0	0
	Look	0.5	1	0.5	0
	Price	0	1	0	0.5
Zoidberg	Taste	1	1	1	1
	Aroma	1	1	1	1
	Look	1	1	1	1
	Price	0	0.5	0	0.5

Table 13. Five triadic fuzzy concepts generated from Table 12.

	1	2	3	4	5
Steak	1	0	1	1	1
Salad	0	1	1	0	1
Vegetable	0	0	1	0	1
Wings	1	0	1	0.5	1
Taste	1	1	1	1	1
Aroma	1	1	1	0.5	1
Look	0.5	1	1	1	1
Price	0	0.5	0	0	1
Fry	1	0	0	1	0
Bender	1	0	0	1	0
Leila	0	1	0	0	0
Zoidberg	1	1	1	1	0

an optimal factorization. The elements of F are called (optimal) factors. Then $O(X,Y,R) \cap A(X,Y,R) \subseteq F$ are called mandatory factors, where $O(X,Y,R)$ and $A(X,Y,R)$ are the sets of object and attribute concepts, respectively.

The idea of finding factor concepts is based on the set covering problem.

Example 10. Let $U = \{1,2,3,4,5,6,7,8,9,10\}$, $V = \{2,4,6,8,10\}$, $P = (\{1,2\}, \{2,3\}, \{4,5\}, \{6,7,8\}, \{9,10\}, \{1,3,5\}, \{2,4\}, \{4,6\}, \{8,9,10\})$.

- $C = \{\{1,2\}, \{8,9,10\}\}$ is not a covering of V because $\bigcup C \neq V$.
- $C = \{\{1,2\}, \{2,3\}, \{4,5\}, \{6,7,8\}, \{9,10\}\}$ is a covering of V because $\bigcup C = V$. But C is not minimal because there exist coverings of V which contain smaller number of sets.
- $C = \{\{2,4\}, \{6,7,8\}, \{9,10\}\}$ is a minimal covering of V because $\bigcup C = V$ and no other covering has smaller numbers of sets than 3.
- $C = \{\{2,4\}, \{4,6\}, \{8,9,10\}\}$.

♦

Example 11. As an example, a context shown in Table 14 can be considered, with the following abbreviations: g (gas and dust), y (young stars), o (old stars), s (spiral arms), b (bulge), m (minimal star formation). The formal concepts generated from Table 14 are shown in Table 15. The matrix shown in Table 14 can be decomposed into a Boolean matrix $\bigcup_{(A,B)\in F(A\times B} = I$ such that $|F| \leq |Y|$. ♦

Definition 27. (*Mandatory concepts*) These are object and attribute concepts, investigated as follows: $O(X,Y,R) = (\{c_1,c_2,c_3,c_4\}$ and $A(X,Y,R) = (\{c_1,c_2,c_4,c_5,c_6\}$.

The object concepts in Table 15 are $O(X,Y,R) \cap O(X,Y,R) = \{c_1,c_2,c_4\}$. The object concepts are those formal concepts which we are looking for the analysis. We can observe that these concepts $\{c_1,c_2,c_4\}$ do not cover the incidence induced by the objects 5 and 6. These objects can be covered by c_3, c_5 and c_6, as shown in Table 15. However, the obtained set of concepts with c_5 and c_6 would not be a minimal subset with respect to cardinality. Finally, optimal factor F includes $(\{c_1,c_2,c_3,c_4\}$, which decompose the matrices (7×6) shown in Table 13 into two 7×4 and 4×7 matrices to be analyzed in a 4-dimensional space of factors instead of describing the galaxies in a 7-dimensional space. Recently, some applications of factor concepts have been shown in FCA with a fuzzy setting also (Belohlavek *et al.*, 2013a; 2011b; Bartl *et al.*, 2011; Ignatov *et al.*, 2015; Yao *et al.*, 2012) as well as in graph theory (Helmi *et al.*, 2014).

Table 14. Formal context of Galaxy types and their properties.

Galaxies	g	y	o	s	b	m
1. Milky Way	×	×	×	×	×	
2. Virgo A			×			×
3. M 82	×	×			×	
4. M 83		×				
5. M 85	×	×	×	×	×	
6. M 102			×		×	
7. M 105			×			×

4.8. FCA with incomplete data.

In this section, we provide a discussion on handling incomplete data (Simiński, 2012).

Definition 28. (*Incomplete context*) (*Krajca* et al., *2012*; *Li* et al., *2013a*; *Simiński, 2012*) An incomplete **L**-context is a *triplet* $= (X,Y,R)$, where X and Y are sets and R: $X \times Y \to L$ such that $R \subseteq U \cup \{0,1\}$. An ordinary context is the completion of a given relation.

Example 12. For illustration, we have considered an incomplete context shown in Table 16, where u_1 and u_2 represent the unknown values, and $u_1 \leq u_2$. Three possible contexts are shown in Tables 17–19. Their corresponding lattices are shown in Figs. 6–8. ♦

Definition 29. (*Incomplete fuzzy context*) Let $U = \{u_1,u_2,\ldots,u_k\}$ be the set of variables and $V (\subseteq 2^U)$ a set of assignments representing known dependencies between the variables. Then we can find the minimal residuated lattice $\mathbf{K}\,(U \cup L)$ for the set of admissible assignments V. An incomplete L-context with variables $\{u_1,u_2,\ldots,u_k\}$

Table 15. Formal concepts generated from the context shown in Table 14.

C_i	Concept	Descriptions
C_0	$(\oslash, Y)$	empty concept
C_1	$(\{1,4\}, \{g,y,o,s,b\})$	spiral galaxy
C_2	$(\{2,7\}, \{o,m\})$	elliptic galaxy
C_3	$(\{1,4,5,6\}, \{o,b\})$	lenticular galaxy
C_4	$(\{1,3,4\}, \{g,y,b\})$	irregular galaxy
C_5	$(\{1,2,4,5,6,7\}, \{o\})$	galaxy with old stars
C_6	$(\{1,3,4,5,6\}, \{b\})$	galaxy with bulge
C_7	$(X, \oslash)$	universal concept

Table 16. Incomplete formal context.

	y_1	y_2	y_3	y_4	y_5
x_1			×	×	
x_2	u_1	×	u_2	×	
x_3	×	×	×		
x_4		×			

is a formal context $(X, Y, \tilde{R})$, where $\tilde{R}$ can take values from L and U: $\tilde{R}(X \times Y) \subseteq U \cup L$. This means that the formal context contains only elements of L and the variables. Hence, for this purpose we can define a map for v: $U \to \mathbf{L}$, where $\mathbf{L}$ is a residuated lattice.

Table 17. Possible complete formal context for Table 16.

	y_1	y_2	y_3	y_4	y_5
x_1			×	×	
x_2		×		×	
x_3	×	×	×		
x_4		×			

Table 18. Second possible complete formal context for Table 16.

	y_1	y_2	y_3	y_4	y_5
x_1			×	×	
x_2		×	×	×	
x_3	×	×	×		
x_4		×			

Table 19. Third possible complete formal context for Table 16.

	y_1	y_2	y_3	y_4	y_5
x_1			×	×	
x_2	×	×	×	×	
x_3	×	×	×		
x_4		×			

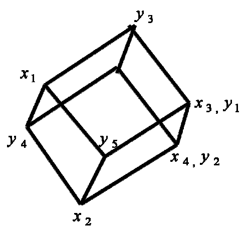


Fig. 6. Concept lattice for Table 17.

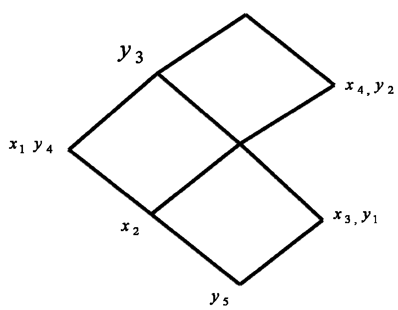


Fig. 7. Concept lattice for Table 18.

Table 20. Incomplete fuzzy formal context.

	y_1	y_2	y_3
x_1	0.5	0.0	0.5
x_2	u_1	1.0	0.0
x_3	0.0	u_2	0.5
x_4	0.0	1.0	1.0

Example 13. For illustration, an incomplete context shown in Table 20 is considered containing values from an L context $= \{0.0, 0.5, 1.0\}$, and the set of variables u_1, u_2 varies between 0.0, 0.5 and 1.0. Hence, the context can take values from $\{0.0, 0.5, 1.0\}$ and represent them as a complete fuzzy context. ♦

5. Applications of FCA

This section summarizes the applications of FCA reported in the literature after 2011. Tables 21–24 provide this summary. From these tables we can conclude that FCA has attracted applications in several domains due to its potential of knowledge discovery (Aswani Kumar, 2011a; 2011b; Aswani Kumar and Singh, 2014), representation (Iordache, 2011; Poelmans *et al.*, 2013a; 2014), reasoning (Rainer and Ganapati, 2011; Ruairi, 2013; Sebastien *et al.*, 2013) and the decision context (Li *et al.*, 2011a; 2011b; Shao *et al.*, 2014) which contains another tuple called a set of decision attributes (Yang *et al.*, 2011a). Ontology engineering is an another research direction regarding relations between individuals and classes. FCA has been applied to identify important groups of individuals that responded similarly to peer-identified experts (Alqadah and Bhatnagar, 2012; Codocedo *et al.*, 2012; Chen *et al.*, 2011; Formica, 2012; Fowler, 2013; Junli *et al.*, 2013; Senatore and Pasi, 2013; Tadrat *et al.*, 2012; Tho *et al.*, 2006). Recently, several researchers have shown the application of formal concepts in description logic for improving the knowledge representation task (Atif *et al.*, 2014; Borgwardt and Penaloza, 2014; Distel, 2012; Denniston *et al.*, 2013; Pei *et al.*, 2013; Wu *et al.*, 2012). Description logic discounts the structural representation of knowledge consisting of

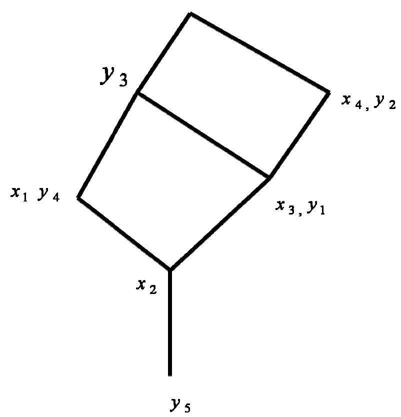


Fig. 8. Concept lattice for Table 19.

the terminological part (TBox) and the assertional part (ABox). Subsequently, hierarchical order visualization between the entity and its relations in a conceptual graph using FCA has been discussed (Croitorua *et al.*, 2012; Li and Guo, 2013; Nguyen *et al.*, 2011; Nguyen and Yamamoto, 2012; Yu *et al.*, 2013; Annapurna and Aswani Kumar, 2013).

FCA has been used for handling relational structures in the source code and dependency between software parts in several aspects including RBAC (Priss, 2011; 2012; Priss *et al.*, 2012; 2013). Role based access control (RBAC) provides the role of a user in the IT systems with specific permissions like read or write. Designing RBAC using FCA has been discussed by Aswani Kumar (2013). The technique to identify unexpected and potential effects caused by software changes and their impact analysis using FCA is developed by Korei (2013) and Li *et al.* (2013c).

A gene expression dataset is a many-valued context in which each row corresponds to a gene and each column to a sample, and the attribute (expression) values indicate the abundance of mRNA in a sample (Muszyński and Osowski, 2013), `http://indianalgae.co.in`. Hence the patterns of gene data have been studied after the scaled context using FCA by Kaytoue *et al.* (2011a). FCA has been applied for mining the common hypermethylated genes between breast cancer subtypes by Amin *et al.* (2012) and Bouaud *et al.* (2013). Endres *et al.* (2012) have applied FCA to read the semantic information obtained from fMRI Bold responses using FCA. The ingredients of FCA with mathematical morphology and description logics have been combined for image processing tasks by Atif *et al.* (2014). We observe that some of the researchers have tried to analyze the sentiments of people using FCA (emotions, love, preference) (Li and Tsai, 2013; Antoni *et al.*, 2014). The word opinion or preference shows two sides: one is acceptation and another is non-acceptation, which may mold the concept lattice for bipolar information visualization (Singh and Aswani Kumar, 2014).

"Big data" and their analysis attracted the attention of some researchers using FCA to find the pattern structure and its visualization (Biao *et al.*, 2012; Kuznetsov, 2013; Radvansky *et al.*, 2013). Subsequently, in cloud computing, allocating resources to users using FCA has been discussed by Sarnovsky *et al.* (2012).

Table 21. Some important applications of FCA in the KDD process and ontology engineering.

KDD process	Research goal
Alcalde *et al.*, 2012c	Finding temporal patterns
Alqadah and Bhatnagar, 2012	Mining similar concepts
Aswani Kumar, 2011b	Knowledge discovery
Aswani Kumar, 2012	Rule mining
Belohlavek *et al.*, 2011a	IPAQ questionnaires
Belohlavek *et al.*, 2013b	Background knowledge
Dau, 2013	Analyzing a triple store
Fowler, 2013	Order in taxonomy
Galitsky *et al.*, 2013	Pattern on parse thickets
Macko, 2013	Fuzzy FCA
Missaoui and Kwuida, 2011	Triadic rules
Nguyen *et al.*, 2011	Mathematical search
Nguyen and Yamamoto, 2012	Learning from graph
Li *et al.*, 2011b	Symbolic data analysis
Pavlovic, 2012	Quantitative data analysis
Rouane *et al.*, 2013	Multi relational data
Li and Tsai, 2013	Sentiments analysis
Trabelsi *et al.*, 2012	Analyzing folksonomies
Vityaev *et al.*, 2012	Probabilistic concepts
Watmough, 2014	ERP analysis
Yang *et al.*, 2011b	Decision-making
Zhao and Liu, 2011	Complex systems
Zhang *et al.*, 2012	Frequent concepts
Tang *et al.*, 2015	Chemical structure
Ontology engineering	**Research goal**
Alqadah and Bhatnagar, 2012	Mining similar concepts
Chen *et al.*, 2011	Merging domain ontology
Dau, 2013	Analyzing triple store
Formica, 2012	Semantic web search
Formica, 2013	Similarity reasoning
Fowler, 2013	Ontology investigation
Ilvovsky and Klimushki, 2013	Duplicate ontology
Junli *et al.*, 2013	Merging ontology
Macko, 2013	Friendly ontology
De Maio *et al.*, 2012b	E-learning
De Maio *et al.*, 2014	Ontological structure
Tadrat *et al.*, 2012	Case based reasoning
Tho *et al.*, 2006	Fuzzy ontology

6. Conclusions

In this paper we aimed at analyzing the current research trends in FCA based on innovations reported in more than 350 papers published after 2011. We can observe that FCA has received significant attention of researchers for knowledge discovery and representation tasks. Subsequently, FCA is extended into different applications of data analysis. In this paper we have analyzed some of these extensions and augmentation of FCA with illustrative examples. The first categorized domain is granular based computing of formal concepts to describe their importance. Other domains discuss the mathematics behind FCA with a fuzzy setting, an interval-valued fuzzy setting, possibility theory, a rough setting, a triadic, factor and incomplete context to apply these extensions in the appropriate context for knowledge processing tasks.

Table 22. Some important applications of FCA in text mining, information retrieval and linguistics.

Text mining	Research goal
Belohlavek and Macko, 2011	Selecting some concepts
Ferjani *et al.*, 2012	Feature extractions
Formica, 2012	Semantic web
Galitsky *et al.*, 2013	Finding patterns on parse thickets
Hamrouni *et al.*, 2013	Finding some frequent itemset
Li and Guo, 2013	Investigating formal query
De Maio *et al.*, 2014	Text mining
Muangprathub *et al.*, 2013	Classification
Li and Tsai, 2013	Text mining
Information retrieval	**Research goal**
Aswani Kumar *et al.*, 2012	Information retrieval
Alqadah and Bhatnagar, 2012	Similar concepts
Bloch, 2011	Bipolar information
Eklund *et al.*, 2012	Similar concepts
Chen *et al.*, 2011	Domain ontology
Codocedo *et al.*, 2012	Finding cousins
Neznanov and Kuznetsov, 2013	FCART tool
Poshyvanyk *et al.*, 2012	Concept location
Priss, 2006	Application in information sciences
Senatore and Pasi, 2013	Finding correlations
Li and Tsai, 2013	Opinion classification
Zerarga and Djouadi, 2013	Information retrieval
Linguistics	**Research goal**
Alcalde *et al.*, 2011	Linguistic proposition
Bloch, 2011	Linguistics representation
Chen *et al.*, 2011	Wordnet system
Croitorua *et al.*, 2012	Linguistics analysis
Muangprathub *et al.*, 2013	Classification
Priss, 2005	Linguistics application
Yu *et al.*, 2013	Analyzing verbs

Table 23. Some important applications of FCA in security analysis, web services, social network analysis and software engineering.

Security analysis	Research goal
Aswani Kumar, 2013	Role based access control
Aufaure and Grand, 2013	Social network analysis
Cook and Coombs, 2004	Military intelligence
Du and Hai, 2013	Mining web page
Elzinga *et al.*, 2010	Terrorist threat assessment
Poelmans *et al.*, 2013c	Criminal trajectories
Priss, 2011	Unix system monitoring
Romanov *et al.*, 2012	Detect anomalies
Web services	**Research goal**
Qin *et al.*, 2013	Impact analysis
De Maio *et al.*, 2012a	E-learning
Priss *et al.*, 2013	Software assessment
Rouane *et al.*, 2013	Mining from multi relational data
Watmough, 2014	ERP analysis
Tho *et al.*, 2006	Web retrieval
Zhang *et al.*, 2013a; 2013b	Extracting data from web database
Social network analysis	**Research goal**
Aufaure and Grand, 2013	FCA in social network analysis
Cook and Coombs, 2004	Network analysis using FCA
Elzinga *et al.*, 2010	Terrorist threat assessment by FCA
Li *et al.*, 2013c	Call graph for network
Poelmans *et al.*, 2013c	Criminal trajectory network analysis
Wang *et al.*, 2012	Wireless sensor network
Software engineering	**Research goal**
Helen *et al.*, 2013	Energy saving model using FCA
Priss *et al.*, 2012	Learning process
Rouane *et al.*, 2013	Relational concept analysis
Sarnovsky *et al.*, 2012	Distributed S/W analysis

Acknowledgment

The first and the second author acknowledge the financial support from NBHM, DAE, Government of India, under the grant 2/48(11)/2010-R&D II/10806. Further, the first and the third author acknowledge the financial support from the *Mobile Cloud Computing* research project funded by the Malaysian Ministry of Higher Education under the University of Malaya High Impact Research Grant UM.C/625/1/HIR/MOE/FCSIT/03. The authors also thank the anonymous reviewers for their useful suggestions and comments.

References

Akram, M. and Dudek, W.A. (2011). Interval valued fuzzy graphs, *Computers Mathematics with Applications* **61**(2): 289–299.

Alcalde, C., Burusco, A., Fuentes-González, R. and Zubia, I. (2011). The use of linguistic variables and fuzzy propositions in the L-fuzzy concept theory, *Computers and*

Table 24. Some important applications of FCA in bioinformatics, image processing and psychology.

Bioinformatics	Research goal
Amin *et al.*, 2012	Breast cancer analysis using FCA
Belohlavek *et al.*, 2013b	Study of taxonomy behavior using FCA
Gonzalez Calabozo *et al.*, 2011	Gene expression data analysis
Fan *et al.*, 2013	Analysis of Chinese medicine using FCA
Kaytoue *et al.*, 2011b	Mining patterns in Gene expression data
Junli *et al.*, 2013	Merging geo-ontology
Image processing	**Research goal**
Atif *et al.*, 2014	Image processing
Bloch, 2011	Morphology and its link to FCA
Croitorua *et al.*, 2012	Linguistics analysis
De Maio *et al.*, 2014	Image processing using fuzzy FCA
Endres *et al.*, 2012	Analyzing the semantic of fMRI brain recording
Sawase *et al.*, 2009	Visualizing huge image data using FCA
Xu *et al.*, 2012	MapReduce framework using FCA
Psychology	**Research goal**
Antoni *et al.*, 2014	Preference analysis
Li and Tsai, 2013	Opinion classification
Poelmans, 2011	Domestic violence
Priss, 2006	Information science
Spoto *et al.*, 2010	Computerized psychological assessment using FCA

Mathematics with Applications **62**(8): 3111–3122.

Alcalde, C., Burusco, A. and Fuentes-González, R. (2012a). Analysis of certain L-fuzzy relational equations and the study of its solutions by means of the L-fuzzy concept theory, *International Journal of Uncertainty, Fuzziness and Knowlege-Based Systems* **20**(1): 21–40.

Alcalde, C., Burusco, A. and Fuentes-González, R. (2012b). Composition of L-fuzzy contexts, *Proceedings of the 10th ICFCA 2012, Leuven, Belgium*, pp. 1–14 .

Alcalde, C., Burusco, A. and Fuentes-González, R. (2012c). The study of fuzzy context sequences, *International Journal of Computational Intelligence Systems* **6**(3): 518–529.

Alcalde, C., Burusco, A. and Fuentes-González, R. (2015). The use of two relations in L-fuzzy contexts, *Information Sciences* **301**: 1–14.

Alqadah, F. and Bhatnagar, R. (2012). Similarity measures in formal concept analysis, *Annals of Mathematics and Artificial Intelligence* **61**(3): 245–256.

Amin, I.I., Kassim, S.K., Hassanien, A.E. and Hefny, H.A. (2012). Formal concept analysis for mining hypermethylated genes in breast cancer tumor subtypes, *Proceedings of 12th ISDA, 2012, Kochi, India*, pp. 764–769.

Annapurna, J. and Aswani Kumar, Ch. (2013). Exploring attribute with domain knowledge in formal concept analysis, *Journal of Computing and Information Technology* **21**(2): 109–123.

Antoni, L., Krajci, S., Kridlo, O., Macek, B. and Piskova, L. (2014). On heterogeneous formal contexts, *Fuzzy Sets and Systems* **234**: 22–33.

Aswani Kumar, Ch. (2011a). Reducing data dimensionality using random projections and fuzzy K-means clustering, *International Journal of Intelligent Computing and Cybernetics* **4**(3): 353–365.

Aswani Kumar, Ch. (2011b). Knowledge discovery in data using formal concept analysis and random projections, *International Journal of Applied Mathematics and Computer Science* **21**(4): 745–756, DOI: 10.2478/v10006-011-0059-1.

Aswani Kumar, Ch. (2012). Fuzzy clustering-based formal concept analysis for association rules mining, *Applied Artificial Intelligence* **26**(3): 274–301.

Aswani Kumar, Ch. (2013). Designing role-based access control using formal concept analysis, *Security and Communication Networks* **6**(3): 373–383.

Aswani Kumar, Ch., Radvansky, M., Fuentes-Gonzlez, R. and Annapurna, J. (2012). Analysis of a vector space model, latent semantic indexing and formal concept analysis for information retrieval, *Cybernetics and Information Technologies* **12**(1): 34–48.

Aswani Kumar, Ch., Dias, S.M. and Vieira, N.J. (2015a). Knowledge reduction in formal contexts using non-negative matrix factorization, *Mathematics and Computers in Simulation* **109**: 46–63.

Aswani Kumar, Ch., Ishwaryaa, M.S. and Loo, C.K. (2015b). Formal concept analysis approach to cognitive functionalities of bidirectional associative memory, *Biologically Inspired Cognitive Architectures* **22**: 20–33, DOI:10.1016/j.bica.2015.04.003.

Aswani Kumar, Ch. and Singh, P.K. (2014). Knowledge representation using formal concept analysis: A study on concept generation, *in* B.K. Tripathy and D.P. Acharjya (Eds.), *Global Trends in Knowledge Representation and Computational Intelligence*, IGI Global Publishers, Hershey, PA, pp. 306–336.

Aswani Kumar, Ch. and Srinivas, S. (2010). Concept lattice reduction using fuzzy K-means clustering, *Expert Systems with Application* **37**(3): 2696–2704.

Atif, J., Hudelot, C. and Bloch, I. (2014). Explanatory reasoning for image understanding using formal concept analysis and description logics, *IEEE Transactions on Systems, Man, and Cybernetics A* **44**(4): 552–570.

Aufaure, M.A. and Grand, B.L. (2013). Advances in FCA-based applications for social networks analysis, *International Journal of Conceptual Structures and Smart Applications* **1**(1): 73–89.

Babin, M.A. and Kuznetsov, S.O. (2012). Approximating concept stability, *in* F. Domenach *et al.* (Eds.), *Proceedings of the 10th International Conference, ICFCA 2012*, Lecture Notes in Computer Science, Vol. 7278, Springer, Berlin/Heidelberg, pp. 7–15.

Babin, M.A. and Kuznetsov, S.O. (2013). Computing premises of minimal cover of functional dependencies is intractable, *Discrete Applied Mathematics* **161**(6): 742–749.

Bartl, E., Rezankova, H. and Sobisek, L. (2011). Comparison of classical dimensionality reduction methods with novel approach based on formal concept analysis, *in* J.T. Yao *et al.* (Eds.), *Rough Set and Knowledge Technology*, Lecture Notes in Computer Science, Vol. 6954, Springer, Berlin/Heidelberg, pp. 26–35.

Bazhanov, K. and Obiedkov, S. (2014). Optimizations in computing the Duquenne-Guigues basis of implications, *Annals of Mathematics and Artificial Intelligence* **70**(1): 5–24.

Belohlavek, R. (2012). Optimal decompositions of matrices with entries from residuated lattices, *Annals of Mathematics and Artificial Intelligence* **22**(6): 1405–1425.

Belohlavek, R., Baets, B.D. and Konecny, J. (2014). Granularity of attributes in formal concept analysis, *Information Sciences* **260**(5): 149–170.

Belohlavek, R., Glodeanu, C. and Vychodil, V. (2013a). Optimal factorization of three-way binary data using triadic concepts, *Order* **30**(2): 437–454.

Belohlavek, R., Kostak, M. and Osicka, P. (2013b). Formal concept analysis with background knowledge: A case study in paleobiological taxonomy of belemnites, *International Journal of General Systems* **42**(4): 426–440.

Belohlavek, R., and Macko, J. (2011). Selecting important concepts using weights, *in* P. Valtchev *et al.* (Eds.), *Formal Concept Analysis*, Lecture Notes in Computer Science, Vol. 6628, Springer, Berlin/Heidelberg, pp. 65–80.

Belohlavek, R., Sigmund, E. and Zacpal, J. (2011a). Evaluation of IPAQ questionnaires supported by formal concept analysis, *Information Sciences* **181**(10): 1774–1786.

Belohlavek, R., Osicka, P. and Vychodil, V. (2011b). Factorizing three-way ordinal data using triadic formal concepts, *in* H. Christiansen *et al.* (Eds.), *Flexible Query Answering Systems*, Lecture Notes in Computer Science, Vol. 7022, Springer, Berlin/Heidelberg, pp. 400–411.

Belohlavek, R., and Osicka, P. (2012a). Triadic fuzzy Galois connections as ordinary connections, *IEEE International Conference on Fuzzy Systems, Brisbane, Australia*, pp. 1–6.

Belohlavek, R. and Osicka, P. (2012b). Triadic concept lattices of data with graded attributes, *International Journal of General Systems* **41**(2): 93–108.

Belohlavek, R., and Trnecka, M. (2013). Basic level in formal concept analysis: Interesting concepts and psychological ramifications, *Proceedings of the 23rd International Joint Conference on Artificial Intelligence, Beijing, China*, pp. 1233–1239.

Belohlavek, R. and Vychodil, V. (2005). Fuzzy attribute logic: Entailment and non-redundant basis, *11th International Fuzzy Systems Association World Congress, Tsinghua, China*, pp. 622–627.

Belohlavek, R. and Vychodil, V. (2012). Formal concept analysis and linguistic hedges, *International Journal of General Systems* **41**(5): 503–532.

Biao, X., Ruairi, de F., Eric, R. and Micheal, F. (2012). Distributed formal concept analysis algorithms based on an iterative map reduce framework, *in* F. Domenach *et al.* (Eds.), *Formal Concept Analysis*, Lecture Notes in Computer Science, Vol. 7278, Springer, Berlin/Heidelberg, pp. 292–308.

Bloch, I. (2011). Lattices of fuzzy sets and bipolar fuzzy sets and mathematical morphology, *Information Sciences* **181**(10): 2002–2015.

Borgwardt, S. and Penaloza, R. (2014). Consistency reasoning in lattice-based fuzzy description logics, *International Journal of Approximate Reasoning* **55**(9): 1917–1938.

Bouaud, J., Messai, N., Laouenan, C., Mentre, F. and Seroussi, B. (2013). Elicitating patient patterns of physician non-compliance with breast cancer guidelines using formal concept analysis, *Studies in Health Technology and Informatics* **180**: 471–481.

Butka, P., Pocs, J. and Pocsova, J. (2012). Use of concept lattices for data tables with different types of attributes, *Journal of Information and Organizational Sciences* **36**(1): 1–12.

Chen, J., Lia, J., Lin, Y., Lin, G. and Ma, Z. (2015). Relations of reduction between covering generalized rough sets and concept lattices, *Information Sciences* **304**: 16–27.

Chen, R.C., Bau, C.T. and Yeh, C.J. (2011). Merging domain ontologies based on the WordNet system and fuzzy formal concept analysis techniques, *Applied Soft Computing* **11**(2): 1908–1923.

Ciobanu, G. and Vaideanu, C. (2014). Similarity relations in fuzzy attribute-oriented concept lattices, *Fuzzy Sets and Systems* **275**: 88–109.

Codocedo, V., Taramasco, C. and Astudillo, H. (2011). Cheating to achieve formal concept analysis over a large formal context, *Proceedings of the 11th International Conference on Concept Lattices and Their Applications, Kosice, Slovakia*, pp. 349–362.

Codocedo, V., Lykourentzou, I. and Napoli, A. (2012). Semantic querying of data guided by formal concept analysis, *Formal Concept Analysis for Artificial Intelligence, Nancy, France*.

Cook, T.M. and Coombs, M. (2004). Using formal concept analysis (FCA) to model and represent counterdeception analytic tasks, *Proceedings of the 13th International Conference on Behavior Representation in Modeling and Simulation, Arlington, VA, USA*, pp. 7–8.

Croitorua, M., Orenb, N., Milesc, S. and Luckc, M. (2012). Graphical norms via conceptual graphs, *Knowledge-Based Systems* **29**: 31–43.

Dau, F. (2013). Towards scalingless generation of formal contexts from an ontology in a triple stores, *International Journal of Conceptual Structures and Smart Applications* **1**(1): 18–38.

Davey, B.A. and Priestley, H.A. (2002). *Introduction to Lattices and Order*, Cambridge University Press, Cambridge.

De Maio, C., Fenza, G., Loia, V. and Senatore, S. (2012a). Hierarchical web resources retrieval by exploiting fuzzy formal concept analysis, *Information Processing and Management* **48**(3): 399–418.

De Maio, C., Fenza, G., Gaeta, M., Loia, V., Orciuoli, F. and Senatore, S. (2012b). RSS-based e-learning recommendations exploiting fuzzy FCA for knowledge modeling, *Applied Soft Computing* **12**(1): 113–124.

De Maio, C., Fenza, G., Gallo, M., Loia, V. and Senatore, S. (2014). Formal and relational concept analysis for fuzzy-based automatic semantic annotation, *Applied Intelligence* **40**(1): 153–174.

Denniston, J.T., Melton, A. and Rodabaugh, S.E. (2013). Formal concept analysis and lattice-valued Chu systems, *Fuzzy Sets and Systems* **216**: 52–90.

Dias, S.M., Zarate, L.E. and Vieira, N.J. (2013). Extracting reducible knowledge from ANN with JBOS and FCANN approaches, *Expert Systems with Applications* **40**(8): 3087–3095.

Dias, S.M., and Vieira, N.J. (2013). Applying the JBOS reduction method for relevant knowledge extraction, *Expert Systems with Applications* **40**(5): 1880–1887.

Dias, S.M. and Vieira, N.J. (2015). Concept lattices reduction: Definition, analysis and classification, *Expert Systems with Applications* **42**(20): 7084–7097, DOI: 10.1016/j.eswa.2015.04.044

Distel, F. (2012). Adapting fuzzy formal concept analysis for fuzzy description logics, *Proceedings of CLA, Fuengirola, Spain*, pp. 163–174.

Djouadi, Y. (2011). Extended Galois derivation operators for information retrieval based on fuzzy formal concept lattice, *in* S. Benferhat *et al.* (Eds.), *Scalable Uncertainty Management,* Lecture Notes in Computer Science, Vol. 6929, Springer, Berlin/Heidelberg, pp. 346–358.

Djouadi, Y. and Prade, H. (2009). Interval-valued fuzzy formal concept analysis, *in* J. Rauch *et al.* (Eds.), *Foundations of Intelligent System*, Lecture Notes in Artificial Intelligence, Vol. 5722, Springer, Berlin/Heidelberg, pp. 592–601.

Doerfel, S., Jaschke, R. and Stumme, G. (2012). Publication analysis of the formal concept analysis community, *in* F. Domenach *et al.* (Eds.), *Formal Concept Analysis*, Lecture Notes in Computer Science, Vol. 7278, Springer, Berlin/Heidelberg, pp. 77–95.

Du, Y. and Hai, Y. (2013). Semantic ranking of web pages based on formal concept analysis, *Journal of Systems and Software* **86**(1): 187–197.

Dubois, D. and Prade, H. (2012). Possibility theory and formal concept analysis: Characterizing independent sub-contexts, *Fuzzy Sets and Systems* **196**: 4–16.

Endres D., Adam, R., Giese. M.A. and Noppeney, U. (2012). Understanding the semantic structure of human fMRI brain recordings with formal concept analysis, *in* F. Domenach *et al.* (Eds.), *Formal Concept Analysis*, Lecture Notes in Computer Science, Vol. 7278, Springer, Berlin/Heidelberg, pp. 96–111.

Eklund, P., Ducrou, J. and Dau, F. (2012). Concept similarity and related categories in information retrieval using formal concept analysis, *International Journal of General Systems* **41**(8): 826–846.

Elzinga, P., Viaene, S., Poelmans, J., Dedene, G. and Morsing, S. (2010). Terrorist threat assessment with formal concept analysis, *Proceedings of the 2010 IEEE International Conference on Intelligence and Security Informatics, Vancouver, BC, Canada*, pp. 77–82.

Fan, F., Hong, W., Song, J., Jing, J. and Ji, S. (2013). A visualization method for Chinese medicine knowledge discovery based on formal concept analysis, *ICIC Express Letters* **4**(3): 801–808.

Ferjani, F., Elloumi, S., Jaoua, A., Ben Yahia, S., Ismail, S. and Ravan, S. (2012). Formal context coverage based on isolated labels: An efficient solution for text feature extraction, *Information Sciences* **188**: 198–214.

Formica, A. (2012). Semantic web search based on rough sets and fuzzy formal concept analysis, *Knowledge-Based Systems* **26**(3): 40–47.

Formica, A. (2013). Similarity reasoning for the semantic web based on fuzzy concept lattices: An informal approach, *Information Systems Frontiers* **15**(3): 511–520.

Fowler, M. (2013). The taxonomy of a Japanese stroll garden: An ontological investigation using formal concept analysis, *Axiomathes* **13**(1): 43–59.

Fu, H. and Mephu Nguifo, E. (2004). A parallel algorithm to generate formal concepts for large data, *in* P. Eklund (Ed.), *Concept Lattices*, Lecture Notes in Computer Science, Vol. 2961, Springer, Berlin/Heidelberg, pp. 394–401.

Ganter, B. and Glodeanu, C.V. (2012). Ordinal factor analysis, *in* F. Domenach *et al.* (Eds.), *Formal Concept Analysis*, Lecture Notes in Computer Science, Vol. 7278, Springer, Berlin/Heidelberg, pp. 128–139.

Ganter, B. and Meschke, C. (2011). A formal concept analysis approach to rough data tables, *in* H. Sakai *et al.* (Eds.), *Rough Sets, Fuzzy Sets, Data Mining and Granular Computing*, Lecture Notes in Computer Science, Vol. 6600, Springer, Berlin/Heidelberg, pp. 37–61.

Ganter, B. and Wille, R. (1999). *Formal Concept Analysis: Mathematical Foundation*, Springer-Verlag, Berlin.

Galitsky, B.A., Ilvovsky, D., Strok, F. and Kuznetsov, S.O. (2013). Improving text retrieval efficiency with pattern structures on parse thickets, *Proceedings of FCAIR 2013, Moscow, Russia*, pp. 6–21.

Glodeanu, C.V. (2011). Factorization with hierarchical classes analysis and formal concept analysis, *in* P. Valtchev *et al.* (Eds.), *Formal Concept Analysis*, Lecture Notes in Computer Science, Vol. 6628, Springer, Berlin/Heidelberg, pp. 107–118.

Glodeanu, C.V. (2012). Attribute dependency in fuzzy setting, *Proceedings of CLA 2012, Fuengirola, Spain*, pp. 127–138.

Glodeanu, C.V. and Ganter, B. (2012). Applications of ordinal factor analysis, *in* P. Cellier *et al.* (Eds.), *Formal Concept Analysis*, Lecture Notes in Computer Science, Vol. 7880, Springer, Berlin/Heidelberg pp. 109–124.

Gonzalez Calabozo, J.M., Pelaez-Moreno, C. and Valverde-Albacete, F.J. (2011). Gene expression array exploration using K-formal concept analysis, *in* P. Valtchev and R. Jäschke (Eds.), *Proceedings of the 9th International Conference ICFCA 2011*, Lecture Notes in Computer Science, Vol. 6628, Springer, Berlin/Heidelberg, pp. 119–134.

Hamrouni, T., Ben Yahia, S. and Mephu Nguifo, E. (2013). Looking for a structural characterization of the sparseness measure of (frequent closed) itemset contexts, *Information Sciences* **222**: 343–361.

Helen, Z., David, J. and Zhao, X.J. (2013). Construction of new energy-saving building materials based on formal concept analysis methods, *Advanced Materials Research* **738**: 133–136.

Helmi, B.H., Rahmani, A.T. and Pelikan, M. (2014). A factor graph based genetic algorithm, *International Journal of Applied Mathematics and Computer Science* **24**(3): 621–633, DOI: 10.2478/amcs-2014-0045.

Ignatov, D.I., Kuznetsov, S.O., Magizov, R.A. and Zhukov, L.E. (2011). From triconcepts to triclusters, *in* S.O. Kuznetsov *et al.* (Eds.) *Rough Sets, Fuzzy Sets, Data Mining and Granular Computing*, Lecture Notes in Computer Science, Vol. 6743, Springer, Berlin/Heidelberg, pp. 257–264.

Ignatov, D.I., Gnatyshak, D.V., Kuznetsov, S.O. and Mirkin, B.G. (2015). Triadic formal concept analysis and triclustering: Searching for optimal patterns, *Machine Learning* **101**(1): 271–302, DOI:10.1007/s10994-015-5487-y.

Ilvovsky, D. and Klimushkin, M. (2013). FCA-based search for duplicate objects in ontologies, *Proceedings of FCAIR, Moscow, Russia*, pp. 36–46.

Iordache, O. (2011). Modeling multi-level systems, *Understanding Complex Systems* **70**: 143–163.

Junli, L., Zongyi, H. and Qiaoli, Z. (2013). An entropy-based weighted concept lattice for merging multi-source geo-ontologies, *Entropy* **15**(6): 2303–2318.

Kaiser, T.B. and Schmidt, S.E. (2013). A macroscopic approach to FCA and its various fuzzifications, *in* F. Domenach *et al.* (Eds.), *Formal Concept Analysis*, Lecture Notes in Computer Science, Vol. 7278, Springer, Berlin/Heidelberg, pp. 140–147.

Kang, X., Li, D., Wang, S. and Qu, K. (2012a). Formal concept analysis based on fuzzy granularity base for different granulations, *Fuzzy Sets and Systems* **203**: 33–48.

Kang, X., Li, D., Wang, S. and Qu, K. (2012b). Rough set model based on formal concept analysis, *Information Sciences* **222**: 611–625.

Kaytoue, M., Kuznetsov, S.O., Napoli, A. and Polaillon, G. (2011a). Symbolic data analysis and formal concept analysis, *XVIIIeme Rencontres de la Societe Francophone de Classification—SFC, Orléans, France*, pp. 1–4.

Kaytoue, M., Kuznetsov, S.O., Napoli, A. and Duplessis, S. (2011b). Mining gene expression data with pattern structures in formal concept analysis, *Information Sciences* **181**: 1989–2001.

Krajca, P, Outrata, J. and Vychodil, V. (2008). Parallel recursive algorithm for FCA, *Proceedings of CLA, Olomouc, Czech Republic*, pp. 71–82.

Krajca, P., Outrata, J. and Vychodil, V. (2012). Concept lattices of incomplete data, *in* F. Domenach *et al.* (Eds.), *Formal Concept Analysis*, Lecture Notes in Computer Science, Vol. 7278, Springer, Berlin/Heidelberg, pp. 180–194.

Korei, A. (2013). Applying formal concept analysis in machine-part grouping problems, *Proceedings of the 11th International Symposium on Applied Machine Intelligence and Informatics 2013, Herl'any, Slovakia,* pp. 197–200.

Kuznetsov, S.O. (2013). Fitting pattern structures to knowledge discovery in big data, *in* P. Cellier *et al.* (Eds.), *Formal Concept Analysis*, Lecture Notes in Computer Science, Vol. 7880, Springer, Berlin/Heidelberg, pp. 254–266.

Kuznetsov, S.O. and Obiedkov, S.A. (2002). Comparing performance of algorithms for generating concept lattices, *Journal of Experimental and Theoretical Artificial Intelligence* **14**(2–3): 189–216.

Kuznetsov, S.O. and Poelmans, J. (2013). Knowledge representation and processing with formal concept analysis, *Wiley Interdisciplinary Reviews: Data Mining and Knowledge Discovery* **3**(3): 200–215.

Langdon, W.B., Yoo, S. and Harma, M. (2011). Formal concept analysis on graphics hardware, *Proceedings of CLA, Nancy, France*, pp. 413–416.

Lei, Y. and Tian, J. (2012). Concepts with negative-values and corresponding concept lattices, *Proceedings of the 9th International Conference on Fuzzy Systems and Knowledge Discovery, Sichuan, China*, pp. 1005–1008.

Li, J., Changlin, M. and Yuejin, L. (2011a). A heuristic knowledge-reduction method for decision formal contexts, *Computers and Mathematics with Applications* **61**(4): 1096–1106.

Li, J., Changlin, M. and Yuejin, L. (2011b). Knowledge reduction in decision formal contexts, *Knowledge-Based Systems* **24**(5): 709–715.

Li, J., Mei, C. and Lv, Y. (2012a). Knowledge reduction in real decision formal contexts, *Information Sciences* **189**(5): 191–207.

Li, J., Mei, C. and Lv, Y. (2012b). Knowledge reduction in formal decision contexts based on an order-preserving mapping, *International Journal of General Systems* **41**(5): 143–161.

Li, J., Mei, C. and Lv, Y. (2013a). Incomplete decision contexts: Approximate concept construction, rule acquisition and knowledge reduction, *International Journal of Approximate Reasoning* **54**(1): 149–165.

Li, J., Mei, C., Aswani Kumar, Ch. and Lv, Y. (2013b). On rule acquisition in decision formal contexts, *International Journal of Machine Learning and Cybernetics* **4**(6): 721–731.

Li, B., Suna, X. and Leungc, H. (2013c). Combining concept lattice with call graph for impact analysis, *Advances in Engineering Software* **53**: 41–43.

Li, J., Mei, C., Xu, W. and Qian, Y. (2015). Concept learning via granular computing: A cognitive viewpoint, *Information Sciences* **298**: 447–467.

Li, M.Z. and Guo, L. (2013). Formal query systems on contexts and a representation of algebraic lattices, *Information Sciences* **239**: 72–74.

Li, M.Z. and Mi, J.S. (2013). The strong direct product of formal contexts, *Information Sciences* **226**: 47–67.

Li, S.T. and Tsai, F.C. (2013). A fuzzy conceptualization model for text mining with application in opinion polarity classification, *Knowledge-Based Systems* **39**: 23–33.

Ma, J.M. and Zhang, W.X. (2013). Axiomatic characterizations of dual concept lattices, *International Journal of Approximate Reasoning* **54**(5): 690–697.

Macko, J. (2013). User-friendly fuzzy FCA, *in* P. Cellier *et al.* (Eds.), *Proceedings of the 11th International Conference ICFCA 2013*, Lecture Notes in Computer Science, Vol. 7880, Springer, Berlin/Heidelberg, pp. 156–171.

Mariano, F.L., Asuncion, G.P. and Mari Carmen, S.F. (2013). Methodological guidelines for reusing general ontologies, *Data and Knowledge Engineering* **86**: 242–275.

Martin, T.P., Abd Rahim, N.H. and Majidian, A. (2013). A general approach to the measurement of change in fuzzy concept lattices, *Soft Computing* **17**(12): 2223–2234.

Martin, T. and Majidian, A. (2013). Finding fuzzy concepts for creative knowledge discovery, *International Journal of Intelligent Systems* **28**(1): 93–114.

Massanet, S., Mayor, G., Mesiar, R. and Torrens, J. (2013). On fuzzy implications: An axiomatic approach, *International Journal of Approximate Reasoning* **54**(9): 1471–1482.

Medina, J. (2012a). Relating attribute reduction in formal, object-oriented and property-oriented concept lattices, *Computers and Mathematics with Applications* **64**(6): 1992–2002.

Medina, J. (2012b). Multi-adjoint property-oriented and object-oriented concept lattices, *Information Sciences* **190**: 95–2006.

Medina, J. and Ojeda-Aciego, M. (2012). On multi-adjoint concept lattices based on heterogeneous conjunctors, *Fuzzy Sets and Systems* **208**: 95–110.

Missaoui, R. and Kwuida, L. (2011). Mining triadic association rules from ternary relations, *in* P. Valtchev and R. Jäschke (Eds.), *Proceedings of the 9th International Conference ICFCA 2011*, Lecture Notes in Computer Science, Vol. 6628, Springer, Berlin/Heidelberg, pp. 204–218.

Muangprathub, J., Boonjing, V. and Pattaraintakorn, P. (2013). A new case-based classification using incremental concept lattice knowledge, *Data and Knowledge Engineering* **83**: 39–53.

Muszyński, M. and Osowski, S. (2013). Data mining methods for gene selection on the basis of gene expression arrays, *International Journal of Applied Mathematics and Computer Science* **24**(3): 657–668, DOI: 10.2478/amcs-2014-0048.

Neznanov, A. and Kuznetsov, S.O. (2013). Information retrieval and knowledge discovery with FCART, *in* S.O. Kuznetsov *et al.* (Eds.), *Proceedings of FCAIR*, Vol. 977, Moscow, pp. 74–82.

Nguyen, T.T., Hui, S.C and Chang, K. (2011). A lattice-based approach for mathematical search using formal concept analysis, *Expert Systems with Applications* **39**(5): 5820–5828.

Nguyen, V.A. and Yamamoto, A. (2012). Learning from graph data by putting graphs on the lattice, *Expert Systems with Applications* **39**(12): 11172–11182.

Obiedkov, S. (2012). Modeling preferences over attribute sets in formal concept analysis, *in* F. Domenach *et al.* (Eds.), *Proceedings of the 10th International Conference ICFCA 2012*, Lecture Notes in Computer Science, Vol. 7278, Springer, Berlin/Heidelberg, pp. 227–243.

Outrata, J. and Vychodil, V. (2012). Fast algorithm for computing fixpoints of Galois connections induced by object-attribute relational data, *Information Sciences* **185**(1): 114–127.

Pavlovic, D. (2012). Quantitative concept analysis, *in* F. Domenach *et al.* (Eds.), *Formal Concept Analysis*, Lecture Notes in Computer Science, Vol. 7278, Springer, Berlin/Heidelberg, pp. 260–277.

Pedrycz, W. (2013). *Granular Computing Analysis and Design of Intelligent Systems*, CRC Press, Boca Raton, FL.

Pei, Z., Ruan, D., Meng, D. and Liu, Z. (2013). Formal concept analysis based on the topology for attributes of a formal context, *Information Sciences* **236**: 66–82.

Pocs, J. (2012). On possible generalization of fuzzy concept lattices using dually isomorphic retracts, *Information Sciences* **210**: 89–98.

Poelmans, J. (2011). Formally analyzing the concepts of domestic violence, *Expert Systems with Applications* **38**(4): 3116–3130.

Poelmans, J., Ignatov, D.I., Kuznetsov, S.O. and Dedene, G. (2013a). Formal concept analysis in knowledge processing: A survey on models and techniques, *Expert Systems with Applications* **40**(16): 6601–6623.

Poelmans, J., Kuznetsov, S.O., Ignatov, D.I. and Dedene, G. (2013b). Formal concept analysis in knowledge processing: A survey on applications, *Expert Systems with Applications* **40**(16): 6538–6560.

Poelmans, J., Elzinga, P. and Dedene, G. (2013c). Retrieval of criminal trajectories with an FCA-based approach, *in* O. Kuznetsov *et al.* (Eds.), *Proceedings of FCAIR*, Vol. 977, Moscow, pp. 83–94.

Poelmans, J., Ignatov, D.I., Kuznetsov, S.O. and Dedene, G. (2014). Fuzzy and rough formal concept analysis: A survey, *International Journal of General Systems* **43**(2): 105–134.

Poshyvanyk, D., Gethers, M. and Marcus, A. (2012). Concept location using formal concept analysis and information retrieval, *ACM Transactions on Software Engineering and Methodology* **21**(4), Article No. 23, DOI:10.1145/2377656.2377660.

Priss, U. (2005). Linguistic applications of formal concept analysis, *in* B. Ganter *et al.* (Eds.), *Formal Concept Analysis: Foundations and Applications*, Lecture Notes in Computer Science, Vol. 3626, Springer, Berlin/Heidelberg, pp. 149–160.

Priss, U. (2006). Formal concept analysis in information science, *Annual Review of Information Science and Technology* **40**(1): 521–543.

Priss, U. (2011). Unix systems monitoring with FCA, *in* S. Andrews *et al.* (Eds.), *Conceptual Structures for Discovering Knowledge*, Lecture Notes in Artificial Intelligence, Vol. 6828, Springer, Berlin/Heidelberg, pp. 243–256.

Priss, U. (2012). Concept lattices and median networks, *Proceedings of CLA, Derby, UK*, pp. 351–354.

Priss, U., Peter, R. and Jensen, N. (2012). Using FCA for modelling conceptual difficulties in learning processes, *in* S. Andrews *et al.* (Eds.), *Conceptual Structures for Discovering Knowledge*, Vol. 6828, Springer, Berlin/Heidelberg, pp. 161–173.

Priss, U., Jensen, N. and Rod, O. (2013). Using conceptual structures in the design of computer-based assessment software, *in* H.D. Pfeiffer *et al.* (Eds.), *Conceptual Structures for Discovering Knowledge,* Lecture Notes in Artificial Intelligence, Vol. 7735, Springer, Berlin/Heidelberg, pp. 193–209.

Qin, X., Liu, K. and Tang, S. (2013). Fuzzy FCA-based web service discovery, *Journal of Information and Computational Science* **9**(17): 5477–5484.

Rainer, B. and Ganapati, P. (2011). Formal concept analysis: Ranking and prioritization for multi-indicator systems, *Environmental and Ecological Statistics* **5**: 117–133.

Radvansky, M., Sklenar, V. and Snasel, V. (2013). Evaluation of stream data by formal concept analysis, *in* M. Pechenizkiy and M. Wojciechowski (Eds.), *New Trends in Databases and Information Systems*, Advances in Intelligent Systems and Computing, Vol. 185, Springer, Berlin/Heidelberg pp. 131–140.

Romanov, V., Poluektova, A. and Sergienko, O. (2012). Adaptive EIS with business rules discovered by formal concept analysis, *in* C. Moller and S. Chaudhry (Eds.), *Reconceptualizing Enterprise Information Systems*, Lecture Notes in Business Information Processing, Vol. 105, Springer, Berlin/Heidelberg, pp. 105–117.

Rouane, H.M., Huchard, M., Napoli, A. and Valtchev, P. (2013). Relational concept analysis: Mining concept lattices from multi-relational data, *Annals of Mathematics and Artificial Intelligence* **67**(1): 81–108.

Ruairi, de F. (2013). Formal concept analysis via atomic priming, *in* P. Cellier *et al.* (Eds.), *Formal Concept Analysis*, Lecture Notes and Computer Science, Vol. 7880, Springer, Berlin/Heidelberg, pp. 92–108.

Saquer, J. and Deogun, J.S. (2001). Concept approximations based on rough sets and similarity measures, *International Journal of Applied Mathematics and Computer Science* **11**(3): 655–674.

Sarmah, A.K., Hazarika, S.M. and Sinha, S.K. (2015). Formal concept analysis: Current trends and directions, *Artificial Intelligence Review* **44**: 47–86, DOI:10.1007/s10462-013-9404-0.

Sarnovsky, M., Butka, P. and Pocsova, J. (2012). Cloud computing as a platform for distributed fuzzy FCA approach in data analysis, *Proceedings of the IEEE 16th International Conference on Intelligent Engineering Systems, Lisbon, Portugal*, pp. 291–296.

Sawase, K., Nobuhara, H. and Bede, B. (2009). Visualizing huge image databases by formal concept analysis, *Studies in Computational Intelligence* **182**: 291–296.

Sebastien, N., Fabien, P., Lotfi, L. and Rosine, C. (2013). The agree concept lattice for multidimensional database analysis, *in* P. Valtchev and R. Jäschke (Eds.), *Formal Concept Analysis*, Lecture Notes and Computer Science, Vol. 6628, Springer, Berlin/Heidelberg, pp. 219–234.

Senatore, S. and Pasi, G. (2013). Lattice navigation for collaborative filtering by means of (fuzzy) formal concept analysis, *Proceedings of the 28th Annual ACM Symposium on Applied Computing, Coimbra, Portugal*, pp. 920–926.

Shao, M.W., Leung, Y. and Wu, W.Z. (2014). Rule acquisition and complexity reduction in formal decision contexts, *International Journal of Approximate Reasoning* **55**(1): 259–274.

Simiński, K. (2012). Neuro-rough-fuzzy approach for regression modelling from missing data, *International Journal of Applied Mathematics and Computer Science* **22**(2): 461–476, DOI:10.2478/v10006-012-0035-4.

Singh, P.K. and Aswani Kumar, Ch. (2012a). Interval-valued fuzzy graph representation of concept lattice, *Proceedings of the 12th ISDA, Kochi, India*, pp. 604–609.

Singh, P.K. and Aswani Kumar, Ch. (2012b). A method for decomposition of fuzzy formal context, *Procedia Engineering* **38**: 1852–1857.

Singh, P.K. and Aswani Kumar, Ch. (2014). Bipolar fuzzy graph representation of concept lattice, *Information Sciences* **288**: 437–448.

Singh, P.K. and Aswani Kumar, Ch. (2015a). A note on computing the crisp order context of a fuzzy formal context for knowledge reduction, *Journal of Information Processing Systems* **11**(2): 184–204.

Singh, P.K. and Aswani Kumar, Ch. (2015b). Analysis of composed contexts through projection, *International Journal of Data Analysis Techniques and Strategies*, (in press).

Singh, P.K., Aswani Kumar, Ch. and Li, J. (2015a). Concepts reduction in formal concept analysis with fuzzy setting using Shannon entropy, *International Journal of Machine Learning and Cybernetics*, DOI: 10.1007/s13042-014-0313-6.

Singh, P.K., Aswani Kumar, Ch. and Jinhai, Li (2015b). Knowledge representation using interval-valued fuzzy formal concept lattice, *Soft Computing*, DOI: 10.1007/s00500-015-1600-1.

Singh, P.K. and Gani, A. (2015). Fuzzy concept lattice reduction using Shannon entropy and Huffman coding, *Journal of Applied Non-Classical Logics* **25**(2): 101–119, DOI: 10.1080/11663081.2015.1039857.

Slezak, D. (2012). Rough sets and FCA-Scalability challenges, *in* F. Domenach *et al.* (Eds.), *Formal Concept Analysis*, Lecture Notes and Computer Science, Vol. 7378, Springer, Berlin/Heidelberg, p. 6.

Spoto, A., Stefanutti, L. and Vidotto, G. (2010). Knowledge space theory, formal concept analysis, and computerized psychological assessment, *Behavior Research Methods* **42**(1): 342–350.

Tadrat, J., Boonjing, V. and Pattaraintakorn, P. (2012). A new similarity measure in formal concept analysis for case-based reasoning, *Expert Systems with Applications* **39**(1): 967–972.

Tang, P., Huia, S.C. and Fong, C.M.A. (2015). A lattice-based approach for chemical structural retrieval, *Engineering Applications of Artificial Intelligence* **39**: 215–222.

Tho, Q.T., Hui, S.C. and Cao, T.H. (2006). Automatic fuzzy ontology generation for semantic web, *IEEE Transactions on Knowledge and Data Engineering* **18**(6): 842–856.

Trabelsi, C., Jelassi, N. and Yahia, S.B. (2012). Scalable mining of frequent tri-concepts from Folksonomies, *in* P.-N. Tan *et al.* (Eds.), *Advances in Knowledge Discovery and Data Mining*, Lecture Notes and Computer Science, Vol. 7302, Springer, Berlin/Heidelberg, pp. 231–242.

Vityaev, E.E., Demin, A.V. and Ponomaryov, D.K. (2012). Probabilistic generalization of formal concepts, *Programming and Computer Software* **38**(5): 219–230.

Wang, T.Z and Xu, H.S. (2011). Constructing domain ontology based on fuzzy set and concept lattice, *Applied Mechanics and Materials* **63–64**: 715–718.

Wang, X. and Li, G. (2012). A similarity measure model based on rough concept lattice, *in* Y. Wu (Ed.), *Software Engineering and Knowledge Engineering: Theory and Practice*, Advances in Intelligent and Soft Computing, Vol. 114, Springer, Berlin/Heidelberg, pp. 99–103.

Wang, Y., Zhang, J. and Xu, H. (2012). The design of data collection methods in wireless sensor networks based on formal concept analysis, *in* D. Jin and S. Lin (Eds.), *Advances in Computer Science and Information Engineering*, Advances in Intelligent and Soft Computing, Vol. 169, Springer, Berlin/Heidelberg, pp. 33–38.

Watmough, M. (2014). Discovering the hidden semantics in enterprise resource planning data through formal concept analysis, *Studies in Computational Intelligence* **495**: 291–314.

Wille, R. (1982). Restructuring lattice theory: An approach based on hierarchies of concepts, *in* I. Rival (Ed.), *Ordered Sets*, Reidel, Dordrecht/Boston, MA, pp. 445–470.

Wu, L., Qiua, D. and Mi, J.S. (2012). Automata theory based on complete residuated lattice-valued logic: Turing machines, *Fuzzy Sets and Systems* **208**(12): 43–66.

Wu, W.Z., Leung, Y. and Mi, J.S. (2009). Granular computing and knowledge reduction in formal contexts, *IEEE Transactions on Knowledge and Data Engineering* **21**(10): 1461–1474.

Xu, B., Frein, R.D., Robson, E. and Foghlu, M.O. (2012). Distributed formal concept analysis algorithms based on an iterative MapReduce framework, *in* F. Domenach *et al.* (Eds.), *Formal Concept Analysis*, Lecture Notes in Computer Science, Vol. 7278, Springer, Berlin/Heidelberg, pp. 292–308.

Xu, W. and Li, W. (2015). Granular computing approach to two-way learning based on formal concept analysis in fuzzy datasets, *IEEE Transactions on Cybernetics* **46**(2): 366–379, DOI: 10.1109/TCYB.2014.2361772.

Yan, H., Zou, C., Liu, J. and Wang, Z. (2015). Formal concept analysis and concept lattice: Perspectives and challenges, *International Journal of Autonomous and Adaptive Communications Systems* **8**(1): 81–96.

Yang, H. (2011). Formal concept analysis based on rough set theory and a construction algorithm of rough concept lattice, *in* H. Deng *et al.* (Eds.), *Emerging Research in Artificial Intelligence and Computational Intelligence*, Communications in Computer and Information Science, Vol. 237, Springer, Berlin/Heidelberg, pp. 239–244.

Yang, H.Z., Yee, L. and Shao, M.W. (2011a). Rule acquisition and attribute reduction in real decision formal contexts, *Soft Computing* **15**(6): 1115–1128.

Yang, Y.P., Shieh, H.M., Tzeng, G.Z., Yen, L. and Shao, M.W. (2011b). Combined rough sets with flow graph and formal concept analysis for business aviation decision-making, *Journal of Intelligent Information Systems* **36**(3): 347–366.

Yao, Y. (2004). A comparative study of formal concept analysis and rough set theory in data analysis, *in* S. Tsumoto *et al.* (Eds.), *Rough Sets and Current Trends in Computing*, Lecture Notes in Artificial Intelligence, Vol. 3066, Springer, Berlin/Heidelberg, pp. 59–66.

Yao, Y., Mi, J., Li, Z. and Xie, B. (2012). The construction of fuzzy concept lattices based on (θ, σ)-fuzzy rough approximation operators, *Fundamenta Informaticae* **111**(1): 33–45.

Yu, J., Hong, W., Li, S., Zhang, T. and Shao, M.W. (2013). A new approach of word sense disambiguation and knowledge discovery of English modal verbs by formal concept analysis, *International Journal of Innovative Computing, Information and Control* **9**(3): 1189–1200.

Zerarga, L. and Djouadi, Y. (2013). Interval-valued fuzzy extension of formal concept analysis for information retrieval, *in* T. Huang *et al.* (Eds.), *Neural Information Processing,* Lecture Notes in Computer Science, Vol. 7663, Springer, Berlin/Heidelberg, pp. 608–615.

Zhai, Y., Li, D. and Qu, K. (2012). Probability fuzzy attribute implications for interval-valued fuzzy set, *International Journal of Database Theory and Application* **5**(4): 95–108.

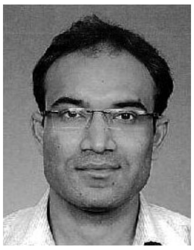
Zhai, Y., Li, D. and Qu, K. (2013). Fuzzy decision implications, *Knowledge-Based Systems* **37**: 230–236.

Zhang, S., Guo, P., Zhang, J., Wang, X. and Pedrycz, W. (2012). A completeness analysis of frequent weighted concept lattices and their algebraic properties, *Data and Knowledge Engineering* **81–82**: 104–117.

Zhang, L., Zhang, H., Shen, X. and Yin, L. (2013a). A bottom-up algorithm of vertical assembling concept lattices, *International Journal of Data Mining and Bioinformatics* **7**(3): 229–244.

Zhang, Z., Du, J. and Yin, L. (2013b). Formal concept analysis approach for data extraction from a limited deep web database, *Journal of Intelligent Information Systems* **41**(2): 1–24.

Zhao, J. and Liu, L. (2011). Construction of concept granule based on rough set and representation of knowledge-based complex system, *Knowledge-Based Systems* **24**(6): 809–815.



Prem Kumar Singh is a postdoc fellow at the Faculty of Computer Science and Information Technology, University of Malaya, Malaysia. He holds a Ph.D. degree in computer science from VIT University, India. His research interests include formal concept analysis and graph theory. He has published more than 15 refereed research papers so far in various international conferences and journals.

Cherukuri Aswani Kumar is a professor at the Network and Information Security Division, School of Information Technology and Engineering, VIT University, Vellore, India. He holds a Ph.D. degree in computer science from VIT University. His current research interests are formal concept analysis and machine intelligence. He has published 70 refereed research papers so far in various conferences and journals. He is a senior member of the ACM and is associated with other professional bodies including the ISC, CSI, ISTE.

Abdullah Gani is a professor at the Faculty of Computer Science and Information Technology, University of Malaya, Malaysia. He holds bachelor and master degrees from the University of Hull, UK, and a Ph.D. from the University of Sheffield, UK. His research interests include machine learning and mobile cloud computing. He has published more than 100 research papers in various conferences and journals.

Received: 25 April 2014
Revised: 11 December 2014
Re-revised: 10 April 2015
Accepted: 26 May 2015