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TRANSIENT FLOW IN GAS NETWORKS: TRAVELING WAVES

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In the context of gas transportation, analytical solutions are helpful for the understanding of the underlying dynamics governed by a system of partial differential equations. We derive traveling wave solutions for the one-dimensional isothermal Euler equations, where an affine linear compressibility factor is used to describe the correlation between density and pressure. We show that, for this compressibility factor model, traveling wave solutions blow up in finite time. We then extend our analysis to networks under appropriate coupling conditions and derive compatibility conditions for the network nodes such that the traveling waves can travel through the nodes. Our result allows us to obtain an explicit solution for a certain optimal boundary control problem for the pipeline flow.

Keywords: traveling wave, isothermal Euler equations, compressibility factor, gas network, blow-up, optimal control.

1. Introduction

The transportation of gas plays an important role in today's energy market. For the control of the gas flow in a pipeline system, a model for the flow is essential. A well-established model is given by one-dimensional isothermal Euler equations, a hyperbolic system of two partial differential equations (PDEs) (see Osiadacz, 1984; 1989; Banda *et al.*, 2006). In the case of ideal gas the correlation between pressure and density is given by the constant speed of sound. This is not true in the more realistic case of the so-called real gas, where the speed of sound is not constant. Instead, a pressure dependent compressibility factor (the so-called z-factor) appears in the model. Like de Almeida *et al.* (2014) or Starling and Savidge (1992), we consider a compressibility factor that depends on the pressure in an affine linear way.

For the understanding of the PDE model and for the evaluation of discretizations, analytical solutions of the PDEs are extremely useful. The steady state solutions on gas networks are already available; see the works of Gugat *et al.* for the case of ideal gas (2015) and for an affine linear z-factor (2016). In the application of gas networks, the transient solutions are of particular interest (see Osiadacz, 1984). In this paper we provide analytical transient solutions for the PDE model. To be precise, we construct solutions that move with constant speed, that is we study traveling waves. It is well known that for many PDE systems, traveling waves can be constructed, for example for the Korteweg–de Vries equation (Whitham, 1974). Traveling waves on networks have already been considered for various models (see, e.g., Caputo and Dutykh, 2014; Mugnolo *et al.*, 2014). Our result allows us to obtain an explicit solution for a certain optimal boundary control problem for the pipeline flow. This is useful not only for the understanding of the system, but also to test the performance of numerical schemes.

It turns out that the isothermal Euler equations with affine linear z-factor generate traveling waves that blow up at a finite critical time. The reason is that as the wave is traveling, the pressure is strictly increasing until it reaches a critical value where the expression for the z-factor has a root. At this point the model no longer makes sense physically and the density of the solution blows up to infinity. Note that while the solution blows up after a finite time, depending on the initial state, the time interval where the solution exists can be arbitrarily long. Thus, depending on the parameter settings, we provide analytical reference solutions for arbitrarily long time intervals. In fact, in the operation of gas pipeline networks there are usually upper bounds $\bar{v} \in (0, \infty)$ on the admissible velocities v, that is, state constraints

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$|v| \leq \bar{v}$ have to be valid everywhere at all times, for example, in order to avoid noise pollution by vibrating pipes. As $\bar{v} \to 0$, the blow up time for the traveling waves tends to infinity.

Interestingly, our construction shows that for all velocities $v \in (-\infty, \infty)$, waves that travel with the velocity v exist. This is due to the nonlinearity of the model and in contrast, for example, to the linear wave equation where the wave speed is a unique fixed constant. Let us also note that before the blow up time, the constructed solutions are smooth. For the case of ideal gas, global smooth product solutions and traveling waves that do not blow up are presented in (Gugat and Ulbrich, 2017). This is in contrast to the blow up after finite time that occurs for the real gas model that is considered in this paper.

We also consider the solutions on networks under standard coupling conditions. We establish compatibility conditions for the network data that allow us to extend the traveling waves to the network.

This paper is structured as follows. In Section 2 we state the model of the isothermal Euler equations and introduce a nonconstant compressibility factor. In Section 3 we derive analytical traveling wave solutions and show that they blow up in finite time. In Remark 2, we present the solution of a certain optimal boundary control problem. The extension to networks by standard coupling conditions is described in Section 4. We consider a network with three edges for which we obtain compatibility conditions on the network data. We also consider a more general case of a star-shaped network with N edges and discuss networks with cycles. At the end of the paper, we present an example that fulfills these conditions and illustrates the behavior of the solution.

2. Isothermal Euler equations with friction

The isothermal gas flow through a pipeline can be described by the conservation of mass and the balance of momentum. Together they form a quasilinear system of hyperbolic balance laws known as the isothermal Euler equations with friction. Let a length $L > 0$ be given. For $t > 0$, $x \in [0, L]$, we have

$$\begin{cases} \partial_t \rho + \partial_x q = 0, \\ \partial_t q + \partial_x \left(p + \dfrac{q^2}{\rho} \right) = -\dfrac{1}{2}\theta \dfrac{q|q|}{\rho}. \end{cases} \quad \text{(ISO)}$$

There are three functions of time and space: the *mass flow* per cross sectional area q, the *density* ρ and the *pressure* p. The constant $\theta := \lambda/D$ is defined through the *friction factor* $\lambda > 0$ and the *pipe diameter* $D > 0$. We model the compressibility factor as suggested by the American Gas Association (Starling and Savidge, 1992) using

$$z(p) := 1 + \alpha p, \quad \text{(AGA)}$$

where $\alpha < 0$ constitutes a negative constant. It is sufficiently accurate within the network operating range (de Almeida *et al.*, 2014). We use the state equation for real gas given by

$$p = \hat{T}\rho z(p), \quad (1)$$

where $\hat{T} := RT$ with the specific gas constant $R > 0$ and the temperature $T > 0$.

Liu *et al.* (2005) proposed a method for the detection of leaks in gas pipelines that was based upon a semilinear model for the flow that is obtained from (ISO) by deleting the term q^2/ρ. Note that (ISO) is similar to the Saint-Venant equations for the flow in open channels (see Dos Santos Martins *et al.*, 2012). The stability of an irrigation canal system was studied by Bounit (2003).

3. Traveling wave solutions

In this section, we present traveling wave solutions for the isothermal Euler equations. The following theorem gives explicitly solutions where the speed $v := q/\rho$ is constant in time and space. In the representation, the Lambert W function W_0 appears that has been introduced by Lambert (1758) as the inverse function of the function $f : (-1, \infty) \mapsto (-1/e, \infty)$, $x \mapsto x \exp(x)$ (see also Corless *et al.*, 1996; Veberic, 2010).

Theorem 1. (Traveling waves) *Choose a speed $v \neq 0$. If $v > 0$, choose a real constant $C > 1$ and if $v < 0$, choose*

$$C > 1 + \frac{\theta v^2}{2\hat{T}}L.$$

Define the critical time

$$t_{crit} = \begin{cases} \dfrac{2\hat{T}}{\theta|v|^3}(C - 1) & \text{if } v > 0, \\[2mm] \dfrac{2\hat{T}}{\theta|v|^3}(C - 1) + \dfrac{L}{v} & \text{if } v < 0. \end{cases} \quad (2)$$

For $x \in [0, L]$ and $t \in [0, t_{crit})$ define

$$g(t, x) := W_0\left(-\exp\left(\frac{\theta|v|^3}{2\hat{T}}t - \frac{\theta v|v|}{2\hat{T}}x - C\right)\right). \quad (3)$$

For $\alpha < 0$ as in (AGA) consider the initial boundary value problem with the partial differential equation (ISO), *the initial conditions*

$$\rho(0, x) = \frac{1}{\alpha\hat{T}}\frac{g(0, x)}{1 + g(0, x)} \quad \text{for } x \in [0, L], \quad \text{(IC1)}$$

$$q(0, x) = \frac{v}{\alpha\hat{T}}\frac{g(0, x)}{1 + g(0, x)} \quad \text{for } x \in [0, L], \quad \text{(IC2)}$$

and the boundary conditions

$$\rho(t,0) = \frac{1}{\alpha\,\hat{T}}\,\frac{g(t,0)}{1+g(t,0)} \quad for\ t \in (0,\,t_{crit}), \quad \text{(BC1)}$$

$$q(t,L) = \frac{v}{\alpha\,\hat{T}}\,\frac{g(t,L)}{1+g(t,L)} \quad for\ t \in (0,\,t_{crit}),. \quad \text{(BC2)}$$

Then for $(t,\,x) \in [0,\,t_{crit}] \times [0,\,L]$,

$$\rho(t,x) = \frac{1}{\alpha\,\hat{T}}\,\frac{g(t,x)}{1+g(t,x)}, \tag{4}$$

$$q(t,x) = \frac{v}{\alpha\,\hat{T}}\,\frac{g(t,x)}{1+g(t,x)}, \tag{5}$$

is a classical solution of the initial boundary value problem (ISO), (IC1), (IC2), *and* (BC1), (BC2). *Moreover, it is the unique traveling wave solution that satisfies* (IC1), (IC2) *and the partial differential equation* (ISO). *We have*

$$p(t,x) = \frac{1}{\alpha}\,g(t,x). \tag{6}$$

Remark 1. Note that the pressure p and the density ρ remain positive as long as the solution does not blow up, since $\alpha < 0$ and g has values in $(-1,0)$.

If $v > 0$, for fixed t the pressure $p(t,\,x)$ as defined in (6) is strictly decreasing along the pipe. Note that the case $v < 0$ becomes symmetric to the case $v > 0$ if the orientation of the interval is switched, that is, $x \in [0,\,L]$ is replaced by $L - x$ and C is replaced by $C + \frac{\theta\,v\,|v|\,L}{2\hat{T}}$.

Theorem 1 implies that for all velocities $v \neq 0$, traveling waves exist. The corresponding solutions for $v = 0$ are steady states with $q = 0$ and constant pressure and density.

Remark 2. (*Optimal boundary control problem*) For a given constant velocity v, consider the optimal boundary control problem

$$\min \int_0^T \int_0^L \left| \frac{q(t,x)}{\rho(t,x)} - v \right|^2 \,\mathrm{d}x\,\mathrm{d}t$$

subject to (ISO), (IC1), (IC2), and the boundary conditions $p(t,\,0) = u_1(t)$ and $p(t,\,L) = u_2(t)$, where the aim is to keep the gas velocity constant.

Suppose that there exist controls u_1 and u_2 such that

$$\frac{q(t,x)}{\rho(t,x)} - v = 0.$$

Then $q = \rho\,v$. Due to the first equation in (ISO) this implies that ρ satisfies the transport equation $\partial_t\rho + v\,\partial_x\rho = 0$. Since $q = \rho\,v$, this implies that q and ρ are a traveling waves solution that satisfies (ISO) and (IC1), (IC2). Theorem 1 states that if $T < t_{crit}$, these conditions already determine uniquely the solution defined in (4) and (5). Hence the unique optimal controls are $u_1(t) = g(t,0)/\alpha$ and $u_2(t) = g(t,L)/\alpha$ with g as defined in (3).

Proof. (Theorem 1) Let

$$z = \frac{\theta|v|^3}{2\hat{T}}t - \frac{\theta v|v|}{2\hat{T}}x.$$

The domain of W_0 is $(-1/e, \infty)$. This means in (3) we need

$$-\exp(z - C) > -\exp(-1)$$

or

$$z - C < -1.$$

Setting $t = 0$, we see that

$$z = -\frac{\theta v|v|}{2\hat{T}}x$$

and deduce that $C > 1 + z$ for all $x \in [0,\,L]$ is equivalent to the conditions for the choice of C stated in Theorem 1. This ensures that the function g is well defined for $t = 0$ and $x \in [0,\,L]$. Thus the solutions are well-defined locally around $t = 0$. In fact, they are well defined as long as $g(x,t) > -1$. However, due to the definition (2) of t_{crit} for $t < t_{crit}$ we have $z - C < -1$, hence $-\exp(z - C) > -\exp(-1)$ and thus $W_0(-\exp(z - C)) > -1$. Thus the solutions are well defined for $t \in [0,\,t_{crit})$.

It is obvious that the functions given by (4) and (5) satisfy the initial conditions (IC1), (IC2) and the boundary conditions (BC1), (BC2). It remains to verify that they also satisfy (ISO). The definition of g yields

$$\partial_t g(t,\,x) + v\,\partial_x g(t,\,x) = 0. \tag{7}$$

Since $q = v\rho$, this implies that the first equation in (ISO) holds. Now we check that also the second equation in (ISO) holds. We substitute the pressure and its space derivative by a function of the density via the state equation (1) and obtain

$$p = \frac{\hat{T}\rho}{1 - \alpha\hat{T}\rho}, \quad \partial_x p = \frac{\hat{T}\partial_x\rho}{(1 - \alpha\hat{T}\rho)^2}.$$

Since $q/\rho = v$ is constant, with the second equation in (ISO) (the balance of momentum), this leads to

$$\partial_t q + \partial_x\left(p + \frac{q^2}{\rho}\right)$$

$$= \partial_t q + \left[\frac{\hat{T}}{(1 - \alpha\hat{T}\rho)^2} - \frac{q^2}{\rho^2}\right]\partial_x\rho + \frac{2q}{\rho}\partial_x q$$

$$= \partial_t q + \left[\frac{\hat{T}}{(1 - \alpha\hat{T}\rho)^2} - v^2\right]\partial_x\rho + 2\,v\,\partial_x q.$$

Since $q = v\,\rho$, we have $\partial_x q = v\,\partial_x\rho$ and obtain

$$\partial_t q + \partial_x\left(p + \frac{q^2}{\rho}\right)$$

$$= v\,\partial_t\rho + \left[\frac{\hat{T}}{(1 - \alpha\hat{T}\rho)^2} + v^2\right]\partial_x\rho.$$

Due to (4) and (7), in terms of g this yields

$$\partial_t q + \partial_x \left(p + \frac{q^2}{\rho} \right)$$

$$= -v^2 \frac{1}{\alpha \hat{T}} \frac{1}{(1+g)^2} \partial_x g$$

$$+ \left[\hat{T}(1+g)^2 + v^2 \right] \frac{1}{\alpha \hat{T}} \frac{1}{(1+g)^2} \partial_x g$$

$$= \frac{1}{\alpha} \partial_x g.$$

Hence the second equation in (ISO) holds if g satisfies the ordinary differential equation

$$\frac{1}{\alpha} \partial_x g = -\frac{1}{2} \theta v \, |v| \, \frac{1}{\alpha \hat{T}} \frac{g}{1+g}.$$

After multiplication by α, separation of variables yields

$$\int \left(1 + \frac{1}{g} \right) \mathrm{d}g = D - \frac{\theta v \, |v|}{2 \hat{T}} x$$

with a real constant D. By integration we obtain

$$g + \ln(|g|) = D - \frac{1}{2} \theta v \, |v| \, \frac{1}{\hat{T}} x.$$

By applying the exponential function this yields

$$|g| \exp(g) = \exp \left(D - \frac{\theta v \, |v|}{2 \hat{T}} x \right).$$

Since $g < 0$, due to the definition of the Lambert W function, the function g as defined in (3) satisfies this equation.

For initial and boundary data with a sufficiently small C^1–norm, the uniqueness of the solution of the initial boundary value problem follows from the theory of semi-global classical solutions (Li, 2010).

Suppose that a classical traveling wave solution of (ISO) is given by $\rho(t, x) = f_1(v \, t - x)$ and $q(t, x) = f_2(v \, t - x)$. Then f_1 and f_2 are defined on the interval $[-L, v \, t_{\mathrm{crit}})$. The first equation in (ISO) implies $v \, f_1'(s) = f_2'(s)$ for all $s \in [-L, v \, t_{\mathrm{crit}})$. The initial conditions (IC1), (IC2) imply that $f_2(x) = v \, f_1(x)$ for all $x \in [-L, 0]$. Hence $f_2(0) = v \, f_1(0)$. Since $f_2' = v \, f_1'$ on $[0, t_{\mathrm{crit}})$, this yields $f_2 = v \, f_1$ on $[0, v \, t_{\mathrm{crit}})$. By (IC1) and (IC2), the values of f_1 and f_2 on the interval $[-L, 0]$ are uniquely determined. Since $f_2 = v \, f_1$, the second equation in (ISO) yields the ordinary differential equation

$$f_1' = \frac{1}{2 \hat{T}} \theta \, v \, |v| \, f_1 \, (1 - \alpha \, \hat{T} \, f_1)^2. \qquad (8)$$

The value of $f_1(0)$ is determined by (IC1). Hence f_1 is uniquely determined as the solution to the initial value problem with the corresponding initial condition and the

differential equation (8) on the maximal interval of the existence of this solution, which is $[0, v \, t_{\mathrm{crit}})$. Since $f_2 = v \, f_1$, this implies that the traveling wave solution is uniquely determined by (IC1), (IC2) and (ISO). ∎

The next lemma states that for the model that we consider here all non stationary traveling waves blow up in finite time.

Lemma 1. *Assume that $v \neq 0$. There exists a finite time t_{crit} such that the solution $\rho(t, x)$ as in (4) goes to infinity as t approaches t_{crit} for either $x = 0$ or $x = L$.*

Proof. Clearly, the blow-up occurs for $g(x, t) \to -1$. This happens for

$$-\exp \left(\frac{\theta |v|^3}{2 \hat{T}} t - \frac{\theta v |v|}{2 \hat{T}} x - C \right) \to -\exp(-1).$$

Hence, since the exponential function is injective, we consider

$$\frac{\theta |v|^3}{2 \hat{T}} t - \frac{\theta v |v|}{2 \hat{T}} x - C = -1.$$

Solving for t yields

$$t = \frac{x}{v} + \frac{2 \hat{T}}{\theta \, |v|^3} (C - 1).$$

The critical time t_{crit} is the minimum of the right-hand side over $[0, L]$. Depending on the sign of v, it is attained at either $x = 0$ or $x = L$, and we retain t_{crit} as defined in (2). The choice of C in Theorem 1 implies that we have $t_{\mathrm{crit}} > 0$. ∎

4. Coupling conditions and compatibility conditions for networks

We consider the flow on networks that are given by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with the set of nodes $\mathcal{V}$ and the set of edges $\mathcal{E}$. For a given node $u \in \mathcal{V}$ the set of ingoing edges (where the end L_e of the interval $[0, L_e]$ that corresponds to the edge e is at the node u) is denoted by $\mathcal{E}_+(u)$ and the set of outgoing edges (where the end 0 is at the node u) is denoted by $\mathcal{E}_-(u)$. On each edge e we denote the corresponding quantities by an index e. For each node $u \in \mathcal{V}$, and $t > 0$ we have the *conservation of mass*, i.e.,

$$\sum_{e \in \mathcal{E}_+(u)} D_e^2 \, q_e(t, L_e) = \sum_{f \in \mathcal{E}_-(u)} D_f^2 \, q_f(t, 0). \qquad (9)$$

Furthermore, for every node u we have the *continuity of pressure* for all $e, f \in \mathcal{E}_+(u)$, $g, h \in \mathcal{E}_-(u)$,

$$p_e(t, L_e) = p_f(t, L_f) = p_g(t, 0) = p_h(t, 0). \qquad (10)$$

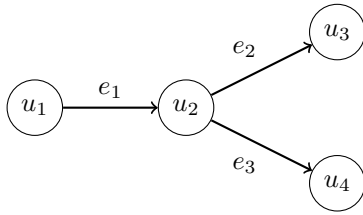Condition (9) is also known as Kirchhoff's law (cf. Kirchhoff, 1847).

Fig. 1. Three-pipe network.

Now our aim is to develop nontrivial algebraic compatibility conditions for a small network that allow the traveling waves to travel through the nodes without a change in their structure such that the coupling conditions (9) and (10) hold.

4.1. n-Pointed star. Consider a star graph with one ingoing and $n - 1$ outgoing edges as in Fig. 1 for $n = 3$ and in Fig. 2 for $n = 8$. The index $i = 1, 2, \ldots, n$ refers to the edge e_i and is used on constants and functions. The ingoing edge is denoted by the index 1. We assume that the constants α, R, T are equal on each edge. The following lemma contains conditions on the system parameters and the wave speeds that allow the waves to travel through the nodes.
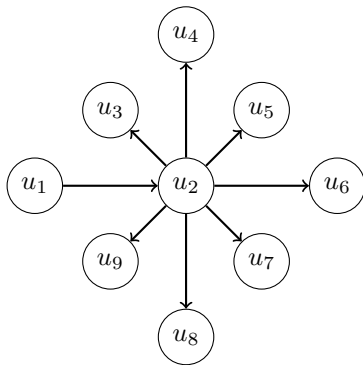

Fig. 2. Eight-pointed star.

Lemma 2. *The traveling wave solutions on the n-pointed star with one ingoing edge fulfill the Kirchhoff conditions* (9) *and* (10) *if and only if the following three conditions hold:*

(a) $\quad C_1 + \dfrac{\theta_1 v_1 |v_1|}{2\hat{T}} L_1 = C_i \quad$ *for $i = 2, \ldots, n$,*

(b) $\quad \theta_1 |v_1|^3 = \theta_i |v_i|^3 \quad$ *for $i = 2, \ldots, n$,*

(c) $\quad \text{sign}(v_1) \dfrac{D_1^2}{\sqrt[3]{\theta_1}} = \displaystyle\sum_{i=2}^{n} \text{sign}(v_i) \dfrac{D_i^2}{\sqrt[3]{\theta_i}}.$

Proof. The continuity condition (10) at the middle node states that

$$p_1(t, L_1) = p_i(t, 0), \quad \forall i = 2, \ldots, n.$$

Hence, the injectivity of the Lambert W function and the exponential function implies

$$\frac{\theta_1 |v_1|^3}{2\hat{T}} t - \frac{\theta_1 v_1 |v_1|}{2\hat{T}} L_1 - C_1$$
$$= \frac{\theta_i |v_i|^3}{2\hat{T}} t - C_i, \quad \forall i = 2, \ldots, n.$$

For $t = 0$, this leads to (a). In turn, $t > 0$ yields (b). The first Kirchhoff condition (9) implies

$$D_1^2 v_1 = \sum_{i=2}^{n} D_i^2 v_i, \tag{11}$$

which can be rewritten with (b) to get (c).

Also, (a)–(c) imply the Kirchhoff conditions (9) and (10). By multiplying (b) with t and adding the result to (a), we retain (10). From (b) and (c) we can get (11) and by multiplying with the density we arrive at (9). ∎

Remark 3. Condition (a) ensures the continuity of pressure at $t = 0$. Condition (b) controls the growth of the solution with time such that the continuity is preserved. Equation (c) ensures the conservation of mass. Note that conditions (a), (b) and (c) are independent since the C_i's only appear in (a) and the D_i's only appear in (c).

Example 1. We choose $\hat{T} = 1, \alpha = -0.1, \theta_1 = 0.1^3, \theta_2 = 0.2^3, \theta_3 = 0.1^3, L_1 = 8, L_2 = 10, L_3 = 12$. Inserting the friction values in (c) yields $D_1^2 = 0.5 D_2^2 + D_3^2$. The choice $D_1 = 1, D_2 = 1, D_2 = 1/\sqrt{2}$ is valid. Furthermore, we use $v_1 = 10$, which gives, by (b),

$$|v_2| = \sqrt[3]{\frac{\theta_1}{\theta_2}} |v_1| = 5, \quad |v_3| = \sqrt[3]{\frac{\theta_1}{\theta_3}} |v_1| = 10.$$

For $C_1 = 1$ condition (a) requires C_2 and C_3 to be

$$C_3 = C_2 = C_1 + \frac{\theta_1 v_1 |v_1|}{2\hat{T}} L_1 = 1.4.$$

The critical time on each edge is given by

$$t_{\text{crit}}^i = \frac{2\hat{T}}{\theta_i |v_i^3|} (C_i - 1);$$

see (2). For our data we calculate $t_{\text{crit}}^{e_1} = 2, t_{\text{crit}}^{e_2} = 11.2, t_{\text{crit}}^{e_3} = 2.8$. This means the state will blow up first on edge e_1 in node u_1.

Figure 6 shows the three edges of the graph depicted at the x-y-plane in different shades of gray. The z-axis corresponds to the value of the density in a logarithmic

scale. The flow is just the density scaled by the constants v_i. Therefore, it has a similar behavior, but it is not continuous through the node u_2. The pictures are snapshots at different times up to 95% of the calculated critical time $t_{\text{crit}}^{e_1} = 2$. ♦

Remark 4. It is a feature of the real gas model (1) that, while the density blows up, the pressure does not. This can be seen directly from (6) with the knowledge that the W_0 function has a real limit at the point -1. Compare Fig. 6 to Fig. 5, which shows the pressure solution for Example 1. Although in principle the behavior seems similar, the blow up at node u_1 does not occur. However, remember that the function $g/(1+g)$ is only defined up to the critical time. Note that (AGA) only makes sense for $p \in [0, -1/\alpha)$. As ρ blows up to infinity, p converges to $-1/\alpha$. In the example we have $-1/\alpha = 10$.

A slight generalization of Lemma 2 can be made by not imposing the number of ingoing edges to be restricted to one. Let us consider now a star network of $1 \le k < n$ ingoing and $n - k$ outgoing edges with respect to the center node as in Fig. 3. The proof is analogous to that for Lemma 2.
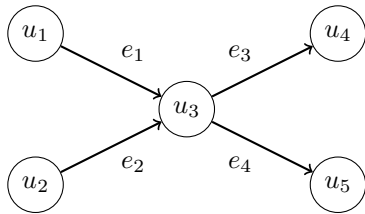

Fig. 3. Four-pointed star.

Corollary 1. *The traveling wave solutions on the n-pointed star with $1 \le k < n$ ingoing edges fulfill the Kirchhoff conditions* (9) *and* (10) *if and only if the following three conditions hold:*

(a) $C_i + \dfrac{\theta_i v_i |v_i|}{2\hat{T}} L_i = C_j$ *for* $i = 1, \ldots, k$

and $j = k + 1, \ldots, n$;

(b) $\theta_i |v_i^3| = \theta_j |v_j^3|$ *for* $i = 1, \ldots, k$

and $j = k + 1, \ldots, n$;

(c) $\displaystyle\sum_{i=1}^{k} \operatorname{sign}(v_i) \frac{D_i^2}{\sqrt[3]{\theta_i}} = \sum_{j=k+1}^{n-k} \operatorname{sign}(v_j) \frac{D_j^2}{\sqrt[3]{\theta_j}}$.

4.2. Networks with cycles. Now we will extend our analysis to networks containing cycles. We consider the example that is depicted in Fig. 4. We state the important observation that there are no traveling wave solutions leading to circular flows.
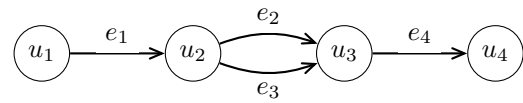

Fig. 4. Network with two parallel pipes.

Lemma 3. *There are no traveling wave solutions of* (ISO) *on the network of Fig. 4 with the Kirchhoff conditions* (9) *and* (10) *that have a nonzero circular flow, that is, there are no traveling wave solutions with*

$$\operatorname{sign}(v_2) = -\operatorname{sign}(v_3). \tag{12}$$

Proof. The compatibility conditions are those of the three-pointed star with center u_1 and the three-pointed star with center u_2. Applying Lemma 2 and condition (a), yields $C_2 = C_3$ for the left star graph while for the right star graph, we have from Corollary 1 and condition (a)

$$C_2 + \frac{\theta_2 v_2 |v_2|}{\hat{T}} = C_3 + \frac{\theta_3 v_3 |v_3|}{\hat{T}}.$$

Therefore, $\theta_2 v_2 |v_2| = \theta_3 v_3 |v_3|$. Because $\theta_2, \theta_3 > 0$, this shows $\operatorname{sign}(v_2) = \operatorname{sign}(v_3)$. Since the last condition is mandatory for the existence of a traveling wave solution on the considered graph, no solution fulfilling (12) can exist with a nonzero circular flow. ∎

Solutions on cycles are therefore only possible if the flow direction for parallel pipes is the same. Then the coupling conditions in each inner node are determined by those of an n-pointed star graph.

5. Conclusion

We have derived explicit analytical traveling wave solutions for the isothermal Euler equations with a nonconstant compressibility factor given by a decreasing affine linear function. We have shown that all traveling wave solutions blow up in finite time if the velocity is nonzero. For zero velocity we obtain constant stationary states. The extension to networks leads to specific compatibility conditions on the data. We have shown that on graphs containing cycles, there can be no traveling wave solutions with circular flow. The analytical solutions provide a useful test for numerical methods for the isothermal Euler equations on networks. The finite time blow-up of traveling waves that occurs in the model for real gas with an affine linear compressibility factor also illustrates the mathematical difficulties that are generated by this model if the pressure becomes too large. In the practical operation of pipeline networks, on account of upper bounds for the admissible pressure, the pressure values where the blow-up occurs are avoided.

We have shown that if the initial state is compatible to the traveling waves, the boundary traces of the traveling
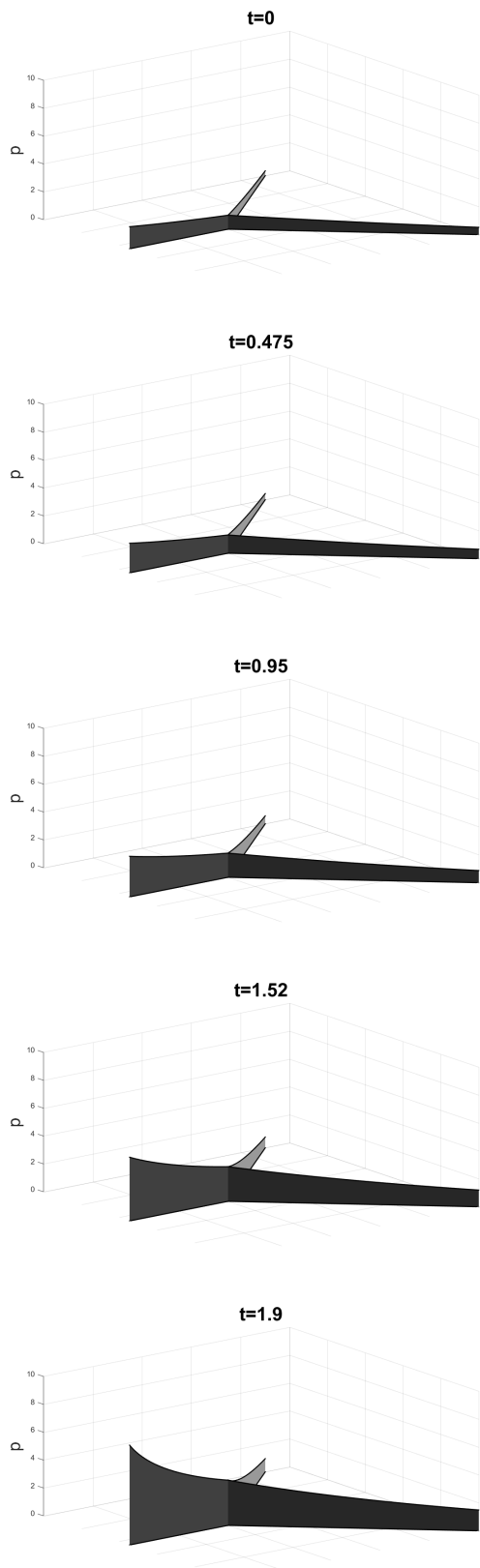
Fig. 5. Pressure in the network at different times. Left edge: e_1, top right edge: e_2, bottom right edge: e_3.
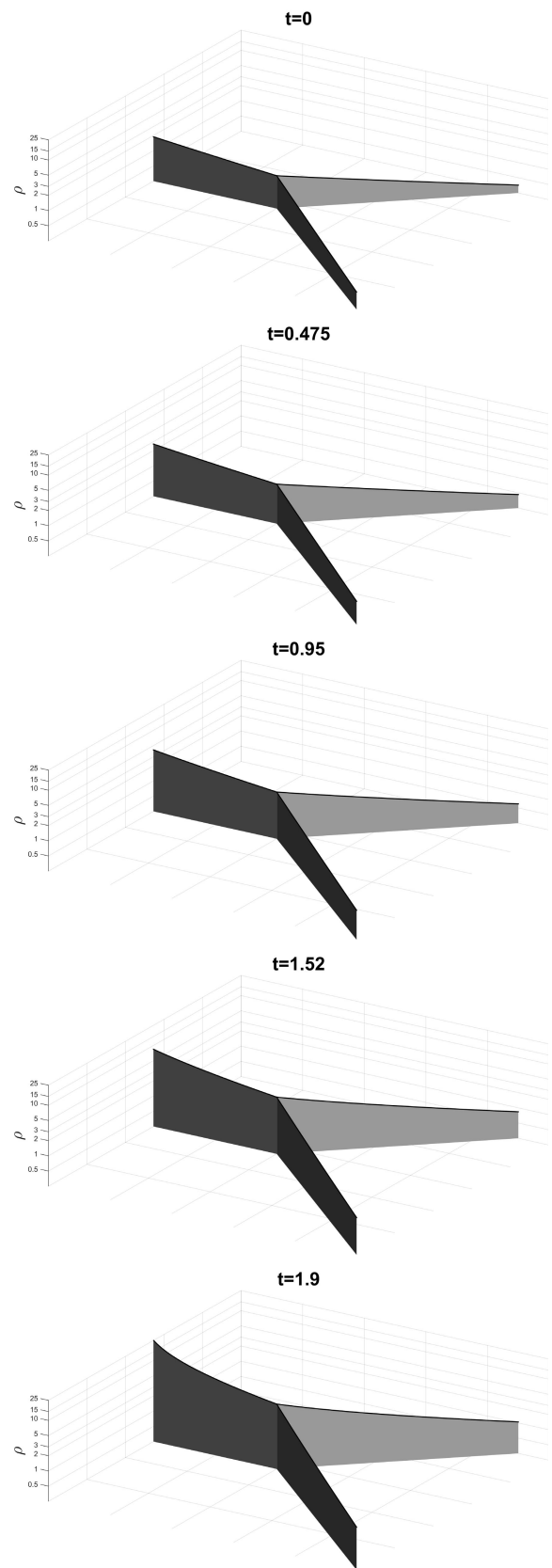
Fig. 6. Density in the network at different times. Left edge: e_1, top right edge: e_2, bottom right edge: e_3. Note that ρ is plotted using a logarithmic scale.

waves appear as the unique optimal controls that solve optimal boundary control problems where the aim is to keep the gas velocity constant. If the compatibility conditions that allow the traveling waves to travel through the nodes are satisfied, the situation on networks is similar.

Acknowledgment

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